

# A Collection of Geometric Nirenberg-Gagliardo Type Inequalities with Extra Terms and Applications

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## Abstract

Functional interpolation inequalities play a huge role in Mathematical Analysis and these inequalities are widely used in other parts of hard analysis such as Partial Differential Equations. In this article, geometric functional Nirenberg-Gagliardo type inequalities are obtained and rigorously proved. The norms that appear depend on the geometry of the domain. Additionally, applications of the Nirenberg-Gagliardo type inequalities are given to obtain estimates involving mollifier type functions which are ubiquitous in Analysis and Partial Differential Equations.

## 1 Introduction

Functional interpolation inequalities play a critical role in Analysis and especially in the hard type Analysis that requires precision and obtaining hard estimates. The work presented in this article is motivated by the works of one of the greatest analysts of the 20th century, Louis Nirenberg [19]. Louis Nirenberg has discovered and worked on some interpolation inequalities during the end of 50s involving the norm of a function  $u$  and the product of the norm or the function with the norm of the first derivative, working in specific function spaces [1]. These types of inequalities are a cornerstone in modern analysis. Louis Nirenberg presented them in a seminal work and in the context of Elliptic type Partial Differential equations, as these inequalities are very useful for obtaining a priori estimates and finding properties of the solution and its higher order derivatives [5], [6], [7], [17]. There are various works in the literature about Nirenberg interpolation inequalities. Around the same period, an Italian analyst named Emilio Gagliardo discovered the interpolation estimates independently [2], and are now referred to as Nirenberg-Gagliardo inequalities [3]. There are various works in literature where the authors derive variants of Nirenberg-Gagliardo inequalities [8], [9], [10], [11], [12]. This work is novel and differs from other works in the following sense: In this article, the author derives Nirenberg-Gagliardo type estimates that depend on the geometry of the domain, and have extra geometric terms and in some cases there are correction terms such as energy type integrals. Furthermore, if the domain is fixed, the constants that

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Received: April 27, 2026; Revised & Accepted: June 22, 2026; Published online: June 29, 2026

2020 Mathematics Subject Classification: 26D10, 46E35, 35A23, 46B70.

Keywords and phrases: geometric analysis, classical analysis, interpolation, functional inequalities, inequalities involving mollifier functions.

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appear in front of the norms can be specified. In most works, specifying the constant  $C$  is not evident. Also the derivation of the inequalities relies on various decompositions of the integrand function and then applying fundamental functional inequalities to yield the results. Another aspect that needs to be indicated is that the Nirenberg-Gagliardo estimates derived in this work are then employed to derive inequalities involving mollifier functions which are commonly used in Partial Differential Equations. More details about such functions can be found in the works of Kurt Friedrichs [14], [15], Haim Brezis [13] and other works in the literature [16]. The inequalities presented here involving mollifier functions are considered new in the literature to the best of the author's knowledge. Consequently, the work is organized as follows: Initially, there is a preliminary section explaining the novelty of the Nirenberg-Gagliardo inequalities obtained in this article. Then, the first part of the article presents the main Nirenberg-Gagliardo type inequalities with rigorous proofs included. The 2nd part of the article contains detailed commentary on the derived inequalities. The 3rd part of the article employs the Nirenberg-Gagliardo type inequalities to derive estimates involving mollifier type functions. The last part of the article presents the graphs of the estimates involving mollifier functions to verify the theoretical results.

## 2 Novelty of the Nirenberg-Gagliardo Geometric Type Inequalities

The inequalities obtained in this article are structurally new in the literature, compared to the inequalities given L. Nirenberg [1], [7] and by Emilio Gagliardo [2]. The inequalities given by the previous authors consider a fixed domain, and the coefficients in the norms at the right hand side of the inequality are difficult and tedious to specify. Additionally, the authors do not consider extra energy terms arising at the right hand side of the inequality. Also, considering the [11], the authors do not consider any energy extra terms in the inequality. Also, taking into notice the work [8], the authors do not consider extra energy terms, the domain is fixed and they incorporate Fourier Analysis tools in their work. Concerning the work [10], the authors study various Nirenberg-Gagliardo type inequalities in various spaces such as Lorenz, BMO and the domain considered is fixed ( $N$  dimensional Euclidean space), the structure of the inequalities is different, and extra terms are not embedded at the right hand side of the inequality. Also, checking classical works of Brezis [3], [13], no geometry dependence is discussed and no extra functional terms are involved with geometric dependence. The inequalities presented in this article also rely on decomposition of the integrand function in various ways and then employing analytical inequalities which are standard in the literature of the mathematical analysis. All the inequalities presented here are structurally new involving geometry dependence and extra energy terms. However, the most interesting ones are (2), (3), (4), (5), (6), (7). All the inequalities presented here are complementary to the original Nirenberg-Gagliardo inequalities and the geometry dependence and geometric terms could be useful in geometric analysis where the region of integration depends on spatial coordinates.

