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On Rational Contractions and Fixed Points in m-Hemi-Metric Spaces

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Abstract

In this paper, we investigate rational contraction mappings in the setting of m-hemi-metric spaces. Unlike the classical rational contractions formulated in metric, G-metric, and b-metric spaces, the proposed contractive condition is established directly within the framework of m-hemi-metrics by exploiting their multi-point distance structure. Under this nonlinear rational contractive condition, we prove an existence and uniqueness theorem for fixed points in h-complete m-hemi-metric spaces. Several examples are presented to illustrate the applicability of the proposed contraction. As an application, we establish the existence and uniqueness of continuous solutions for a class of nonlinear Fredholm integral equations by means of the obtained fixed point theorem. The results extend Banach-type fixed point theory to the setting of m-hemi-metric spaces and contribute to the study of nonlinear problems in generalized distance spaces.

1 Introduction

Fixed point theory is one of the fundamental branches of nonlinear analysis and provides indispensable tools for the study of differential equations, integral equations, optimization, dynamical systems, economics, engineering, computer science, and mathematical modelling. Since the celebrated Banach contraction principle [3], extensive research has been devoted to extending fixed point results from classical metric spaces to more general distance structures. These extensions have significantly broadened the applicability of fixed point methods by providing suitable frameworks for nonlinear problems that cannot be adequately treated within ordinary metric spaces.

The development of generalized distance spaces has progressed through several important stages. Wilson [27] introduced semi-metric spaces, whereas Gähler [16] proposed the notion of 2-metric spaces. Subsequently, Lal and Singh [23], Chi and Thuy [7], and Lahiri *et al.* [22] established various fixed point

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results in 2-metric spaces. Bakhtin [2] initiated the study of quasi-metric spaces, followed by important contributions due to Dhage [13], Branciari [4], Das [8], and many other researchers, leading to a rich variety of generalized metric-type spaces.

Among the most important generalized distance structures are partial metric spaces introduced by Matthews [24], modular metric spaces due to Chistyakov [6], and G-metric spaces introduced by Mustafa and Sims [25]. These spaces have generated extensive research activity in fixed point theory, as reflected in the works of Jleli and Samet [19, 20], George *et al.* [17], Ding *et al.* [14], Fadail *et al.* [15], Abtahi *et al.* [1], Chandok *et al.* [5], and the survey article of Khamsi [21], which provides an overview of generalized metric spaces and their applications in nonlinear analysis.

The geometric theory of multi-point distance functions has also attracted considerable attention. Deza and Rosenberg [12] investigated the structure of m-hemimetrics from a combinatorial perspective, while Deza and Deza [10, 11] presented a comprehensive treatment of generalized distance functions and their geometric and combinatorial properties. Recently, Ozturk and Radenović [26] introduced the concept of m-hemi-metric spaces and established Banach-type fixed point theorems, thereby providing an abstract framework in which the distance depends simultaneously on multiple points. This framework extends several classical generalized distance spaces and offers a natural setting for the development of fixed point theory.

Parallel to the development of generalized distance spaces, rational contractive conditions have become an important topic in fixed point theory. Unlike the classical Banach contraction, rational contractions incorporate nonlinear combinations of distance functions and are capable of treating mappings that do not satisfy ordinary Lipschitz-type contractive conditions. Fundamental contributions in this direction include the rational contraction principles introduced by Dass and Gupta [9] and Jaggi [18], together with numerous subsequent extensions in metric, partial metric, b-metric, and other generalized metric spaces.

However, the existing rational contractive conditions have been formulated for distance functions involving one or two variables. Their proofs rely heavily on the geometric properties of the underlying spaces and cannot be directly extended to the recently introduced m-hemi-metric framework, where the distance depends on multiple variables and satisfies the m-simplex inequality. Consequently, new analytical techniques are required to establish fixed point results in this setting.

The principal contribution of the present paper is the introduction of a new rational contractive condition formulated intrinsically in terms of the m-hemi-metric. Unlike existing approaches, the proposed contraction is defined directly by the multi-point distance function without reducing the problem to an associated ordinary metric. This enables the development of a Banach-type fixed point theory that is genuinely adapted to the geometry of m-hemi-metric spaces.

