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Parameter Dependent Refinements for the Sinc Hyperbolic Function and Applications

Abd Raouf Chouikha

Universite Paris-Sorbonne, Paris-Nord, Villetaneuse, 93430, France
e-mail: juju153@yahoo.com

Abstract

In this paper, we prove for $x \in (0, \infty)$ the double inequality for the Sinc hyperbolic function

$$\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{r}{2}} + 3\,(\cosh x)^{\frac{r}{3}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x} < \cosh x\left(\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3}{2}r} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

Both bounds converge to $\frac{\sinh x}{x}$ as r approaches zero from the right.

Similar double inequalities are provided for the Huygens and the Wilker functions as well as for the Lazarević function.

1 Introduction

The hyperbolic version of the Mitrinović-Adamović inequality is given as for $x \neq 0$

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 < \frac{\cosh x + 2}{3}. \tag{1}$$

The first one was obtained by Lazarević and the second inequality is called hyperbolic Cusa-Huygens inequality.

There are many results on the refinement of the inequalities (1) in the literature. E.g., see:

Recall that Bhayo and Sandor [1] Theorem 1.5 proved the following for $x \geq 0$

$$\exp\left(\frac{1}{2}\left(\frac{x}{\tanh x} - 1\right)\right) < \frac{\sinh x}{x} < \exp\left(\left(\frac{x}{\tanh x} - 1\right)\right).$$

In addition, [1] Theorem 3.4 states:

$$\left(\frac{1 + \cosh x}{2}\right)^{2/3} < \frac{\sinh x}{x} < \left(\frac{1 + \cosh x}{2}\right).$$

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Here is another bound provided by [5] for $x \geq 0$

$$\left(\frac{1+2\cosh x}{3}\right)^{1/2} < \frac{\sinh x}{x}.$$

On the other hand, [11] Theorem 2.2 proved that for $x \neq 0$

$$\cosh x < \left(\frac{\sinh x}{x}\right)^3 - \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2.$$

This paper can be seen, in a way, as the hyperbolic version of [9]. Indeed, we will present new bounds for the Sinc hyperbolic function as well as for the Huygens and the Wilker hyperbolic functions in terms of one parameter functions $f(r,x)$ defined for $r \geq 0$ and for $x \in (0,\infty)$. It can be made arbitrarily close to $\frac{\sinh x}{x}$ and can outperform known bounds for small parameters. These bounds appear to be finer than older

$$\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}} + 3\,(\cosh x)^r\right)^{\frac{1}{3r}} < \left(\frac{\sinh x}{x}\right) <$$

$$\cosh x\left(\frac{1}{3} + \frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}}(\cosh x)^{-\frac{3r}{2}}\right)^{\frac{1}{r}}. \tag{2}$$

More precisely, notice that when r tends to 0, one proves

$$\lim_{r\to 0}\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}} + 3\,(\cosh x)^r\right)^{\frac{1}{3r}} =$$

$$\lim_{r\to 0}\cosh x\left(\frac{1}{3} + \frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{9r}{2}}(\cosh x)^{-\frac{3r}{2}}\right)^{\frac{1}{r}} = \left(\frac{\sinh x}{x}\right).$$

Therefore, if we bound $\frac{\sinh x}{x}$ by two functions $f(x)$ and $g(x)$:

$$f(x) < \left(\frac{\sinh x}{x}\right) < g(x),$$

then there exists r_0 such that for $r < r_0$ one has

$$f(x) < \left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}} + 3\,(\cosh x)^r\right)^{\frac{1}{3r}},$$

$$g(x) > \cosh x\left(\frac{1}{3} + \frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{9r}{2}}(\cosh x)^{-\frac{3r}{2}}\right)^{\frac{1}{r}}.$$

Thus (2) appears to be the finest bound for the Sinc function.

We also prove analogous bounds for Huygens and Wilker functions as well as for the Lazarević function.

2 Main Results

Next results give bounds for $\frac{\tanh x}{x}$ as well as for the Sinc function $\frac{\sinh x}{x}$.

