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A Dynamical Analysis on the Solutions of Time-Fractional Rosenau-Korteweg de Vries-Regularized Long Wave Equation via Approximate Analytical Method

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Abstract

Wave propagation in fluids, optical fibers and plasma physics are explained by complex math equations that include dispersion and dissipation effects. This study looks at the solutions for a specific equation that includes dispersion and regularization effects. To understand the effects of the Rosenau-Korteweg de Vries-regularized long wave equation, we used a new method called the approximate analytical method (AAM). We also proved that our solution is unique. The method uses the Caputo derivative, which helps describe how things change over time and memory effects. The solution is a series that shows how the wave behaves and converges quickly without needing to simplify or discretize the equation. We also drew plots and graphs to visualize the wave dynamics. Our method describes the behavior and accuracy of solutions and the algorithm shows that it can handle problems with nonlinearity and memory effects. We tested our method with simulations and 3D visualizations, which show that it is effective and accurate compared to results. The results confirm that our method can be used to understand fractional systems suggesting its use in many areas of science and technology.

1 Introduction

Nonlinear waves are a thing that happens and it is very important in many areas of study. We know that nonlinear chaos, natural things and real world problems can be studied using the concept of classical calculus but it has some limits. That is why fractional calculus came into being three hundred years ago thanks to the ideas of two famous mathematicians, Leibniz and L'Hospital. Fractional calculus is a

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theory that is growing fast and it is the study of derivatives and integrals of any order. One of the main advantages of calculus is that it gives us solutions between intervals, which helps us understand the results more clearly and it also describes nonlinear phenomena in a very specific way [1–3].

The mathematical models or problems that have properties like solitons, chaos and random motions can be studied well using fractional calculus theory. Many researchers have contributed to the development of this theory. They have proposed their own fractional derivatives and integrals such as Riemann-Liouville, Caputo-Fabrizio and Atangana-Baleanu. We used the Caputo operator, which is very important because it shows non-locally bounded and smoother behavior than other fractional operators. It is very good for value problems and modeling systems with initial conditions. The linear and nonlinear partial and ordinary differential equations of order have many applications in areas like physics [4], electrodynamics [5], astrophyics [6], human diseases [7], and so on [8–10]. The main good things about differential equations are that they have properties like hereditary property, memory effects and uncertainty property. That is why fractional differential equations have become very popular in modeling complicated processes in real world challenges. Some researchers have solved differential equations of fractional order using different methods. Zhang et al. [11] studied the Emden-Fowler equations of order and they used the variational iteration technique and Shehu transformation to get the solutions. Ganie et al. [12] used the Elzaki transform and the Adomian decomposition method to solve some differential equations. Alshehry et al. [13] used the Laplace residual power series approach to obtain solutions of fractional differential equations. Many analytical and numerical techniques have been applied to solve differential equations of fractional order [10, 15].

Korteweg and de Vries proposed the Korteweg-de Vries equation to describe wave behavior, which's very important in understanding ion-acoustic waves in plasmas and acoustic waves in harmonic crystals [16,17]. In the ten years, many researchers have studied the Rosenau-Korteweg-de Vries-regularised long wave equation, which is one of the partial differential equations. This equation has been studied in areas using different methods, such as the Haar wavelet method and the Chebyshev polynomials [18–23]. We consider the Rosenau-KdV-RLW equation [24, 25], which is given by

$$u_\varsigma \gamma(u_{\xi\xi})_\varsigma + (u_{\xi\xi\xi\xi})_\varsigma + \beta u_{\xi\xi\xi} + u_\xi + \mu {u^2}_\xi = 0 \tag{1}$$

with initial and boundary conditions

$$\begin{aligned} &u(\xi,0) = f(\xi), \quad u(a,\varsigma) = g_1(\varsigma), \quad u(b,\varsigma) = g_2(\varsigma), \\ &u_\xi(a,\varsigma) = g_3(\varsigma), \quad u_\xi(b,\varsigma) = g_4(\varsigma), \qquad a \le \xi \le b. \end{aligned} \tag{2}$$

If $\gamma = \mu = 0$ the equation becomes the KdV equation and when $\beta = \mu = 0$ it reduces to the Rosenau equation.

In this paper, we study the time-Rosenau-KdV-RLW equation of fractional order α, which is

$$\mathcal{D}_\varsigma^\alpha u_\varsigma \Gamma(u_{\xi\xi})_\varsigma + (u_{\xi\xi\xi\xi})_\varsigma + \beta u_{\xi\xi\xi} + u_\xi + \mu {u^2}_\xi = 0, \tag{3}$$

where $\varsigma \in (0,\infty]$ and $a \le \xi \le b$ are the temporal and spatial terms. $\mathcal{D}_t^\alpha$ is the Caputo fractional derivative and γ, β and μ are advection and dispersion constants.

We use a semi-analytical approach called approximate analytical method to solve this equation. It is also very simple and effective in solving highly nonlinear situations. This method has been used by researchers to solve different differential systems and models in diverse fields. The novelty of this work lies in the application of the Approximate Analytical Method to the time-fractional Rosenau–KdV–RLW equation. Although AAM has been successfully employed for various nonlinear fractional differential equations, its implementation for the present model has not been reported previously. The method provides accurate series solutions with reduced computational complexity. This study shows the dynamics of the proposed equation using the considered method with the aid of the Caputo operator. The results demonstrate the accuracy and effectiveness of the proposed method and highlight the capability of the considered fractional derivative to capture the underlying dynamics of the model. Thabet et al. [26] studied Korteweg-de Vries (KdV) equations of non-integer order to find the analytical travelling wave solutions with the aid of AAM. The fractional Navier-Stokes model has been examined using AAM which yields the numerical solutions with the aid of the Liouville-Caputo operator by Umar et al [27]. Chethan et al. [7] investigated the solid tumour invasion model of fractional order via AAM and formulated the numerical solutions in series form with the aid of the Caputo fractional operator. Kumar et al. [4] applied AAM to solve the foam drainage equation and explored the dynamical behaviour of foams using time-fractional derivative. This article is divided into sections: Section 2 reviews some notions and properties related to the Laplace transform and fractional calculus. The algorithm of the projected method and its analytical solution, for fractional partial differential equations is seen in Section 3. Section 4 provides examples of the Rosenau-KdV-RLW equation solved using the AAM. The description of results and graphical representation are portrayed in Section 5 and concluded with our findings in Section 6.