### 3 Main Geometric Nirenberg-Gagliardo Type Inequalities with Rigorous Proofs

In this section, the main geometric Nirenberg type inequalities are introduced and they are rigorously proved. The domain of integration is geometry dependent where the bounds of integration are functions of space coordinates, and the function  $u$  is assumed to be smooth, well behaved function belonging to  $C^\infty$  class of functions. Furthermore, the norms appearing in the inequalities are geometry dependent and we assume that the norms are well behaved and exist in the sense of a closed formula. The inequalities are proved using Functional Analysis machinery, avoiding monotonicity, series expansion or convexity properties. The approach used is the following: The first step is doing appropriate integral decomposition, by balancing the integrand function. Then there is heavy reliance on Schwarz, Hölder and Minkowski functional inequalities to derive the interpolation inequalities. The triangle inequality is also important tool to split the integrand function.

**Theorem 3.1.**

$$\begin{aligned} \|u\|_{L_1((a(x),b(x)))} &\leq 2\|u\|_{L_2((a(x),b(x)))}\|u'\|_{L_2((a(x),b(x)))} + \|u\|_{L_2((a(x),b(x)))}\sqrt{b(x) - a(x)}, \\ u &= u(x_1), \quad u' = u'(x_1) \\ 0 &\leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x),b(x))). \end{aligned} \tag{1}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned} \|u\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u| \, dx_1 \\ &= \int_{a(x)}^{b(x)} |u(u' + 1 - u')| \, dx_1 \\ &= \int_{a(x)}^{b(x)} |uu' + u(1 - u')| \, dx_1 \\ \text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |uu'| \, dx_1 + \int_{a(x)}^{b(x)} |u(1 - u')| \, dx_1 \\ \text{Schwarz inequality} &\leq \left( \int_{a(x)}^{b(x)} u^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} u'^2 \, dx_1 \right)^{1/2} + \\ &\quad + \left( \int_{a(x)}^{b(x)} u^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (1 - u')^2 \, dx_1 \right)^{1/2} \\ &= \|u\|_{L_2((a(x),b(x)))}\|u'\|_{L_2((a(x),b(x)))} + \|u\|_{L_2((a(x),b(x)))} \|1 - u'\|_{L_2((a(x),b(x)))} \\ \text{Minkowski inequality} &\leq 2\|u\|_{L_2((a(x),b(x)))}\|u'\|_{L_2((a(x),b(x)))} + \|u\|_{L_2((a(x),b(x)))}\sqrt{b(x) - a(x)}. \end{aligned}$$

□

**Theorem 3.2.**

$$\begin{aligned} \|u\|_{L_1((a(x),b(x)))} &\leq \frac{\sqrt{3}}{3} \sqrt{b(x)^3 - a(x)^3} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\ &+ \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (x_1 u'(x_1))^2 dx_1 \right)^{1/2} \right), \\ &u = u(x_1), \quad u' = u'(x_1) \\ &0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x), b(x))). \end{aligned} \quad (2)$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned} \|u\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u| dx_1 \\ &= \int_{a(x)}^{b(x)} |u(x_1 u' + 1 - x_1 u')| dx_1 \\ &= \int_{a(x)}^{b(x)} |x_1 u u' + u(1 - x_1 u')| dx_1 \\ \text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |x_1 u u'| dx_1 + \int_{a(x)}^{b(x)} |u(1 - x_1 u')| dx_1 \\ \text{Schwarz and Hölder inequality} &\leq \left( \int_{a(x)}^{b(x)} |x_1|^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} |u|^4 dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} |u'|^4 dx_1 \right)^{1/4} + \\ &+ \left( \int_{a(x)}^{b(x)} |u|^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} |1 - x_1 u'|^2 dx_1 \right)^{1/2} \\ \text{Minkowski inequality} &\leq \frac{\sqrt{3}}{3} \sqrt{b(x)^3 - a(x)^3} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\ &+ \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (x_1 u'(x_1))^2 dx_1 \right)^{1/2} \right). \end{aligned}$$