Theorem 1.1 *For $x \in (0, \infty)$, the function $\frac{\tanh x}{x}$ satisfies the following double inequality for any $r > 0$*

$$\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{1}{r}} < \left(-2 + 3\left(\frac{\tanh x}{x}\right)^{\frac{1}{3}r}\right)^{\frac{1}{r}} <$$

$$\frac{\tanh x}{x} < \left(\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3}{2}r} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

Moreover, the equalities are attained when r tends to 0.

Theorem 1.2 *For $x \in (0, \infty)$, the Sinc function satisfies the following double inequality for any $r > 0$*

$$(i) \quad \left(-2\left(\frac{\sinh x}{x}\right)^{-\frac{r}{2}} + 3\right)^{\frac{1}{r}} < \left(-2 + 3\left(\frac{\sinh x}{x}\right)^{\frac{r}{3}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x} < \left(\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

$$(ii) \quad \cosh x\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{1}{r}} < \frac{\sinh x}{x} < \left(\frac{(\cosh x)^r}{3} + \frac{2(\cosh x)^{\frac{1}{2}r}}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}}\right)^{\frac{1}{r}}.$$

Moreover, the equalities are attained when r tends to 0.

Remark 1.2.1 It seems that the following is true:

$$\cosh x\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{1}{r}} < \left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{r}{2}} + 3\,(\cosh x)^{\frac{r}{3}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x}.$$

If so, we then can improve Theorem 1.2.

We may also derive bounds for the Huygens function.

Theorem 1.3 *For $x \in (0, \infty)$, the following double inequality holds for any $r > 0$*

$$(2\cosh x + 1)\left(3 - 2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r}\right)^{\frac{1}{r}} < \frac{2\sinh x + \tanh x}{x} <$$

$$(2\cosh x + 1)\left(\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3}{2}r} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

Moreover, the equalities are attained when r tends to 0.

In the same way one obtains bounds for the Wilker function.

Theorem 1.4 *For* $x \in (0, \infty)$, *the following double inequality holds for any* $r > 0$

$$(\cosh x)^2 \left(-2 \left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{2}{r}} + \left(-2 \left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{1}{r}} <$$

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} <$$

$$(\cosh x)^2 \left(\frac{2}{3} \left(\frac{\tanh x}{x}\right)^{\frac{3}{2}r} + \frac{1}{3}\right)^{\frac{2}{r}} + \left(\frac{2}{3} \left(\frac{\tanh x}{x}\right)^{\frac{3}{2}r} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

Moreover, the equalities are attained when r *tends to* 0.

Concerning the Lazarević inequality, we prove the following:

Theorem 1.5 *For* $x \in (0, \infty)$, *the following double inequality holds for any* $r > 0$

$$(i) \quad \left(-2\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^{-\frac{r}{2}} + 3\right)^{\frac{1}{r}} < \left(-2 + 3\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^{\frac{r}{3}}\right)^{\frac{1}{r}} <$$

$$\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3 < \left(\frac{2}{3}\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^{\frac{3r}{2}} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

For $x \in (0, \frac{\pi}{2})$, *the following double inequality holds for any* $r > 0$

$$(ii) \quad \left(3 - 2\left(\frac{\sinh x}{x}\right)^{-\frac{3r}{2}} (\cosh x)^{\frac{r}{2}}\right)^{\frac{1}{r}} < \frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3 <$$

$$\left(\frac{1}{3} + \frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{9r}{2}} (\cosh x)^{-\frac{3r}{2}}\right)^{\frac{1}{r}}.$$

Moreover, the equalities are attained when r *tends to* 0.

3 Proofs

The following lemmas will be useful in the sequel [4], [10]:

Lemma 3.1 *For* $x \in (0, \infty)$ *and* $p, q \geq 0$, *the following inequalities hold:*

$$(i) \quad \left(\frac{\sinh x}{x}\right)^p > 1 + \frac{px^2}{6} + \left(\frac{1}{120}p + \frac{1}{72}p(p-1)\right)x^4 +$$

$$\left(\frac{1}{5040}p + \frac{1}{720}p(p-1) + \frac{1}{1296}p(p-1)(p-2)\right)x^6,$$

$$(ii) \quad 1-\frac{qx^2}{3}+\left(\frac{2q}{15}+\frac{q(q-1)}{18}\right)x^4+\left(-\frac{17q}{315}-\frac{2q(q-1)}{45}-\frac{q(q-1)(q-2)}{162}\right)x^6<\left(\frac{\tanh x}{x}\right)^q,$$