2 Preliminaries

Definition 2.1. *For a function $f(t) \in C_\mu$ $(\mu \geq -1)$, the Riemann-Liouville (RL) fractional integral is specified by $I_\varsigma^\alpha f(t)$ and it is demacrated [29] as*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (u-v)^{\alpha-1} f(v)dv, \qquad (4)$$
$$I_t^0 f(t) = f(t).$$

Definition 2.2. *The definition of the fractional derivative of function $f \in C_{-1}^r$ is defined as follows in terms of Caputo sense [30] :*

$$\mathcal{D}_t^\gamma f(t) = \begin{cases} \frac{d^r f(t)}{dt^r}, & \gamma = r \in \mathbb{N}, \\ \frac{1}{\Gamma(r-\gamma)} \int_0^\varepsilon (u-v)^{n-\gamma-1} f^{(r)}(v)dv, & r-1 < \gamma \leqslant r,\ r \in \mathbb{N}. \end{cases} \qquad (5)$$

The notation $\mathbb{N}$ represents the set of natural numbers.

Theorem 2.3. *Let $b > -1$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1, \alpha_2 \geq 0$. Then, the RL fractional partial integral operator*

${}_0J_t^{\alpha}$ *satisfies the following properties for the function* $f(x,t) \in C_{\mu}, \mu > -1$,

$$\begin{aligned} {}_0J_t^{\alpha_1}{}_0J_t^{\alpha_2} f(t) &= {}_0J_t^{\alpha_1+\alpha_2} f(t), \\ {}_0J_t^{\alpha_1}{}_0J_t^{\alpha_2} f(t) &= {}_0J_t^{\alpha_2}{}_0J_t^{\alpha_1} f(t), \\ {}_0J_t^{\alpha} t^b &= \frac{\Gamma(b+1)}{\Gamma(b+\alpha+1)} t^{\alpha+b}. \end{aligned} \tag{6}$$

Theorem 2.4. *Let* $\alpha, t \in \mathbb{R}$, $t \geqslant 0, n-1 < \alpha < \mathbb{N}$. *Then*

$$\begin{aligned} &D_t^{\alpha} J_t^{\alpha} f(t) = f(t), \\ &D_t^{\alpha} J_t^{\alpha} f(t) = f(t) - \sum_{0}^{m-1} \frac{t^k}{k!} \frac{\partial^k f(0^+)}{\partial t^k}. \end{aligned} \tag{7}$$

3 Fundamental Idea of the AAM

We consider the following linear fractional partial differential equation with some constraints.

The equation is given by

$$\begin{gathered} \mathcal{D}_t^{\gamma} u(\bar{\varphi}t) = f(\bar{\varphi}t) + L\bar{u} + N\bar{u}, \quad m-1 < \gamma < m \in N \\ \frac{\partial^i u(\bar{\varphi}t)}{\partial}, \quad i = 0, 1, 2, 3, \dots m-1, \end{gathered} \tag{8}$$

where $\mathcal{D}_t^{\gamma}$ is the Caputo operator of fractional order γ, $f(\bar{\varphi}t)$ is the source term, L and N represent linear and non-linear operators and $\bar{\varphi} = (\varphi_1\varphi_2 \dots \varphi_n) \in \mathbb{R}^n$. For $u(\bar{\varphi}t) = \sum_{k=0}^{\infty} r^k u(\bar{\varphi}t)$, the linear function $L(u)$ holds the following property:

$$L(u(\bar{x}t)) = L\left(\sum_{k=0}^{\infty} r^k u(\bar{\varphi}t)\right) = \sum_{k=0}^{\infty} r^k L\left(u_k(\bar{\varphi}t)\right). \tag{9}$$

Let $u(\bar{\varphi}t) = \sum_{k=0}^{\infty} u_k(\bar{\varphi}t)$ and $u_{\lambda}(\bar{\varphi}t) = \sum_{k=0}^{\infty} \lambda^k u_k(\bar{\varphi}t)$. Then the non-linear operator $N(u_{\lambda})$ satisfies the below conditions:

$$N(u_{\lambda}) = N\sum_{k=0}^{\infty}\left(\lambda^k u_k\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\delta}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right)_{\lambda=0}\right)\lambda^n, \tag{10}$$

where λ is the non-zero parameter that lies between 0 and 1. The polynomials $P_n(u_0u_1u_2\dots u_n)$ is defined as follows:

$$P_n(u_0u_1u_2\dots u_n) = \frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\mathrm{n}}\lambda^k u_k\right)\right)_{\lambda=0}. \tag{11}$$

Let $P_n = P_n(u_0u_1u_2\dots u_n)$. In terms of P_n, the term $N(u_{\lambda})$ can be written with the help of the above equation as follows:

$$N(u_{\lambda}) = \sum_{n=0}^{\infty} \lambda^n P_n. \tag{12}$$

Thus the following theorem presents existence of an approximate solution for a NFPDE given in the above equation.

Theorem 3.1. *(Existence Theorem) By defining the functions $f(\bar{\varphi}t)f_i(\bar{\varphi})$ as in the above equation and for $m-1<\gamma<m\in\mathbb{N}$. The solution of the above equation obtained in the series form as follows,*

$$u(\bar{\varphi}t)=f_t^{(-\gamma)}(\bar{\varphi}t)+\sum_{i=0}^{m-1}\frac{t^i}{i!}f_i(\bar{\varphi})+\sum_{i=0}^{m-1}\left[L_t^{-\gamma}u_{(k-1)}+P_{(k-1)t}^{(-\gamma)}\right], \tag{13}$$

where $p_{(k-1)t}^{(-\gamma)}$ and $L_t^{-\gamma}u_{(k-1)}$ are the Riemann-Liouville fractional integral of order γ for P_{k-1} and $L\left(u_{k-1}\right)$ concerning t.