□

**Theorem 3.3.**

$$\begin{aligned} \|u\|_{L_1((a(x),b(x)))} &\leq \frac{\sqrt{2}}{2} \sqrt{\exp(-2a(x)) - \exp(-2b(x))} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\ &+ \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (u' \exp(-x_1))^2 dx_1 \right)^{1/2} \right), \\ &u = u(x_1), \quad u' = u'(x_1) \\ &0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x), b(x))). \end{aligned} \quad (3)$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned} \|u\|_{L_1((a(x), b(x)))} &= \int_{a(x)}^{b(x)} |u| dx_1 \\ &= \int_{a(x)}^{b(x)} |u(\exp(-x_1)u' + 1 - \exp(-x_1)u')| dx_1 \\ &= \int_{a(x)}^{b(x)} |\exp(-x_1)uu' + u(1 - \exp(-x_1)u')| dx_1 \\ \text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |\exp(-x_1)uu'| dx_1 + \int_{a(x)}^{b(x)} |u(1 - \exp(-x_1)u')| dx_1 \\ &\quad \text{Schwarz and Hölder inequality} \\ &\leq \left( \int_{a(x)}^{b(x)} \exp(-2x_1) dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} u^4 dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} u'^4 dx_1 \right)^{1/4} + \\ &\quad + \left( \int_{a(x)}^{b(x)} u^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (1 - \exp(-x_1)u')^2 dx_1 \right)^{1/2} \\ &= \frac{\sqrt{2}}{2} \sqrt{\exp(-2a(x)) - \exp(-2b(x))} \|u\|_{L_4((a(x), b(x)))} \|u'\|_{L_4((a(x), b(x)))} + \\ &\quad + \|u\|_{L_2((a(x), b(x)))} \|1 - \exp(-x_1)u'\|_{L_2((a(x), b(x)))} \\ \text{Minkowski inequality} &\leq \frac{\sqrt{2}}{2} \sqrt{\exp(-2a(x)) - \exp(-2b(x))} \|u\|_{L_4((a(x), b(x)))} \|u'\|_{L_4((a(x), b(x)))} + \\ &\quad + \|u\|_{L_2((a(x), b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (u' \exp(-x_1))^2 dx_1 \right)^{1/2} \right). \end{aligned}$$

□

**Theorem 3.4.**

$$\begin{aligned} \|u\|_{L_1((a(x), b(x)))} &\leq \|u\|_{L_2((a(x), b(x)))} \|u'\|_{L_4((a(x), b(x)))} \|u''\|_{L_4((a(x), b(x)))} + \\ &\quad + \|u\|_{L_2((a(x), b(x)))} \left( \sqrt{b(x) - a(x)} + \|u'u''\|_{L_2((a(x), b(x)))} \right), \\ &\quad u = u(x_1), \quad u' = u'(x_1), \quad u'' = u''(x_1) \\ &\quad 0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x), b(x))) . \end{aligned} \tag{4}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned}
\|u\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u| \, dx_1 \\
&= \int_{a(x)}^{b(x)} |u(u'u'' + 1 - u'u'')| \, dx_1 \\
&= \int_{a(x)}^{b(x)} |uu'u'' + u(1 - u'u'')| \, dx_1 \\
\text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |uu'u''| \, dx_1 + \int_{a(x)}^{b(x)} |u(1 - u'u'')| \, dx_1 \\
&\quad \text{Schwarz and Hölder inequality} \\
&\leq \left( \int_{a(x)}^{b(x)} u^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} u'^4 \, dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} u''^4 \, dx_1 \right)^{1/4} + \\
&\quad + \left( \int_{a(x)}^{b(x)} u^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (1 - u'u'')^2 \, dx_1 \right)^{1/2} \\
&= \|u\|_{L_2((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} \|u''\|_{L_4((a(x),b(x)))} + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \|1 - u'u''\|_{L_2((a(x),b(x)))} \\
\text{Minkowski inequality} &\leq \|u\|_{L_2((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} \|u''\|_{L_4((a(x),b(x)))} + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \|u'u''\|_{L_2((a(x),b(x)))} \right).
\end{aligned}$$

□

**Theorem 3.5.**

$$\begin{aligned}
\|u\|_{L_1((a(x),b(x)))} &\leq \sqrt{\ln \left( \frac{b(x) + 1}{a(x) + 1} \right)} \|u\|_{L_4((a(x),b(x)))} \|u''\|_{L_4((a(x),b(x)))} + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} \frac{u''^2}{x_1 + 1} \, dx_1 \right)^{1/2} \right), \\
&\quad u = u(x_1), \quad u'' = u''(x_1), \\
&\quad 0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x), b(x))). \tag{5}
\end{aligned}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

Proof.