$$(iii) \quad \left(\frac{\log x}{\sinh x}\right)>-\frac{1}{6}x^2+\frac{1}{180}x^4-\frac{1}{2835}x^6+\frac{1}{37800}x^8,$$

$$(iv) \quad \log\left(\frac{\sinh x}{x}\right)<\frac{1}{6}x^2-\frac{1}{180}x^4+\frac{1}{2835}x^6,\ \log\left(\frac{\tanh x}{x}\right)<-\frac{1}{3}x^2+\frac{7}{90}x^4-\frac{62}{2835}x^6.$$

Lemma 3.2 (Proposition 1 of [13]) *Let the series* $f(x)=\sum_{k=1}^{\infty}(-1)^{k+1}A(k)x^{2k}$ *converge for* $x\in(0,c), c\in IR^+$ *. Suppose that the following statements are true:*

(i) If $c<1$, *then the sequence* $A(k), k\in IN$ *is a positive decreasing sequence that converges to* 0.

(ii) If $c\geq 1$, *then the sequence* $A(k)k\in IN$ *is a positive sequence,* $\lim_{k\to\infty}c^{2k}A(k)=0$ *and* $A(k)>c^2A(k+1)$ *for* $k\geq 1$.

Then, for all $x\in(0,c)$ *and for all* $n\in N$ *and* $m\in N$, *we have*

$$\sum_{k=1}^{2n}(-1)^{k+1}A(k)x^{2k}<f(x)<\sum_{k=1}^{2n+1}(-1)^{k+1}A(k)x^{2k}.$$

However, in [12] Theorem 2, the following result has been proved:

Lemma 3.3 *Suppose* f *is a real function on* (a,b) *and* n *is a positive integer such that* $f^k(a+), f^k(b-), k=0,1,2...,n$ *exist and* $f^n(x)$ *is increasing on* (a,b). *Then, for all* $x\in(a,b)$, *the following inequality holds:*

$$\sum_{k=0}^{n-1}\frac{f^k(b-)}{k!}(x-a)^k+\frac{1}{(a-b)^n}\left(f(b-)-\sum_{k=0}^{n-1}\frac{(a-b)^kf^k(a+)}{k!}\right)(x-b)^n$$
$$>f(x)>\sum_{k=0}^{n}\frac{f^k(a+)}{k!}(x-a)^k.$$

Furthermore, if $f^n(x)$ is decreasing on (a,b), then the reversed inequalities hold.

Lemma 3.4 ([9] Lemma 3.3) *Let* $u(x)$ *be a positive* C^1*-function defined for* $x\in(a,b)$ *such that* $u(0)=1$. *Then, for* $r>0$, *the following inequalities hold:*

$$(i) \quad \left(-2\left(u(x)\right)^{-\frac{r}{2}}+3\right)^{\frac{1}{r}}<\left(-2+3\left(u(x)\right)^{\frac{r}{3}}\right)^{\frac{1}{r}}<u(x)<\left(\frac{2}{3}\left(u(x)\right)^{\frac{3r}{2}}+\frac{1}{3}\right)^{\frac{1}{r}}.$$

$$(ii) \quad \left(-2+3\left(u(x)\right)^{\frac{2r}{3}}\right)^{\frac{1}{r}}<(u(x))^2<\left(\frac{2}{3}\left(u(x)\right)^{\frac{3r}{2}}+\frac{1}{3}\right)^{\frac{2}{r}}<\left(\frac{2}{3}\left(u(x)\right)^{3r}+\frac{1}{3}\right)^{\frac{1}{r}}.$$

$$(iii) \quad \left(-2+3\left(u(x)\right)^{r}\right)^{\frac{1}{r}}<(u(x))^3<\left(\frac{2}{3}\left(u(x)\right)^{\frac{3r}{2}}+\frac{1}{3}\right)^{\frac{3}{r}}<\left(\frac{2}{3}\left(u(x)\right)^{\frac{9r}{2}}+\frac{1}{3}\right)^{\frac{1}{r}}.$$

More generally, for $k > 0$, one gets

$$(iv) \quad \left(-2 + 3\,(u(x))^{\frac{kr}{3}}\right)^{\frac{1}{r}} < (u(x))^k < \left(\frac{2}{3}\,(u(x))^{\frac{3kr}{2}} + \frac{1}{3}\right)^{\frac{1}{r}}.$$

Moreover, all these inequalities become equalities when r tends to 0.