Proof. Let the solution $u(\bar{\varphi}t)$ of the above equation be

$$u(\bar{\varphi}t)=\sum_{k=0}^{\infty}u_k(\bar{\varphi},t). \tag{14}$$

Let us consider the given below terms to solve the NFPDE in the equation

$$\mathcal{D}_t^{\gamma}u_\lambda(\bar{\varphi}t)=\lambda\left[f(\bar{\varphi}t)+L\left(u_\lambda\right)+N\left(u_\lambda\right)\right]\quad 0\leq\lambda\leq 1 \tag{15}$$

with initial constraints

$$\frac{\partial^i u_\lambda(\bar{\varphi}0)}{\partial t^i}=g_i(\bar{\varphi})\quad i=0,1,2\ldots,m-1. \tag{16}$$

Suppose that the above equation has the solution in the form

$$u_\lambda(\bar{\varphi}t)=\sum_{k=0}^{\infty}\lambda^k u_k(\bar{\varphi}t). \tag{17}$$

Now using the above equation for the above equation and by the above equation we have

$$u_\lambda(\bar{\varphi}t)=\sum_{i=0}^{m-1}\frac{t^i}{i!}\frac{\partial^i u_\lambda(\bar{\varphi}0)}{\partial t^i}+\lambda_0 t)+L\left(u_\lambda\right)+N\left(u_\lambda\right). \tag{18}$$

The above equation can be written as below by using the equation

$$u_\lambda(\bar{\varphi}t)=\sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\varphi})+\lambda\left[f_t^{(-\gamma)}(\bar{\varphi}t)+J_t^{\gamma}\left[L\left(u_\lambda\right)\right]+J_t^{\gamma}\left[N\left(u_\lambda\right)\right]\right]. \tag{19}$$

Substitute the above equation into the above equation, which gives

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda^k u_k(\bar{\varphi}t)&=\sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\varphi})+\lambda f_t^{(-\gamma)}(\bar{\varphi}t)+J_t^{\gamma}\lambda\sum_{k=0}^{\infty}\left[L\left(\lambda^k u_k\right)\right]\\&+J_t^{\gamma}\lambda\sum_{n=0}^{\infty}\left[\frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right)_{\lambda=0}\right]\lambda^n.\end{aligned} \tag{20}$$

Using the above equation can be reduced to the following form

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda^k v_k(\bar{\varphi},t)&=\sum_{i=0}^{m-1}\frac{t^i}{i!}g_i(\bar{\varphi})+\lambda f_t^{(-\gamma)}(\bar{\varphi}t)\\&+J_t^{\gamma}\lambda\sum_{k=0}^{\infty}\left[L\left(\lambda^k v_k\right)\right]+J_t^{\gamma}\lambda\sum_{n=0}^{\infty}P_n\lambda^n\end{aligned} \tag{21}$$

equating the coefficients of same powers of λ in the above equation we have

$$\begin{aligned} u_0(\bar{\varphi}, t) &= \sum_{i=0}^{m-1} \frac{t^i}{i!} g_i(\bar{\varphi}), \\ u_1(\bar{\varphi}t) &= f_t^{(-\gamma)}(\bar{\varphi}t) + L_t^{(-\gamma)} u_0 + P_{0t}^{(-\gamma)}, \\ u_k(\bar{\varphi}t) &= L_t^{(-\gamma)} u_{k-1} + P_{(k-1)t}^{(-\gamma)}, \quad k = 2, 3, \ldots \end{aligned} \tag{22}$$

Rewriting the the above equation, which gives approximate the solution of the above equation. With the aid of the above equation we have

$$u(\bar{\varphi}t) = \lim_{\lambda \to 1} u_\lambda(\bar{\varphi}t) = u_0(\bar{\varphi}t) + u_1(\bar{\varphi}t) + \sum_{k=2}^{\infty} u_k(\bar{\varphi}, t). \tag{23}$$

We can see that, $\frac{\partial^i u(\bar{\varphi}0)}{\partial t^i} = \lim_{\lambda \to 1} \frac{\partial^i u_\lambda(\bar{\varphi}0)}{\partial t^i} \implies g_i(\bar{\varphi}) = f_i(\bar{\varphi})$. Replacing the above equation ends the proof. □

4 Solution using AAM

In this section, we apply the approximate analytical method (AAM) to some nonlinear time-fractional equations to construct series solutions and to verify consistency with known analytical solutions.

Example 4.1. *Consider the nonlinear time-fractional Rosenau-KdV-RLW equation with* $\gamma = 0$, $\beta = 1$, *and* $\mu = 0.5$ *in the domain* $[-70, 100]$ *is given by,*

$$\mathcal{D}_\varsigma^\alpha u - \gamma(u_{\xi\xi})_\varsigma + (u_{\xi\xi\xi\xi})_\varsigma + \beta u_{\xi\xi\xi} + u_\xi + \mu {u^2}_\xi = 0, \tag{24}$$

subject to the initial condition

$$u(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi). \tag{25}$$

We first rewrite (24) in the form

$$\mathcal{D}_t^\gamma u(\varphi, t) = \gamma(u_{\xi\xi})_\varsigma - (u_{\xi\xi\xi\xi})_\varsigma - \beta u_{\xi\xi\xi} - u_\xi - \mu {u^2}_\xi. \tag{26}$$

With the help of the AAM algorithm [7], assume that the solution of the (24) in the below manner

$$u(\varphi, t) = \sum_{k=0}^{\infty} u_k(\varphi, t). \tag{27}$$

Consider (26) have an approximate solution

$$\mathcal{D}_t^\gamma u(\varphi, t) = \lambda \left[\gamma(u_{\xi\xi})_\varsigma - (u_{\xi\xi\xi\xi})_\varsigma - \beta u_{\xi\xi\xi} - u_\xi - \mu {u^2}_\xi\right]. \tag{28}$$

Suppose that (26) has the series solution as

$$u_\lambda(\varphi, t) = \sum_{k=0}^{\infty} \lambda^k u_k(\varphi, t). \tag{29}$$

Applying the Riemann–Liouville fractional integral J_t^γ on both sides of (28) and using the fractional integral identity associated with (29), we obtain

$$u_\lambda(\varphi, t) = \sin\varphi + \lambda_0 J_t^\gamma\left[\gamma(u_{\xi\xi})_\varsigma - (u_{\xi\xi\xi\xi})_\varsigma - \beta u_{\xi\xi\xi} - u_\xi - \mu {u^2}_\xi\right]. \tag{30}$$

Substituting the series (29) into (30) gives

$$\sum_{k=0}^{\infty} \lambda^k u_k(\varphi, t) = \sin\varphi + \lambda_0 J_t^\gamma\left[\sum_{n=0}^{\infty} \lambda^n \mathcal{A}_n(\varphi, t)\right], \tag{31}$$

where $\mathcal{A}_n$ denotes the Adomian-type polynomial corresponding to

$$\mathcal{N}[u] = \gamma(u_{\xi\xi})_\varsigma - (u_{\xi\xi\xi\xi})_\varsigma - \beta u_{\xi\xi\xi} - u_\xi - \mu {u^2}_\xi,$$

expanded in terms of the components u_k.

