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A Relationship Between the Ahmed Integral and the Lee Integral

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Abstract

This note clearly establishes the relationship between two challenging integrals: the Ahmed integral and the Lee integral. We thus contribute to a deeper understanding of complex integral situations.

1 Introduction

Integral formulas play a fundamental role in analysis and applied mathematics. A comprehensive collection of these formulas can be found in [2]. However, certain notable evaluations, such as the family of integrals introduced by S. Lee in [3], fall outside its scope. In this note, we highlight two prominent examples: the Ahmed integral and the Lee integral. Both exhibit rich analytical properties centered on the arctangent function and have garnered independent attention in the literature.

- The Ahmed integral is given by

$$\int_0^1 \frac{\arctan\left(\sqrt{\frac{2+x^2}{4+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}dx = \frac{\pi^2}{30}.$$

  See [1].

- The Lee integral is given by

$$\int_0^{1/\sqrt{3}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}dx = \frac{\pi^2}{20}.$$

  This result follows directly from [3, Theorem 4], as it belongs to a broader family of integrals studied therein.

The aim of this note is to establish a direct relationship between these two integrals via a sequence of carefully chosen changes of variables and a key property of the arctangent function. The main result and its detailed proof are presented below.

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2 Main Result

Theorem 2.1. *The Lee integral can be used to calculate the Ahmed integral and vice versa, as follows:*

$$\int_0^1 \frac{\arctan\left(\sqrt{\frac{2+x^2}{4+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}dx + \int_0^{1/\sqrt{3}} \frac{\arctan(\sqrt{2-x^2})}{1+x^2}dx = \frac{\pi^2}{12}.$$

Proof. Let us consider the Ahmed integral, as follows:

$$I = \int_0^1 \frac{\arctan\left(\sqrt{\frac{2+x^2}{4+x^2}}\right)}{(1+x^2)\sqrt{2+x^2}}dx.$$

Making the change of variables $2+x^2 = u^2$, so that $x^2 = u^2-2$, $dx = (u/\sqrt{u^2-2})du$, and

$$1+x^2 = u^2-1, \quad \sqrt{2+x^2} = u,$$

we obtain

$$I = \int_{\sqrt{2}}^{\sqrt{3}} \frac{\arctan\left(\frac{u}{\sqrt{2+u^2}}\right)}{(u^2-1)\sqrt{u^2-2}}du.$$

Now, making the change of variables $u = \sqrt{2}\sec\theta$, so that $\sqrt{u^2-2} = \sqrt{2}\tan\theta$ and $du = \sqrt{2}\sec\theta\tan\theta d\theta$, we get

$$I = \int_0^{\arccos\left(\sqrt{2/3}\right)} \frac{\arctan\left(\frac{1}{\sqrt{2-\sin^2\theta}}\right)}{1+\sin^2\theta}\cos\theta d\theta.$$

Subsequently, making the change of variables $t = \sin\theta$, so that $dt = \cos\theta d\theta$, the limits become 0 to $1/\sqrt{3}$, and hence

$$I = \int_0^{1/\sqrt{3}} \frac{\arctan\left(\frac{1}{\sqrt{2-t^2}}\right)}{1+t^2}dt.$$

Using the identity $\arctan(1/x) = \pi/2 - \arctan(x)$ for $x > 0$, we obtain

$$I = \frac{\pi}{2}\int_0^{1/\sqrt{3}} \frac{dt}{1+t^2} - \int_0^{1/\sqrt{3}} \frac{\arctan\left(\sqrt{2-t^2}\right)}{1+t^2}dt.$$

Since

$$\int_0^{1/\sqrt{3}} \frac{dt}{1+t^2} = \arctan t\Big|_0^{1/\sqrt{3}} = \frac{\pi}{6},$$

we get

$$I = \frac{\pi^2}{12} - J,$$

where

$$J = \int_0^{1/\sqrt{3}} \frac{\arctan(\sqrt{2 - t^2})}{1 + t^2} dt.$$

Recognizing this expression as the Lee integral completes the proof of the theorem. □

The non-trivial sequence of changes of variables employed in the proof highlights the deep underlying relationship between the Ahmed integral and the Lee integral.

3 Conclusion

In this note, we have established a direct relationship between the Ahmed integral and the Lee integral using classical calculus and functional techniques. This connection deepens our understanding of their underlying structure and offers a unifying perspective for the study of related integrals and future analytical developments.

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