Proof of Theorem 1.1

We will now separately prove the double inequalities.

(i) Let us prove the right one inequality

$$\frac{\tanh x}{x} \; < \; \left(\frac{1}{3} + \frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3r}{2}}\right)^{\frac{1}{r}}.$$

Since $\frac{\tanh x}{x} < 1$, we have now to prove

$$\left(\frac{\tanh x}{x}\right)^{r} - \frac{1}{3} - \frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3r}{2}} < 0.$$

It is easy to see that the function

$$Z - \frac{1}{3} - \frac{2}{3}\,Z^{3/2} = -\frac{1}{3}(2\sqrt{Z}+1)(\sqrt{Z}-1)^2 < 0,$$

where $Z = \left(\frac{\tanh x}{x}\right)^r < 1$.

(ii) The left inequality of Theorem 1.1

$$\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3\right)^{\frac{1}{r}} < \frac{\tanh x}{x}$$

is equivalent to

$$-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3 < \left(\frac{\tanh x}{x}\right)^{r}.$$

Notice that if we take $Z = \left(\frac{\tanh x}{x}\right)^{\frac{1}{2}}$, then

$$-2\left(\frac{\tanh x}{x}\right)^{-\frac{1}{2}r} + 3 - \left(\frac{\tanh x}{x}\right)^{r} = -\frac{2}{Z} + 3 - Z^2 = -\frac{(Z+2)\,(Z-1)^2}{Z} < 0,$$

since $Z < 1$. This proves the left inequality.

(iii) Turn now to the last part of Theorem 1.1.

Notice, first, that for $r > 0$ the following expression can be written

$$-2\left(\frac{\tanh x}{x}\right)^{\frac{-3r}{2}} + 3\,(\cosh x)^r = \frac{\left(-2\,\left(\frac{\sinh x}{x}\right)^{\frac{-3r}{2}} + 3\,(\cosh x)^{\frac{-r}{2}}\right)}{\left((\cosh x)^{\frac{-3r}{2}}\right)}.$$

Moreover, it is easy to see that

$$-2\left(\frac{\sinh x}{x}\right)^{\frac{-3r}{2}}+3\left(\cosh x\right)^{\frac{-r}{2}}>-2+3\left(\cosh x\right)^{\frac{-r}{2}}>0,$$

then its logarithm is well defined. It yields:

$$\ln\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}}+3\left(\cosh x\right)^{r}\right)^{\frac{1}{3r}}=\frac{\ln\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}}+3\left(\cosh x\right)^{r}\right)}{3r}=$$

$$\frac{1}{3r}\left(3\ln\left(\frac{\sinh x}{x}\right)r\right)+O(r)=\ln\left(\frac{\sinh x}{x}\right)+O(r).$$

Thus

$$\lim_{r\to 0}\left(-2\left(\frac{\tanh x}{x}\right)^{-\frac{3r}{2}}+3\left(\cosh x\right)^{r}\right)^{\frac{1}{3r}}=\frac{\sinh x}{x}.$$

In the same way, let us take the logarithm

$$\ln\cosh x\left(\frac{1}{3}+\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3r}{2}}\right)^{\frac{1}{r}}=\frac{1}{r}\ln\cosh x\left(\frac{1}{3}+\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3r}{2}}\right)=\ln\left(\frac{\sinh x}{x}\right)+O(r).$$

Thus

$$\lim_{r\to 0}\cosh x\left(\frac{1}{3}+\frac{2}{3}\left(\frac{\tanh x}{x}\right)^{\frac{3r}{2}}\right)^{\frac{1}{r}}=\frac{\sinh x}{x}.$$

Proofs of Theorems 1.2, 1.3 and 1.4

These theorems are in fact an easy consequence of Theorem 1.1. Indeed, we may write for Theorem 1.2 (ii)

$$\frac{\sinh x}{x}=\cosh x\frac{\tanh x}{x}.$$

(i) of Theorem 1.2 is derived from (i) Lemma 3.4 where $u(x)=\frac{\sinh x}{x}$.