Equating coefficients of like powers of λ from (31), we obtain the recursive relations

$$\begin{aligned}
&\mathcal{O}(\lambda^0) : u_0(\varphi, t) = \sin\varphi,\\
&\mathcal{O}(\lambda^1) : u_1(\varphi, t) = J_t^\gamma\big[\mathcal{A}_0(\varphi, t)\big],\\
&\mathcal{O}(\lambda^2) : u_2(\varphi, t) = J_t^\gamma\big[\mathcal{A}_1(\varphi, t)\big],\\
&\qquad\qquad \vdots
\end{aligned}$$

Since $u_0(\varphi, t)$ is independent of t, each fractional integral J_t^γ introduces factors of t^γ, $t^{2\gamma}$, etc., divided by Gamma functions. Consequently, the first components read

$$\begin{aligned}
u_0(\varphi, t) &= \sin\varphi,\\
u_1(\varphi, t) &= \frac{t^\gamma}{\Gamma(\gamma+1)} F_1(\varphi),\\
u_2(\varphi, t) &= \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} F_2(\varphi),
\end{aligned}$$

where F_1 and F_2 are trigonometric combinations obtained by inserting u_0 into $\mathcal{N}[u]$ and into the higher-order polynomials $\mathcal{A}_n$.

Summing the series, we obtain the AAM series solution

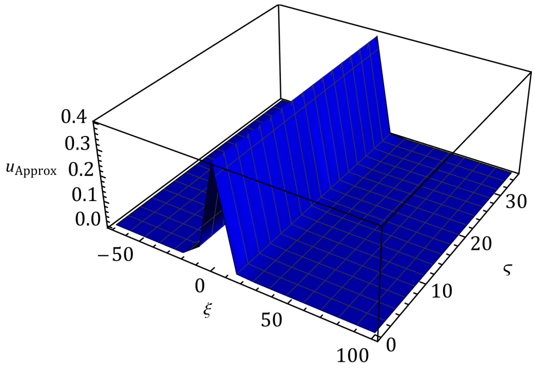
$$u(\varphi, t) = \sum_{k=0}^{\infty} u_k(\varphi, t). \tag{32}$$

Here, the analytical solution for $\alpha = 1$ of (24) is given by,

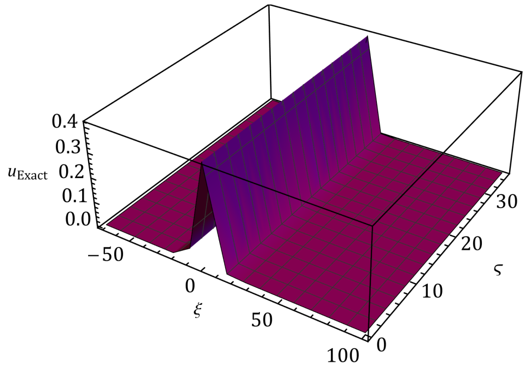
$$u(\xi, \varsigma) = c_1 \operatorname{sech}^4[c_2(\xi - c_3\varsigma)] \tag{33}$$

with

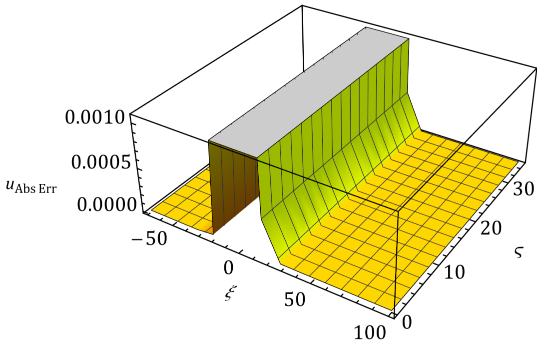
$$\begin{aligned}
c_1 &= -\frac{35}{24} + \frac{35}{312}\sqrt{313},\\
c_2 &= \frac{1}{24}\sqrt{-26 + 2\sqrt{313}},\\
c_3 &= \frac{1}{2} + \frac{1}{26}\sqrt{313}.
\end{aligned}$$



(a)



(b)



(c)

Figure 1: Surface plots of (**a**) achieved series solution, (**b**) the exact solution and (**c**) the absolute error between them at $\alpha = 1$.

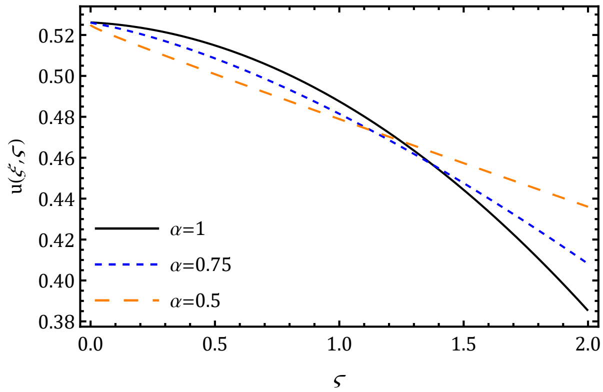


Figure 2: Nature of achieved solution for 24 when $\xi = 0.1$, for distinct α values.

Example 4.2. *Consider the time-fractional Rosenau–KdV–RLW equation with $\gamma = 0$, $\beta = 1$ and $\mu = 0.5$ in the domain $[-40, 60]$:*

$$\mathcal{D}_{\varsigma}^{\alpha} u_{\varsigma} - \gamma (u_{\xi\xi})_{\varsigma} + (u_{\xi\xi\xi\xi})_{\varsigma} + \beta u_{\xi\xi\xi} + u_{\xi} + \mu (u^2)_{\xi} = 0, \tag{34}$$

subject to the initial condition

$$u(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi). \tag{35}$$

For $\gamma = 0$ and $\beta = 1$, equation (34) can be written as

$$\mathcal{D}_\varsigma^\alpha u_\varsigma(\xi, \varsigma) = \mathcal{N}[u(\xi, \varsigma)], \tag{36}$$

where

$$\mathcal{N}[u] = -(u_{\xi\xi\xi\xi})_\varsigma - u_{\xi\xi\xi} - u_\xi - \mu(u^2)_\xi. \tag{37}$$

Introducing the embedding parameter $\lambda \in [0, 1]$ [7], we construct the homotopy

$$\mathcal{D}_\varsigma^\alpha u_\varsigma(\xi, \varsigma) = \lambda \mathcal{N}[u(\xi, \varsigma)], \qquad u(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi). \tag{38}$$