$$\begin{aligned}
 \|u\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u| dx_1 \\
 &= \int_{a(x)}^{b(x)} \left| u \left( \frac{u''}{\sqrt{x_1+1}} + 1 - \frac{u''}{\sqrt{x_1+1}} \right) \right| dx_1 \\
 &= \int_{a(x)}^{b(x)} \left| u \frac{u''}{\sqrt{x_1+1}} + u \left( 1 - \frac{u''}{\sqrt{x_1+1}} \right) \right| dx_1 \\
 &\leq \int_{a(x)}^{b(x)} \left| u \frac{u''}{\sqrt{x_1+1}} \right| dx_1 + \int_{a(x)}^{b(x)} \left| u \left( 1 - \frac{u''}{\sqrt{x_1+1}} \right) \right| dx_1 \\
 &\quad \text{Schwarz and Hölder inequality} \leq \\
 &\left( \int_{a(x)}^{b(x)} \frac{1}{x_1+1} dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} u^4 dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} u''^4 dx_1 \right)^{1/4} + \\
 &\quad + \left( \int_{a(x)}^{b(x)} u^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} \left( 1 - \frac{u''}{\sqrt{x_1+1}} \right)^2 dx_1 \right)^{1/2} \\
 &= \sqrt{\ln \left( \frac{b(x)+1}{a(x)+1} \right)} \|u\|_{L_4((a(x),b(x)))} \|u''\|_{L_4((a(x),b(x)))} + \\
 &\quad + \|u\|_{L_2((a(x),b(x)))} \left\| 1 - \frac{u''}{\sqrt{x_1+1}} \right\|_{L_2((a(x),b(x)))} \\
 \text{Minkowski} &\leq \sqrt{\ln \left( \frac{b(x)+1}{a(x)+1} \right)} \|u\|_{L_4((a(x),b(x)))} \|u''\|_{L_4((a(x),b(x)))} + \\
 &\quad + \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x)-a(x)} + \left( \int_{a(x)}^{b(x)} \frac{u''^2}{x_1+1} dx_1 \right)^{1/2} \right).
 \end{aligned}$$

□

**Theorem 3.6.**

$$\begin{aligned}
 \|u''\|_{L_1((a(x),b(x)))} &\leq \sqrt{\arctan(b(x)) - \arctan(a(x))} \|u''\|_{L_4((a(x),b(x)))} \|u\|_{L_4((a(x),b(x)))} + \\
 &\quad + \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x)-a(x)} + \left( \int_{a(x)}^{b(x)} \frac{u^2}{x_1^2+1} dx_1 \right)^{1/2} \right), \\
 &\quad u = u(x_1), \quad u'' = u''(x_1), \\
 &\quad 0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x),b(x))) .
 \end{aligned} \tag{6}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned}
& \|u''\|_{L_1((a(x),b(x)))} = \int_{a(x)}^{b(x)} |u''| dx_1 \\
&= \int_{a(x)}^{b(x)} \left| u'' \left( \frac{u}{\sqrt{x_1^2+1}} + 1 - \frac{u}{\sqrt{x_1^2+1}} \right) \right| dx_1 \\
&= \int_{a(x)}^{b(x)} \left| u'' \frac{u}{\sqrt{x_1^2+1}} + u'' \left( 1 - \frac{u}{\sqrt{x_1^2+1}} \right) \right| dx_1 \\
&\text{Triangle inequality} \leq \int_{a(x)}^{b(x)} \left| u'' \frac{u}{\sqrt{x_1^2+1}} \right| dx_1 + \\
&\quad + \int_{a(x)}^{b(x)} \left| u'' \left( 1 - \frac{u}{\sqrt{x_1^2+1}} \right) \right| dx_1 \\
&\text{Schwarz and Hölder inequality} \leq \\
&\quad \left( \int_{a(x)}^{b(x)} \frac{1}{x_1^2+1} dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} u^4 dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} u^{m^4} dx_1 \right)^{1/4} + \\
&\quad + \left( \int_{a(x)}^{b(x)} u^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} \left( 1 - \frac{u}{\sqrt{x_1^2+1}} \right)^2 dx_1 \right)^{1/2} \\
&= \sqrt{\arctan(b(x)) - \arctan(a(x))} \|u''\|_{L_4((a(x),b(x)))} \|u\|_{L_4((a(x),b(x)))} + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \left\| 1 - \frac{u}{\sqrt{x_1^2+1}} \right\|_{L_2((a(x),b(x)))} \\
&\text{Minkowski inequality} \leq \\
&\quad \sqrt{\arctan(b(x)) - \arctan(a(x))} \|u''\|_{L_4((a(x),b(x)))} \|u\|_{L_4((a(x),b(x)))} + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} \frac{u^2}{x_1^2+1} dx_1 \right)^{1/2} \right).
\end{aligned}$$

□

**Theorem 3.7.**

$$\begin{aligned}
& \|u'\|_{L_1((a(x),b(x)))} \leq \\
& \frac{\sqrt{2}}{2} \sqrt{b(x) - a(x) - \frac{(\sin(b(x)) - \sin(a(x)))}{2}} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\
& + \|u'\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (u \sin(x_1))^2 dx_1 \right)^{1/2} \right), \\
& 0 \leq a(x) < b(x), \quad u(x_1) \in C^\infty((a(x),b(x))) .
\end{aligned} \tag{7}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