Under suitable h-completeness assumptions, we establish existence and uniqueness theorems for rational contractions in m-hemi-metric spaces. Several illustrative examples, including nonconstant rational contractions, are presented to demonstrate the applicability of the proposed theory. As an application, the obtained results are applied to a class of nonlinear Fredholm integral equations to establish the existence and uniqueness of continuous solutions.

The main contributions of the present paper are summarized as follows.

- A new rational contractive condition is introduced in the framework of m-hemi-metric spaces.
- Existence and uniqueness theorems for rational contractions are proved in h-complete m-hemi-metric spaces.
- The proposed contraction is shown to be fundamentally different from existing rational contractions in metric, partial metric, b-metric, and related generalized metric spaces.
- Several illustrative examples, including nonconstant rational contractions, demonstrate the applicability of the developed theory.
- An application to nonlinear Fredholm integral equations is presented, establishing the existence and uniqueness of continuous solutions.

The remainder of the paper is organized as follows. Section 2 recalls the necessary preliminaries on m-hemi-metric spaces. Section 3 introduces the new rational contraction and develops its basic properties. Section 4 contains the main fixed point theorem together with its proof. Section 5 presents illustrative examples. Section 6 is devoted to an application to nonlinear Fredholm integral equations, and the final section contains concluding remarks.

2 Preliminaries

In this section, we recall several definitions and preliminary results that will be used throughout the paper. The notions of m-hemi-metric spaces, h-convergence, and h-completeness are adopted from the works of Deza and Rosenberg [12] and Öztürk and Radenović [26].

Definition 2.1 (m-Hemi-Metric Space)**.** *Let $m \in \mathbb{N}$ and let H be a nonempty set containing at least $m+2$ distinct elements. A mapping*

$$d_h : H^{m+1} \longrightarrow [0, \infty)$$

*is called an m-*hemi-metric *on H if the following conditions hold for every $h_1, h_2, \ldots, h_{m+2} \in H$.*

(H1) ***(Non-negativity)***

$$d_h(h_1, h_2, \ldots, h_{m+1}) \geq 0.$$

(H2) ***(Identity of indiscernibles)***

$$d_h(h_1, h_2, \ldots, h_{m+1}) = 0$$

if and only if

$$h_1 = h_2 = \cdots = h_{m+1}.$$

(H3) ***(Symmetry)*** *For every permutation* σ *of* $\{1, 2, \ldots, m+1\}$,

$$d_h(h_{\sigma(1)}, h_{\sigma(2)}, \ldots, h_{\sigma(m+1)}) = d_h(h_1, h_2, \ldots, h_{m+1}).$$

(H4) ***(m-Simplex Inequality)***

$$d_h(h_1, h_2, \ldots, h_{m+1}) \leq \sum_{i=1}^{m+1} d_h(h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_{m+2}).$$

Then the pair (H, d_h) *is called an* m-hemi-metric space.

Definition 2.2 (h-Convergence)**.** *Let* (H, d_h) *be an* m*-hemi-metric space and let* $\{h_n\}_{n=1}^{\infty} \subseteq H$.

The sequence $\{h_n\}$ *is said to* h-converge *to a point* $y \in H$ *if*

$$\lim_{n_1, n_2, \ldots, n_m \to \infty} d_h(h_{n_1}, h_{n_2}, \ldots, h_{n_m}, y) = 0.$$

In this case, we write

$$h_n \xrightarrow{h} y.$$

Definition 2.3 (h-Cauchy Sequence)**.** *Let* (H, d_h) *be an* m*-hemi-metric space.*

A sequence $\{h_n\}_{n=1}^{\infty} \subseteq H$ *is called an* h-Cauchy sequence *if*

$$\lim_{n_1, n_2, \ldots, n_{m+1} \to \infty} d_h(h_{n_1}, h_{n_2}, \ldots, h_{n_{m+1}}) = 0.$$

Definition 2.4 (h-Complete m-Hemi-Metric Space)**.** *An* m*-hemi-metric space* (H, d_h) *is said to be* h-complete *if every* h*-Cauchy sequence in* H *is* h*-convergent to some point of* H.

The following examples illustrate important classes of m-hemi-metric spaces and demonstrate that such spaces arise naturally in both finite-dimensional and function spaces. They will also serve to illustrate the applicability of the fixed point results established in the subsequent sections.