We assume a series solution of the form

$$u_\lambda(\xi, \varsigma) = \sum_{k=0}^{\infty} \lambda^k u_k(\xi, \varsigma), \tag{39}$$

with

$$u_0(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi), \qquad u_k(\xi, 0) = 0,\ k \geq 1.$$

Let J_ς^α denote the Riemann–Liouville fractional integral of order α in ς. Applying J_ς^α to (38), and using the Caputo–RL relation for u_ς, we first write

$$u_\varsigma(\xi, \varsigma) = u_\varsigma(\xi, 0) + \lambda_0 J_\varsigma^\alpha \big[\mathcal{N}[u(\xi, \varsigma)]\big]. \tag{40}$$

Substituting the series (39) into (40) and using the linearity of the spatial derivatives, we have

$$u_\xi = \sum_{k=0}^{\infty} \lambda^k (u_k)_\xi, \qquad (u^2)_\xi = \sum_{n=0}^{\infty} \lambda^n B_n(\xi, \varsigma),$$

where B_n are polynomials associated with the nonlinearity $(u^2)_\xi$. Hence (40) becomes

$$\sum_{k=0}^{\infty} \lambda^k u_k(\xi, \varsigma) = c_1 \operatorname{sech}^4(c_2\xi) + \lambda_0 J_\varsigma^\alpha \left[\sum_{n=0}^{\infty} \lambda^n \mathcal{B}_n(\xi, \varsigma)\right], \tag{41}$$

where $\mathcal{B}_n$ collects all order-λ^n contributions from the operator (37).

Equating coefficients of like powers of λ in (41), we obtain the recursive system

$$\begin{aligned}
\mathcal{O}(\lambda^0) &: u_0(\xi, \varsigma) = c_1 \operatorname{sech}^4(c_2\xi), \\
\mathcal{O}(\lambda^1) &: u_1(\xi, \varsigma) = J_\varsigma^\alpha \big[\mathcal{B}_0(\xi, \varsigma)\big], \\
\mathcal{O}(\lambda^2) &: u_2(\xi, \varsigma) = J_\varsigma^\alpha \big[\mathcal{B}_1(\xi, \varsigma)\big], \\
&\vdots
\end{aligned}$$

Thus the AAM series solution for (34)–(35) reads

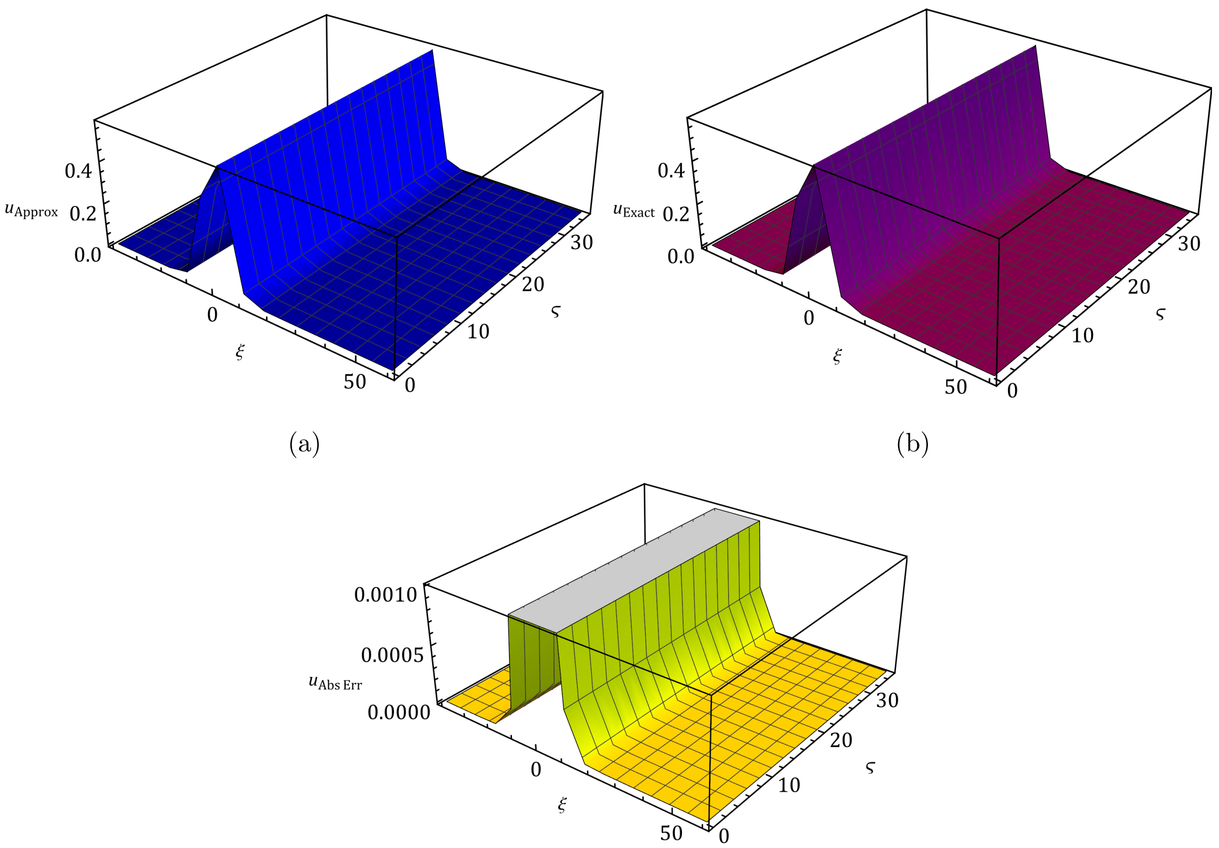
$$u(\xi, \varsigma) = \sum_{k=0}^{\infty} u_k(\xi, \varsigma). \tag{42}$$

The analytical solution for $\alpha = 1$ is given by,

$$u(\xi, \varsigma) = c_1 \operatorname{sech}^4[c_2(\xi - c_3\varsigma)] \tag{43}$$

with

$$\begin{aligned} c_1 &= -\frac{35}{24} + \frac{35}{312}\sqrt{313}, \\ c_2 &= \frac{1}{24}\sqrt{-26 + 2\sqrt{313}}, \\ c_3 &= \frac{1}{2} + \frac{1}{26}\sqrt{313}. \end{aligned}$$



(a) (b) (c)

Figure 3: Surface plots of (**a**) the achieved solution, (**b**) the exact solution, and (**c**) the absolute error between them at $\alpha = 1$.