For Theorem 1.3, we have

$$\frac{2\sinh x+\tanh x}{x}=\frac{2\cosh x\tanh x}{x}+\frac{\tanh x}{x}=(2\cosh x+1)\frac{\tanh x}{x}.$$

For Theorem 1.4, we have

$$\left(\frac{\tanh x}{x}\right)^{2}+\frac{\tanh x}{x}=(\cosh x)^{2}\left(\frac{\tanh x}{x}\right)^{2}+\frac{\tanh x}{x}.$$

Proof of Theorem 1.5

(i) is derived from Lemma 3.4 (i) where $u(x)=\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^{3}$.

To prove (ii) consider the series representations of hyperbolic functions [10] or [4]:

$$\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^r > 1+\frac{1}{15}\,rx^4-\frac{4}{189}\,rx^6+\left(\frac{1}{150}\,r+\frac{1}{450}\,r^2\right)x^8+$$

$$\left(-\frac{68}{31185}\,r-\frac{4}{2835}\,r^2\right)x^{10}+\left(\frac{42842}{58046625}\,r+\frac{5969}{8930250}\,r^2+\frac{1}{20250}\,r^3\right)x^{12},$$

$$-2\left(\frac{\sinh x}{x}\right)^{-\frac{3r}{2}}(\cosh x)^{1/2\,r}+3<(1+\frac{1}{15}\,rx^4-\frac{4}{189}\,rx^6+\left(\frac{1}{150}\,r-\frac{1}{900}\,r^2\right)x^8+$$

$$\left(-\frac{68}{31185}\,r+\frac{2}{2835}\,r^2\right)x^{10}.$$

Therefore, after regrouping and simplifying and using *Maple*, it yields for the left inequality of (i):

$$\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^r-\left(-2\left(\frac{\sinh x}{x}\right)^{-\frac{3r}{2}}(\cosh x)^{\frac{r}{2}}+3\right)>$$

$$\left(\frac{42842}{58046625}\,r+\frac{5969}{8930250}\,r^2+\frac{1}{20250}\,r^3\right)x^{12}-\frac{2}{945}\,x^{10}r^2+\frac{1}{300}\,x^8r^2>0.$$

For the right inequality of (ii) let us estimate

$$\left(\frac{(\sinh x)^3}{\cosh x x^3}\right)^r<1+\frac{1}{15}\,rx^4-\frac{4}{189}\,rx^6+\left(\frac{1}{150}\,r+\frac{1}{450}\,r^2\right)x^8+\left(-\frac{68}{31185}\,r-\frac{4}{2835}\,r^2\right)x^{10},$$

$$\frac{1}{3}+\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{9r}{2}}(\cosh x)^{-\frac{3r}{2}}>1+\frac{1}{15}\,rx^4+\frac{4}{189}\,rx^6+\left(\frac{1}{150}\,r+\frac{1}{300}\,r^2\right)x^8+$$

$$\left(\frac{68}{31185}\,r+\frac{2}{945}\,r^2\right)x^{10}+\left(\frac{42842}{58046625}\,r+\frac{1}{9000}\,r^3+\frac{5969}{5953500}\,r^2\right)x^{12}.$$

After regrouping, simplifying and using *Maple*, it yields

$$\frac{1}{3}+\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{9r}{2}}(\cosh x)^{-\frac{3r}{2}}-\left(\frac{1}{\cosh x}\left(\frac{\sinh x}{x}\right)^3\right)^r>$$

$$\left(\frac{42842}{58046625}\,r+\frac{1}{9000}\,r^3+\frac{5969}{5953500}\,r^2\right)x^{12}-\frac{2}{2835}\,x^{10}r^2+\frac{1}{900}\,x^8r^2>0.$$

(iii) Turn now to the last part of Theorem 1.5.