Example 2.1. *Let* $H \subseteq \mathbb{R}$ *and let* $m \in \mathbb{N}$. *Define*

$$d_h^{(1)} : H^{m+1} \longrightarrow [0, \infty)$$

by

$$d_h^{(1)}(h_1, h_2, \ldots, h_{m+1}) = \sum_{1 \leq i < j \leq m+1} |h_i - h_j|.$$

Then $d_h^{(1)}$ *is an* m*-hemi-metric on* H.

Proof. We verify conditions (H1)–(H4).

(H1) Since each term $|h_i - h_j| \geq 0$,
$$d_h^{(1)}(h_1, \ldots, h_{m+1}) \geq 0.$$

(H2) We have
$$d_h^{(1)}(h_1, \ldots, h_{m+1}) = 0$$
if and only if
$$|h_i - h_j| = 0 \qquad \text{for all } 1 \leq i < j \leq m+1,$$
which is equivalent to
$$h_1 = h_2 = \cdots = h_{m+1}.$$

(H3) Since the defining sum is invariant under every permutation of $h_1, \ldots, h_{m+1}$, $d_h^{(1)}$ is symmetric.

(H4) Let $z \in H$. By the triangle inequality,
$$|h_p - h_q| \leq |h_p - z| + |h_q - z|, \qquad 1 \leq p < q \leq m+1.$$
Summing over all pairs yields
$$d_h^{(1)}(h_1, \ldots, h_{m+1}) \leq m \sum_{i=1}^{m+1} |h_i - z|.$$
Moreover,
$$\sum_{r=1}^{m+1} d_h^{(1)}(h_1, \ldots, \widehat{h_r}, \ldots, h_{m+1}, z) = (m-1) d_h^{(1)}(h_1, \ldots, h_{m+1}) + m \sum_{i=1}^{m+1} |h_i - z|.$$
Since
$$(m-1) d_h^{(1)}(h_1, \ldots, h_{m+1}) \geq 0,$$
it follows that
$$d_h^{(1)}(h_1, \ldots, h_{m+1}) \leq \sum_{r=1}^{m+1} d_h^{(1)}(h_1, \ldots, \widehat{h_r}, \ldots, h_{m+1}, z).$$
Hence the m-simplex inequality holds.

Therefore, $d_h^{(1)}$ satisfies axioms (H1)–(H4) and thus defines an m-hemi-metric on H. □

Example 2.2. *Let $H \subseteq \mathbb{R}$. For*
$$h_1, \ldots, h_{m+1} \in H,$$
define
$$\overline{h} = \frac{1}{m+1} \sum_{i=1}^{m+1} h_i$$

and

$$d_h^{(2)}(h_1, \ldots, h_{m+1}) = \sum_{i=1}^{m+1} |h_i - \overline{h}|.$$

Then $d_h^{(2)}$ *is an* m*-hemi-metric on* H*.*

Proof. We verify conditions (H1)–(H4).

(H1) Since

$$|h_i - \overline{h}| \geq 0, \qquad i = 1, \ldots, m+1,$$

it follows that

$$d_h^{(2)}(h_1, \ldots, h_{m+1}) \geq 0.$$

(H2) If

$$d_h^{(2)}(h_1, \ldots, h_{m+1}) = 0,$$

then

$$|h_i - \overline{h}| = 0, \qquad i = 1, \ldots, m+1,$$

and hence

$$h_1 = h_2 = \cdots = h_{m+1}.$$

The converse is immediate.

(H3) Since the arithmetic mean is invariant under every permutation of $h_1, \ldots, h_{m+1}$, the value of $d_h^{(2)}$ remains unchanged. Hence $d_h^{(2)}$ is symmetric.

(H4) The functional

$$F(x_1, \ldots, x_{m+1}) = \sum_{i=1}^{m+1} \left| x_i - \frac{1}{m+1} \sum_{j=1}^{m+1} x_j \right|$$

is symmetric, positively homogeneous, translation invariant and convex. Therefore, by the convexity property of F,

$$F(h_1, \ldots, h_{m+1}) \leq \sum_{r=1}^{m+1} F(h_1, \ldots, \widehat{h_r}, \ldots, h_{m+2}),$$

where $\widehat{h_r}$ denotes the omission of the r-th coordinate. Equivalently,

$$d_h^{(2)}(h_1, \ldots, h_{m+1}) \leq \sum_{r=1}^{m+1} d_h^{(2)}(h_1, \ldots, \widehat{h_r}, \ldots, h_{m+2}).$$

Thus, the m-simplex inequality holds.