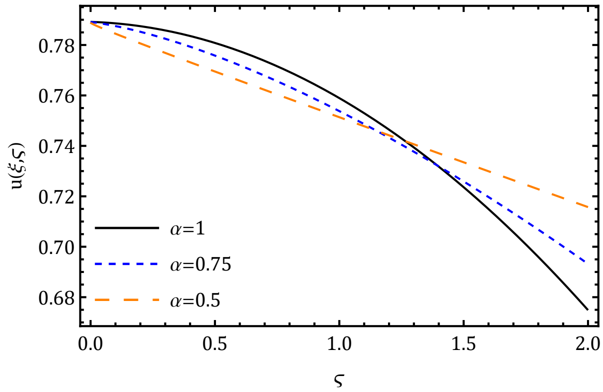


Figure 4: Nature of achieved solution when $\xi = 0.1$, for distinct α values.

Example 4.3. *Consider the time-fractional Rosenau–KdV–RLW equation with* $\gamma = 1$, $\beta = 1$ *and* $\mu = 0.5$ *in the domain* $[-40, 100]$*:*

$$u_\varsigma - \gamma(u_{\xi\xi})_\varsigma + (u_{\xi\xi\xi\xi})_\varsigma + \beta u_{\xi\xi\xi} + u_\xi + \mu(u^2)_\xi = 0, \tag{44}$$

subject to the initial condition

$$u(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi). \tag{45}$$

For $\gamma = 1$ and $\beta = 1$, equation (44) can be rewritten in the evolution form

$$u_\varsigma(\xi, \varsigma) = \mathcal{N}[u(\xi, \varsigma)], \tag{46}$$

where the nonlinear operator $\mathcal{N}$ is given by

$$\mathcal{N}[u] = (u_{\xi\xi})_\varsigma - (u_{\xi\xi\xi\xi})_\varsigma - u_{\xi\xi\xi} - u_\xi - \mu(u^2)_\xi. \tag{47}$$

To apply the AAM [7], we introduce the embedding parameter $\lambda \in [0, 1]$ and construct the homotopy form

$$u_\varsigma(\xi, \varsigma) = \lambda \mathcal{N}[u(\xi, \varsigma)], \qquad u(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi). \tag{48}$$

We seek a solution in the series form

$$u_\lambda(\xi, \varsigma) = \sum_{k=0}^{\infty} \lambda^k u_k(\xi, \varsigma), \tag{49}$$

with

$$u_0(\xi, 0) = c_1 \operatorname{sech}^4(c_2\xi), \qquad u_k(\xi, 0) = 0, \ k \geq 1.$$

Integrating (48) with respect to ς and using the initial condition (45), we obtain the equivalent integral equation

$$u_\lambda(\xi, \varsigma) = u_0(\xi, 0) + \lambda_0 \int_0^\varsigma \mathcal{N}[u_\lambda(\xi, \eta)] \, d\eta. \tag{50}$$

Substituting the series representation (49) into (50), and using the linearity of the spatial derivatives, we obtain

$$u_\xi = \sum_{k=0}^{\infty} \lambda^k (u_k)_\xi, \qquad (u^2)_\xi = \sum_{n=0}^{\infty} \lambda^n B_n(\xi, \varsigma),$$

where B_n denotes the Adomian-type polynomial associated with the nonlinearity $(u^2)_\xi$. Hence (50) can be written as

$$\sum_{k=0}^{\infty} \lambda^k u_k(\xi, \varsigma) = c_1 \operatorname{sech}^4(c_2\xi) + \lambda_0 \int_0^{\varsigma} \left[\sum_{n=0}^{\infty} \lambda^n \mathcal{C}_n(\xi, \eta) \right] d\eta, \tag{51}$$

where each $\mathcal{C}_n$ collects the order-λ^n contributions from (47).

Equating coefficients of like powers of λ in (51), we obtain the recursion

$$\begin{aligned}
\mathcal{O}(\lambda^0) &: u_0(\xi, \varsigma) = c_1 \operatorname{sech}^4(c_2\xi), \\
\mathcal{O}(\lambda^1) &: u_1(\xi, \varsigma) = \int_0^{\varsigma} \mathcal{C}_0(\xi, \eta)\, d\eta, \\
\mathcal{O}(\lambda^2) &: u_2(\xi, \varsigma) = \int_0^{\varsigma} \mathcal{C}_1(\xi, \eta)\, d\eta, \\
&\vdots
\end{aligned}$$

The resulting AAM series solution is given by

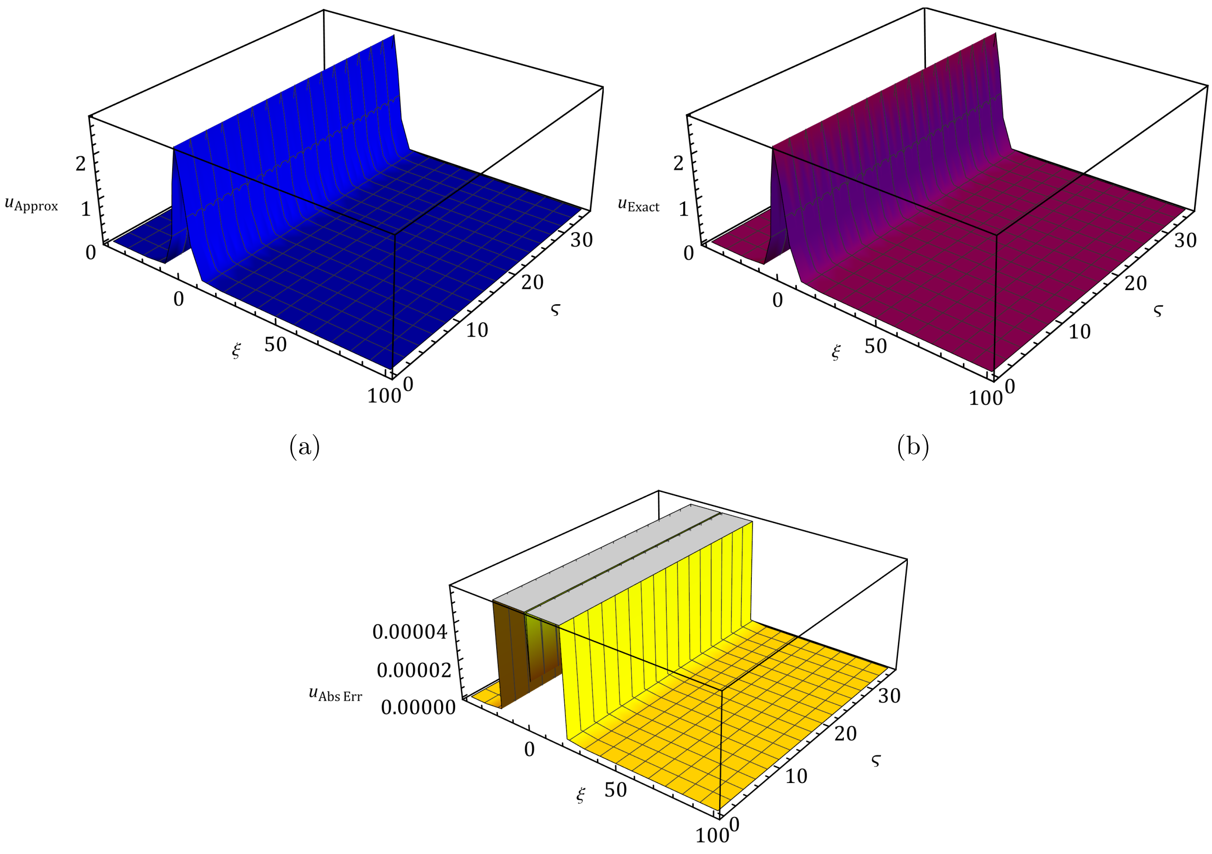
$$u(\xi, \varsigma) = \sum_{k=0}^{\infty} u_k(\xi, \varsigma). \tag{52}$$