Proof.

$$\begin{aligned}
 \|u'\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u'| \, dx_1 \\
 &= \int_{a(x)}^{b(x)} |u'(\sin(x_1)u + 1 - \sin(x_1)u)| \, dx_1 \\
 &= \int_{a(x)}^{b(x)} |u' \sin(x_1)u + u'(1 - \sin(x_1)u)| \, dx_1 \\
 \text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |u' \sin(x_1)u| \, dx_1 + \\
 &\quad + \int_{a(x)}^{b(x)} |u'(1 - \sin(x_1)u)| \, dx_1 \\
 \text{Schwarz and Hölder inequality} &\leq \\
 &\left( \int_{a(x)}^{b(x)} (\sin(x_1))^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (u)^4 \, dx_1 \right)^{1/4} \left( \int_{a(x)}^{b(x)} (u')^4 \, dx_1 \right)^{1/4} + \\
 &\quad + \left( \int_{a(x)}^{b(x)} (u')^2 \, dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (1 - \sin(x_1)u)^2 \, dx_1 \right)^{1/2} \\
 &= \frac{\sqrt{2}}{2} \sqrt{b(x) - a(x) - \frac{(\sin(b(x)) - \sin(a(x)))}{2}} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\
 &\quad + \|u'\|_{L_2((a(x),b(x)))} \|1 - \sin(x_1)u\|_{L_2((a(x),b(x)))} \\
 \text{Minkowski inequality} &\leq \\
 &\frac{\sqrt{2}}{2} \sqrt{b(x) - a(x) - \frac{(\sin(b(x)) - \sin(a(x)))}{2}} \|u\|_{L_4((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))} + \\
 &\quad + \|u'\|_{L_2((a(x),b(x)))} \left( \sqrt{b(x) - a(x)} + \left( \int_{a(x)}^{b(x)} (u \sin(x_1))^2 \, dx_1 \right)^{1/2} \right).
 \end{aligned}$$

□

**Theorem 3.8.**

$$\begin{aligned}
 \|u\|_{L_1((a(x),b(x)))} &\leq 2 \|u\|_{L_2((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))}^2 + \\
 &\quad + \|u\|_{L_2((a(x),b(x)))} \sqrt{b(x) - a(x)}, \\
 0 \leq a(x) < b(x), \quad u(x_1) &\in C^\infty((a(x), b(x))). \tag{8}
 \end{aligned}$$

The functions  $a(x), b(x)$  are smooth, fixed functions of space coordinate  $x$ . The variable  $x_1 \in (a(x), b(x))$ .

*Proof.*

$$\begin{aligned}
\|u\|_{L_1((a(x),b(x)))} &= \int_{a(x)}^{b(x)} |u| dx_1 \\
&= \int_{a(x)}^{b(x)} |u (u'^2 + 1 - u'^2)| dx_1 \\
&= \int_{a(x)}^{b(x)} |uu'^2 + u(1 - u'^2)| dx_1 \\
\text{Triangle inequality} &\leq \int_{a(x)}^{b(x)} |uu'^2| dx_1 + \int_{a(x)}^{b(x)} |u(1 - u'^2)| dx_1 \\
\text{Schwarz inequality} &\leq \left( \int_{a(x)}^{b(x)} (u)^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (u')^4 dx_1 \right)^{1/2} + \\
&\quad + \left( \int_{a(x)}^{b(x)} (u)^2 dx_1 \right)^{1/2} \left( \int_{a(x)}^{b(x)} (1 - u'^2)^2 dx_1 \right)^{1/2} \\
&= \|u\|_{L_2((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))}^2 + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \|1 - u'^2\|_{L_2((a(x),b(x)))} \\
\text{Minkowski inequality} &\leq 2\|u\|_{L_2((a(x),b(x)))} \|u'\|_{L_4((a(x),b(x)))}^2 + \\
&\quad + \|u\|_{L_2((a(x),b(x)))} \sqrt{b(x) - a(x)}.
\end{aligned}$$

□

## 4 Commentary on the Geometric Nirenberg-Gagliardo Type Inequalities

There is a commentary on the functional inequalities derived. The details comments follow below:

- (1) : This is a Nirenberg-Gagliardo type interpolation inequality for the  $L_1$  norm. The right hand side has the product of the norm of the function and its derivative in  $L_2$  with a constant plus the 2 norm of the function times a quantity that depends on the geometry. The norms depend on the geometry, as  $a, b$  are functions of  $x$ . The function  $u$  is assumed to be smooth with infinite order of derivatives. The standard Nirenberg-Gagliardo inequalities have fixed geometry and the constant sometimes is not specified. In this case, this interpolation inequality is geometry-variable and includes additional extra term at the right hand side.
- (2) : This is an interpolation type inequality resembling Nirenberg-Gagliardo type estimate. The 1st quantity at the right hand side has a geometry dependent quantity times the product of norms of the function and its derivative in the 4 norm. The 2nd quantity involves the product of the 2 norm of the function with the sum of two terms. The first term is geometry dependent. The 2nd term is

an energy type integral over a variable region, as the bounds of integration are functions of spatial variable.