A series representation of the left side with respect to r yields

$$\left(3-2\left(\frac{\sinh x}{x}\right)^{-3/2\,r}(\cosh x)^{1/2\,r}\right)^{r^{-1}}=$$

$$\frac{(\sinh x)^3}{x^3\cos x}\left(1-3/4\left(3\,\ln\left(\frac{\sinh x}{x}\right)+\ln\left(\cosh x\right)\right)^2r\right)+O(r^2).$$

This means

$$\lim_{r\to 0}\left(3-2\left(\frac{\sinh x}{x}\right)^{-3/2\,r}(\cosh x)^{1/2\,r}\right)^{r^{-1}}=\left(\frac{\sinh x}{x}\right)^{3}\frac{1}{\cosh x}.$$

For the right side one gets the limited expansion with respect to r

$$\left(1/3+2/3\left(\frac{\sinh x}{x}\right)^{9/2\,r}(\cosh x)^{-3/2\,r}\right)^{r^{-1}}=$$

$$\frac{(\sinh x)^3}{x^3\cosh x}\left(1+1/4\left(3\,\ln\left(\frac{\sinh x}{x}\right)+\ln(\cosh x)\right)^2 r\right)+O(r^2).$$

This means

$$\lim_{r\to 0}\left(1/3+2/3\left(\frac{\sinh x}{x}\right)^{9/2\,r}(\cosh x)^{-3/2\,r}\right)^{r^{-1}}=\frac{(\sinh x)^3}{x^3\cosh x}.$$

4 Links with Older Inequalities

In this section, we will more closely verify why the bounds given by Theorem 1.2 are the most precise concerning the Sinc hyperbolic function. As mentioned in the introduction, we will determine the value of r_0 for certain cases in order that for $r < r_0$ one has

$$f(x)<\left(3-2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}},\ g(x)>\left(\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}}+\frac{1}{3}\right)^{\frac{1}{r}}.$$

To do this, we will return to the examples cited in the introduction and compute, in each case, the difference using *Maple*.

4.1 Inequality of Chen-Sandor [5]

It is easy to ensure that for $x \geq 0$ the following holds:

$$\left(3-2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}}-\left(\frac{1+2\cosh x}{3}\right)^{1/2}>0$$

as soon as $r < \frac{1}{10}$. This means

$$\left(\frac{1+2\cosh x}{3}\right)^{1/2}<\left(3-2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}}<\frac{\sinh x}{x}.$$

4.2 Inequality of Bhayo-Sandor: Theorem 3.4 of [1]

We may easily verify that for $x \geq 0$ the following holds:

$$\left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} - \left(\frac{1}{2} + \frac{1}{2}\cosh x\right)^{\frac{2}{3}} > 0$$

as soon as $r < \frac{1}{70}$, which means

$$\left(\frac{1}{2} + \frac{1}{2}\cosh x\right)^{\frac{2}{3}} < \left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x}.$$

4.3 Left inequality of Bhayo-Sandor: Theorem 1.5 of [1]

It is easy to see that for $x \geq 0$ the following holds:

$$\left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} - \exp\left(\frac{1}{2}\left(\frac{x}{\tanh x} - 1\right)\right) > 0$$

as soon as $r < \frac{1}{70}$, that is to say

$$\exp\left(\frac{1}{2}\left(\frac{x}{\tanh x} - 1\right)\right) < \left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x}.$$

4.4 Right inequality of Bhayo-Sandor: Theorem 1.5 of [1]

We easily verify that for $x \geq 0$ the following holds:

$$\left(\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}} + \frac{1}{3}\right)^{\frac{1}{r}} - \exp\left(\left(\frac{x}{\tanh x} - 1\right)\right) < 0$$

as soon as $r < \frac{1}{150}$, in other words

$$\frac{\sinh x}{x} < \left(\frac{2}{3}\left(\frac{\sinh x}{x}\right)^{\frac{3r}{2}} + \frac{1}{3}\right)^{\frac{1}{r}} < \exp\left(\left(\frac{x}{\tanh x} - 1\right)\right).$$

4.5 Inequality of Wu-Li-Bencze [11]

Thanks to *Maple* it is easy to ensure that for $x \geq 0$ the following holds:

$$\left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} - \cosh x - \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2 > 0$$

as soon as $r < \frac{1}{100}$. This means

$$\cosh x + \frac{12}{5}\left(1 - \frac{x}{\sinh x}\right)^2 < \left(3 - 2\left(\frac{\sinh x}{x}\right)^{\frac{-r}{2}}\right)^{\frac{1}{r}} < \frac{\sinh x}{x}.$$

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