The analytical solution for $\alpha = 1$ is given by,

$$u(\xi, \varsigma) = c_1 \operatorname{sech}^4[c_2(\xi - c_3\varsigma)] \tag{53}$$

with coefficients

$$\begin{aligned}
c_1 &= -\frac{5}{456}(-25 + 13\sqrt{457}), \\
c_2 &= \frac{\sqrt{-13 + \sqrt{457}}}{\sqrt{288}}, \\
c_3 &= \frac{241 + 13\sqrt{457}}{266}.
\end{aligned}$$



(c)

Figure 5: Surface plots of (**a**) the AAM solution, (**b**) the exact solution, and (**c**) the absolute error between them, at $\hbar = -1$ and $\alpha = 1$.

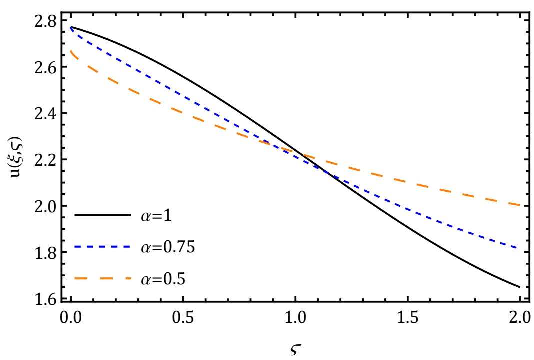


Figure 6: Nature of achieved solution $u(\xi, \varsigma)$ when $\xi = 0.1$, for distinct α values.

5 Results and Discussion

The Approximate Analytical Method has been used to solve the time-fractional Rosenau-KdV-RLW equations with different parameters. The solutions are shown as series that converge quickly and are checked against the solutions when the parameter is one. The graphs and numbers show that this method is effective, reliable and efficient for solving fractional differential equations. For example the behavior of the solution is shown in graphs with the exact solution and the error. The graphs show that the solution from the Approximate Analytical Method is very close to the solution. The error is also very small which means the method is precise and converges well. The solution changes when the fractional-order parameter is changed. When this parameter is small the solution changes slowly. Has strong memory effects.. When the parameter is close to one the solution behaves like the classical integer-order dynamics. This shows that fractional calculus is important for modeling physical systems. The same thing happens with examples. The solution from the Approximate Analytical Method is compared to the solution and the error is shown. The method accurately captures the wave structures and preserves the propagation characteristics of the equation. The error is small which means the method is accurate and stable. The fractional-order parameter affects the behavior of the solution. When the parameter is decreased the wave evolves slowly. Decays non-exponentially. This shows that the fractional model has memory- effects. Overall the graphs and analysis show that the Approximate Analytical Method effectively captures the characteristics of fractional-order Rosenau-KdV-RLW equations. The method incorporates memory effects and nonlocal interactions without discretization or complicated computations. This makes it a reliable framework for solving fractional differential equations in applied mathematics, fluid dynamics, plasma physics and engineering sciences.

6 Conclusion

The Approximate Analytical Method has been used to solve time-fractional Rosenau-KdV-RLW equations. The method combines decomposition with fractional integral operators to generate rapidly convergent series solutions with high accuracy. The solutions are checked against the solutions and the graphs show that they are very close. The error analysis shows that the method provides accurate approximations. The fractional-order parameter plays a role in determining the behavior of the solutions. Lower values produce wave evolution and stronger memory effects. The Approximate Analytical Method has advantages. It avoids discretization, linearization and complicated numerical computations. The method provides an efficient recursive procedure for constructing approximate solutions. It can be used for classes of nonlinear fractional differential equations without introducing unrealistic assumptions. The results show that the Approximate Analytical Method is an reliable framework, for studying nonlinear fractional wave phenomena. Future work can focus on extending the method to dimensional fractional models and other generalized fractional operators. Further studies can also enhance the applicability of the method in world scientific and engineering problems. and also extend our work to complex systems, like higher-dimensional models or models with variable-order fractions. With some optimization and control strategies our method could be more precise and useful. Overall our results show that the AAM is an efficient method, for studying nonlinear fractional differential equations and that it will contribute to the advancement of fractional calculus.

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References

[1] Kilbas, A. A., Srivastava, H. M., & Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*. Elsevier.

[2] Miller, K. S., & Ross, B. (1993). *An introduction to fractional calculus and fractional differential equations*. John Wiley and Sons.

[3] Podlubny, I. (1999). *Fractional differential equations*. Elsevier.