- (3) : This is a functional inequality for the  $L_1$  norm of the function  $u$  in a geometry dependent region. The right member of the inequality consists of two quantities. The first one has the product of a geometry dependent quantity with the product of the norm of  $u$  and its derivative in  $L_4$  norm. The second quantity involves the 2 norm of the function  $u$  times the sum of two terms. The 1st term arises from the geometry. The 2nd term is an energy type integral with geometry variable bounds of integration.
- (4) : This is a functional inequality for the  $L_1$  norm of the function. This norm is bounded by a sum of two quantities. The first one is the product of the norm of the function in 2 norm, the norm of its first derivative in 4 norm and the norm of the second derivative in 4 norm. The second quantity involves a product of the norm of the function in 2 norm with the sum of two terms. The first term is geometry dependent. The 2nd term is the 2 norm of the product of 1st and 2nd derivative of the function  $u$ . It is a Nirenberg type estimate with extra terms.
- (5) : There is an interpolation bound for the norm of the function  $u$  in  $L_1$ . The first quantity involves the product of the square root of logarithmic function depending on the geometry with the product of the norm of function  $u$  and  $u''$  in the  $L_4$ . The 2nd term involves the product of the norm of function  $u$  in  $L_2$  with the sum of two terms. The 1st term is square root of the difference  $b - a$  and the second term is an energy type integral involving the 2nd derivative of the function.
- (6) : This is an interpolation inequality for the  $L_1$  norm of the 2nd derivative of the function. The right member of the inequality consists of two quantities. The first one is the multiplication of a term that is square rooted and the argument involves inverse trigonometric function depending on the variable region and the product of norm of the function with the norm of 2nd derivative in  $L_4$ . The 2nd quantity involves the  $L_2$  norm of the function  $u$  and the sum of two terms. The first term arises from the geometry type domain and the 2nd term is the square root of an energy type integral over a geometric variable domain.
- (7) : This is a functional inequality for the norm of  $u'$  in  $L_1$ . At the right hand side, there are various terms. There is a constant times a square rooted quantity involving trigonometric functions, and the product of the norm of  $u$  and its derivative in  $L_4$  norm. There is also the norm of  $u'$  in  $L_2$  and the sum of two terms, the one stemming from geometry and the second term is an energy type integral involving a trigonometric function.
- (8) : This is a geometric type functional inequality. The left member consists of the  $L_1$  norm of the function  $u$ . The right hand side involves the 2 norm of the function  $u$  and the  $L_4$  norm of the derivative in some positive integer power. There is an additional term that involves the product of the 2 norm of  $u$  and a square root quantity that demonstrates dependence on the geometry.

## 5 Applications of the Geometric Nirenberg-Gagliardo Type Interpolation Inequalities

In this section, explicit estimates are provided involving mollifier type functions of the form

$$m(x) = \exp\left(\pm \frac{1}{1+x^2}\right) \in C^{+\infty}([0, +\infty[),$$

where  $C^{+\infty}([0, +\infty[)$  denotes the class of smooth functions with infinite order of derivatives in the positive real line. All the results are presented below as corollaries of the previously established estimates (1)-(8).

**Corollary 5.1.**

$$\begin{aligned} \phi^2(x) &\leq \phi(x) + \frac{\sqrt{2}}{2} \sqrt{\frac{1}{x^2+1}} \sqrt{1-\phi^2(x)}, \\ \phi(x) &= \exp\left(\frac{-1}{1+x^2}\right), \quad x \in [0, +\infty[. \end{aligned} \quad (9)$$

*Proof.* The result follows by employing (1), picking  $u(x_1) = \exp(-x_1)$ ,  $x_1 \geq 0$ ,  $a(x) = 0$ ,  $b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.2.**

$$\begin{aligned} \psi(x) &\leq 1 + \frac{\sqrt{3}}{6} \left(\frac{1}{1+x^2}\right)^{\frac{3}{2}} \sqrt{\psi^4(x)-1} + \frac{\sqrt{2}}{2} \sqrt{\psi^2(x)-1} \sqrt{\frac{1}{1+x^2}} + \\ &+ \frac{1}{2} \sqrt{\psi^2(x)-1} \sqrt{\psi^2(x) \left(\frac{1}{1+x^2}\right)^2 - \psi^2(x) \left(\frac{1}{1+x^2}\right) + \frac{1}{2} (\psi^2(x)-1)}, \\ \psi(x) &= \exp\left(\frac{1}{1+x^2}\right), \quad x \in [0, +\infty[. \end{aligned} \quad (10)$$