[4] Kumar, C. V. D., Prakasha, D. G., & Turki, N. B. (2025). Exploring the dynamics of fractional-order nonlinear dispersive wave system through homotopy technique. *Open Physics*, *23*(1), 20250128. https://doi.org/10.1515/phys-2025-0128

[5] Tarasov, V. E. (2022). General non-local electrodynamics: Equations and non-local effects. *Annals of Physics*, *445*, 169082. https://doi.org/10.1016/j.aop.2022.169082

[6] Singh, B. K., Baskonus, H. M., Singh, N., Gupta, M., & Prakasha, D. G. (2023). Study of time-fractional nonlinear model governing unsteady flow of polytropic gas. *Axioms*, *12*(3), 285. https://doi.org/10.3390/axioms12030285

[7] Chethan, H. B., Saadeh, R., Prakasha, D. G., Qazza, A., Malagi, N. S., Nagaraja, M., & Sarwe, D. U. (2024). An efficient approximate analytical technique for the fractional model describing the solid tumor invasion. *Frontiers in Physics*, *12*, 1294506. https://doi.org/10.3389/fphy.2024.1294506

[8] Veeresha, P., & Prakasha, D. G. (2019). A novel technique for $(2+1)$-dimensional time-fractional coupled Burgers equations. *Mathematics and Computers in Simulation*, *166*, 324–345. https://doi.org/10.1016/j.matcom.2019.06.005

[9] Kumar, C. D., Prakasha, D. G., Veeresha, P., & Kapoor, M. (2024). A homotopy-based computational scheme for two-dimensional fractional cable equation. *Modern Physics Letters B*, *38*(32), 2450292. https://doi.org/10.1142/S0217984924502920

[10] Veeresha, P., Ilhan, E., Prakasha, D. G., Baskonus, H. M., & Gao, W. (2021). Regarding on the fractional mathematical model of tumour invasion and metastasis. *Computer Modeling in Engineering & Sciences*, *127*(3), 1013–1036. https://doi.org/10.32604/cmes.2021.014988

[11] Zhang, R., Shah, N. A., El-Zahar, E. R., Akgul, A., & Chung, J. D. (2023). Numerical analysis of fractional-order Emden–Fowler equations using modified variational iteration method. *Fractals*, *31*(2), 2340028. https://doi.org/10.1142/S0218348X23400285

[12] Ganie, A. H., AlBaidani, M. M., & Khan, A. (2023). A comparative study of the fractional partial differential equations via novel transform. *Symmetry*, *15*(5), 1101. https://doi.org/10.3390/sym15051101

[13] Alshehry, A. S., Shah, R., Shah, N. A., & Dassios, I. (2022). A reliable technique for solving fractional partial differential equation. *Axioms*, *11*(10), 574. https://doi.org/10.3390/axioms11100574

[14] Veeresha, P., & Prakasha, D. G. (2021). Solution for fractional Kuramoto–Sivashinsky equation using novel computational technique. *International Journal of Applied and Computational Mathematics*, *7*(2), 33. https://doi.org/10.1007/s40819-021-00956-0

[15] Chiranahalli Vijaya, D. K., Doddabhadrappla Gowda, P., & Hadimani, B. (2025). A numerical study on the dynamics of SIR epidemic model through Genocchi wavelet collocation method. *Scientific Reports*, *15*(1), 9780. https://doi.org/10.1038/s41598-025-93820-w

[16] Korteweg, D. J., & De Vries, G. (1895). On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, *39*(240), 422–443. https://doi.org/10.1080/14786449508620739

[17] Zhu, S., & Zhao, J. (2001). The alternating segment explicit–implicit scheme for dispersive equation. *Applied Mathematics Letters*, *14*(6), 657–662. https://doi.org/10.1016/S0893-9659(01)80022-7

[18] Ozer, S., & Kutluay, S. (2005). An analytical–numerical method for solving the Korteweg–de Vries equation. *Applied Mathematics and Computation*, *164*(3), 789–797. https://doi.org/10.1016/j.amc.2004.06.011

[19] Oruc, O., Bulut, F., & Esen, A. (2016). Numerical solution of the KdV equation by Haar wavelet method. *Pramana*, *87*, 1–11. https://doi.org/10.1007/s12043-016-1286-7

[20] Skogestad, J., & Kalisch, H. (2009). A boundary value problem for the KdV equation: Comparison of finite-difference and Chebyshev methods. *Mathematical and Computer Modelling*, *80*(1), 151–163. https://doi.org/10.1016/j.matcom.2009.06.009

[21] Hu, J., Xu, Y., & Hu, B. (2013). Conservative linear difference scheme for Rosenau–KdV equation. *Advances in Mathematical Physics*, *2013*, Article 423718. https://doi.org/10.1155/2013/423718

[22] Avazzadeh, Z., Nikan, O., & Machado, J. A. T. (2020). Solitary wave solutions of the generalized Rosenau–KdV–RLW equation. *Mathematics*, *8*(9), 1601. https://doi.org/10.3390/math8091601

[23] Oruc, O., Esen, A., & Bulut, F. (2024). Numerical solution of the Rosenau–KdV–RLW equation via combination of a polynomial scaling function collocation and finite difference method. *Methods and Applications in the Mathematical Sciences*, *48*(3), 4015–4034. https://doi.org/10.1002/mma.10531

[24] Ozer, S. (2019). Numerical solution of the Rosenau–KdV–RLW equation by operator splitting techniques based on B-spline collocation method. *Numerical Methods for Partial Differential Equations*, *35*(5), 1928–1943. https://doi.org/10.1002/num.22387

[25] Wongsaijai, B., & Poochinapan, K. (2014). A three-level average implicit finite difference scheme to solve equation obtained by coupling the Rosenau–KdV equation and the Rosenau–RLW equation. *Applied Mathematics and Computation*, *245*, 289–304. https://doi.org/10.1016/j.amc.2014.07.075

[26] Thabet, H., Kendre, S., & Peters, J. (2018). Travelling wave solutions for fractional Korteweg–de Vries equations via an approximate–analytical method. *AIMS Mathematics*, *4*, 1203–1222. https://doi.org/10.3934/math.2019.4.1203

[27] Umar, F., Khan, H., Fairouz, T., Evren, H., Dumitru, B., & Haifa, B. J. (2021). New approximate analytical technique for the solution of time fractional fluid flow models. *Advances in Continuous and Discrete Models*, *2021*(1), Article 20. https://doi.org/10.1186/s13662-021-03240-z

[28] Kumar, C. D., Prakasha, D. G., & Amruthalakshmi, M. R. (2025). A comprehensive study on dynamical analysis and numerical simulation of foam drainage equation using time-fractional derivative. *Franklin Open*, *10*, Article 100456. https://doi.org/10.1016/j.fraope.2025.100456

[29] Nonnenmacher, T. F., & Metzler, R. (1995). On the Riemann–Liouville fractional calculus and some recent applications. *Fractals*, *3*, 57–66. https://doi.org/10.1142/S0218348X95000497

[30] Caputo, M. (1969). *Elasticità e dissipazione*. Zanichelli.