*Proof.* The result follows by employing (2), choosing  $u(x_1) = \exp(x_1)$ ,  $x_1 \geq 0$ ,  $a(x) = 0$ ,  $b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.3.**

$$\begin{aligned} \psi(x) &\leq 1 + \frac{\sqrt{2}}{4} \sqrt{\psi^4(x)-1-2\sinh\left(\frac{2}{1+x^2}\right)} + \\ &+ \sqrt{2} \sqrt{\psi^2(x)-1} \sqrt{\frac{1}{1+x^2}}, \\ \psi(x) &= \exp\left(\frac{1}{1+x^2}\right), \quad x \in [0, +\infty[. \end{aligned} \quad (11)$$

*Proof.* The result follows by using (3), choosing  $u(x_1) = \exp(x_1)$ ,  $x_1 \geq 0$ ,  $a(x) = 0$ ,  $b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.4.**

$$\begin{aligned} 1 - \phi(x) &\leq \frac{\sqrt{2}}{2} \sqrt{1 - \phi^4(x) - \phi^2(x) + \phi^6(x)} + \frac{\sqrt{2}}{2} \sqrt{1 - \phi^2(x)}, \\ \phi(x) &= \exp\left(\frac{-1}{1+x^2}\right), \quad x \in [0, +\infty[. \end{aligned} \quad (12)$$

*Proof.* The result follows with the aid of (4), picking  $u(x_1) = \exp(-x_1), x_1 \geq 0, a(x) = 0, b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.5.**

$$\begin{aligned}
 1 - \phi(x) &\leq \frac{1}{2} \sqrt{\ln \left( \frac{x^2 + 2}{x^2 + 1} \right)} \sqrt{1 - \phi^4(x)} + \\
 &+ \frac{\sqrt{2}}{2} \sqrt{1 - \phi^2(x)} \left( \sqrt{\frac{1}{1 + x^2}} + \left( \int_0^{\frac{1}{1+x^2}} \frac{\exp(-2x_1)}{x_1 + 1} dx_1 \right)^{1/2} \right), \\
 \phi(x) &= \exp \left( \frac{-1}{1 + x^2} \right), \quad x \in [0, +\infty[, \quad x_1 \in \left[ 0, \frac{1}{1 + x^2} \right].
 \end{aligned}
 \tag{13}$$

*Proof.* The result is derived with the aid of the inequality (5), choosing  $u(x_1) = \exp(x_1), x_1 \geq 0, a(x) = 0, b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.6.**

$$\begin{aligned}
 1 - \phi(x) &\leq \frac{1}{2} \sqrt{\arctan \left( \frac{1}{1 + x^2} \right)} \sqrt{1 - \phi^4(x)} + \\
 &+ \frac{\sqrt{2}}{2} \sqrt{1 - \phi^2(x)} \left( \sqrt{\frac{1}{1 + x^2}} + \left( \int_0^{\frac{1}{1+x^2}} \frac{\exp(-2x_1)}{1 + x_1^2} dx_1 \right)^{1/2} \right), \\
 \phi(x) &= \exp \left( \frac{-1}{1 + x^2} \right), \quad x \in [0, +\infty[, \quad x_1 \in \left[ 0, \frac{1}{1 + x^2} \right].
 \end{aligned}
 \tag{14}$$

*Proof.* The result is derived with the assistance of inequality (6), picking  $u(x_1) = \exp(-x_1), x_1 \geq 0, a(x) = 0, b(x) = \frac{1}{1+x^2}$ .  $\square$

**Corollary 5.7.**

$$\begin{aligned}
 1 - \phi(x) &\leq \frac{\sqrt{2}}{4} \sqrt{\frac{1}{1 + x^2} - \frac{1}{2} \sin \left( \frac{1}{1 + x^2} \right)} \sqrt{1 - \phi^4(x)} + \\
 &+ \frac{\sqrt{2}}{2} \sqrt{1 - \phi^2(x)} \sqrt{\frac{1}{1 + x^2}} + \frac{1}{2} (1 - \phi^2(x)), \\
 \phi(x) &= \exp \left( \frac{-1}{1 + x^2} \right), \quad x \in [0, +\infty[.
 \end{aligned}
 \tag{15}$$

*Proof.* The result follows with the aid of inequality (7), choosing  $u(x_1) = \exp(-x_1), a(x) = 0, b(x) = \frac{1}{1+x^2}$ , and the property

$$\left( \int_{a(x)}^{b(x)} (u \sin(x_1))^2 dx_1 \right)^{1/2} \leq \left( \int_{a(x)}^{b(x)} u^2 dx_1 \right)^{1/2} = \|u\|_{L_2((a(x), b(x)))},$$

since  $|\sin(x_1)| \leq 1$ .

$\square$

**Corollary 5.8.**

$$1 - \phi(x) \leq \frac{\sqrt{2}}{2} \sqrt{1 - \phi^2(x)} \left( \sqrt{\frac{1}{1+x^2}} + \sqrt{1 - \phi^4(x)} \right), \quad (16)$$

$$\phi(x) = \exp\left(\frac{-1}{1+x^2}\right), \quad x \in [0, +\infty[.$$

*Proof.* The inequality is derived with the aid of estimate (8), choosing  $u(x_1) = \exp(-x_1)$ ,  $x_1 \geq 0$ ,  $a(x) = 0$ ,  $b(x) = \frac{1}{1+x^2}$ .  $\square$

## 6 Graphical Illustration

This section is dedicated to the graphical illustration of the estimates (9) to (16). The blue curve represents the left member of the inequality and the green dotted curve represents the right hand side of the inequality. It is evident that the illustrations verify the rigorous theoretical results.

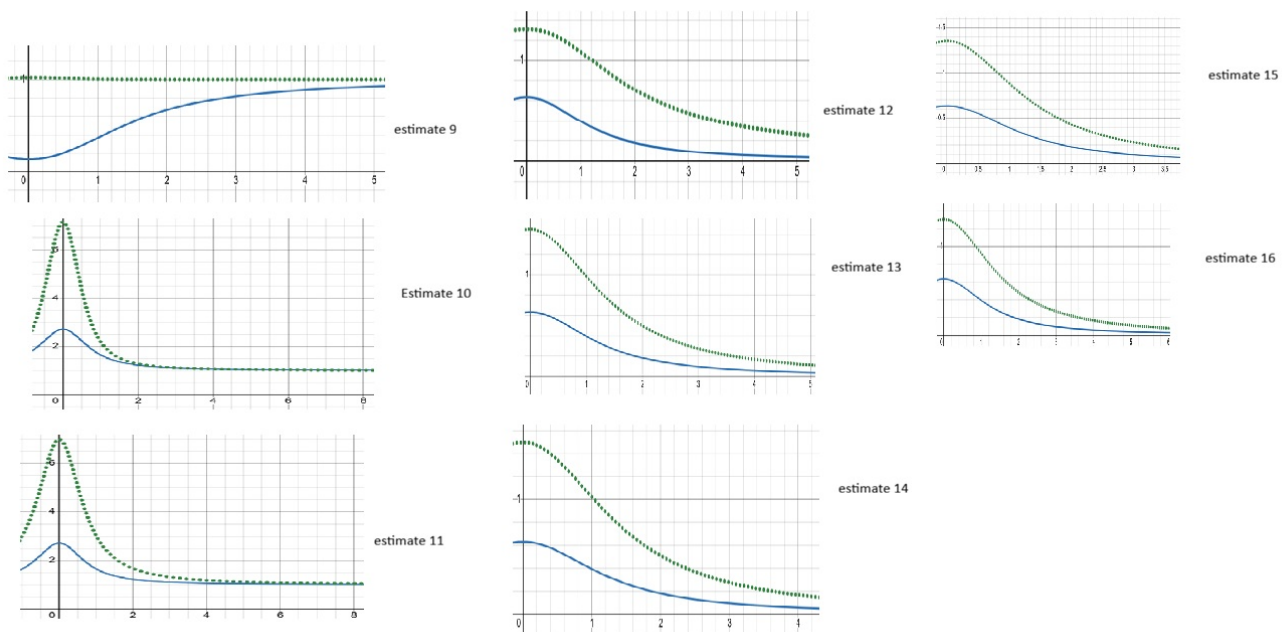


Figure 1: Graphical illustration of the estimates involving mollifier type functions (9) -(16).

## 7 Conclusion

In this article, Nirenberg-Gagliardo type inequalities of geometric flavor have been derived having extra terms. In some cases the extra terms are energy type integrals. The results have been proved rigorously

and using integral decomposition and classical functional inequalities. The approach taken is different from the literature in the sense that most works derive this class of inequalities for fixed domains, with various types of operators, the constants sometimes are not specified and the extra terms appearing are rare. Furthermore, there are no correction terms such as energy integrals. Explicit estimates involving mollifier functions have been yielded, new to the literature to the best knowledge of the author, by employing the derived Nirenberg-Gagliardo type inequalities. This work motivates further mathematical investigations in Analysis. Next aim is to work on variants on Caffarelli-Kohn-Nirenberg interpolation inequalities [4], [18] which are also ubiquitous in Analysis.

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