Hence $d_h^{(2)}$ satisfies (H1)–(H4) and therefore $(H, d_h^{(2)})$ is an m-hemi-metric space. □

Example 2.3. *Let X be a nonempty set and let $C_b(X)$ denote the space of all bounded real-valued functions on X. Define*

$$d_h^{(3)}(f_1, \dots, f_{m+1}) = \sup_{x \in X} \max_{1 \le i < j \le m+1} |f_i(x) - f_j(x)|.$$

Then $d_h^{(3)}$ is an m-hemi-metric on $C_b(X)$.

Proof. We verify conditions (H1)–(H4).

(H1) Since

$$|f_i(x) - f_j(x)| \ge 0$$

for every $x \in X$ and all $1 \le i < j \le m+1$,

$$d_h^{(3)}(f_1, \dots, f_{m+1}) \ge 0.$$

(H2) If

$$d_h^{(3)}(f_1, \dots, f_{m+1}) = 0,$$

then

$$|f_i(x) - f_j(x)| = 0$$

for every $x \in X$ and all i, j. Hence,

$$f_1 = f_2 = \cdots = f_{m+1}.$$

The converse is immediate. Therefore,

$$d_h^{(3)}(f_1, \dots, f_{m+1}) = 0 \iff f_1 = f_2 = \cdots = f_{m+1}.$$

(H3) Since permutations merely relabel the indices,

$$d_h^{(3)}(f_{\sigma(1)}, \dots, f_{\sigma(m+1)}) = d_h^{(3)}(f_1, \dots, f_{m+1})$$

for every permutation σ. Hence $d_h^{(3)}$ is symmetric.

(H4) Assume $m \ge 2$, let $g = f_{m+2} \in C_b(X)$, and for $r = 1, \dots, m+1$, define

$$S_r = d_h^{(3)}(f_1, \dots, \widehat{f_r}, \dots, f_{m+1}, g).$$

Fix $x \in X$, and let

$$M(x) = \max_{1 \le i < j \le m+1} |f_i(x) - f_j(x)|.$$

Choose indices $p < q$ such that

$$M(x) = |f_p(x) - f_q(x)|.$$

The pair (f_p, f_q) appears in every S_r except S_p and S_q. Hence,

$$\sum_{r=1}^{m+1} S_r \geq (m-1)M(x) \geq M(x).$$

Since this inequality holds for every $x \in X$, taking the supremum over X gives

$$d_h^{(3)}(f_1, \ldots, f_{m+1}) \leq \sum_{r=1}^{m+1} d_h^{(3)}(f_1, \ldots, \widehat{f}_r, \ldots, f_{m+1}, g).$$

Thus, the m-simplex inequality holds.

Hence $d_h^{(3)}$ satisfies (H1)–(H4) and therefore

$$(C_b(X), d_h^{(3)})$$

is an m-hemi-metric space. □

The above examples demonstrate that m-hemi-metric spaces arise naturally in both Euclidean spaces and spaces of bounded functions. They provide useful models for the abstract theory developed in the subsequent sections and illustrate the applicability of the fixed point results established in this paper.

3 Rational Contraction in Hemi-metric Spaces

We now extend the classical notion of rational contractions to hemi-metric settings.

Definition 3.1 (Rational Contraction)**.** *Let (H, d_h) be an m-hemi-metric space. A mapping*

$$T : H \to H$$

is called a rational contraction *if there exists a constant*

$$k \in [0, 1)$$

such that for every

$$x_1, x_2, \ldots, x_{m+1} \in H,$$

the following inequality holds:

$$d_h(Tx_1, Tx_2, \ldots, Tx_{m+1}) \leq k \, \frac{d_h(x_1, x_2, \ldots, x_{m+1})}{1 + d_h(x_1, x_2, \ldots, x_{m+1})}.$$

The constant k is called the rational contraction constant*.*

The above contractive condition generalizes the classical Banach contraction principle to the framework of m-hemi-metric spaces. The rational term

$$\frac{d_h(x_1, x_2, \ldots, x_{m+1})}{1 + d_h(x_1, x_2, \ldots, x_{m+1})}$$

remains bounded by 1, thereby providing stronger control when the hemi-metric distance becomes large. Moreover,

$$0 \leq \frac{d_h(x_1, x_2, \ldots, x_{m+1})}{1 + d_h(x_1, x_2, \ldots, x_{m+1})} < 1,$$

which guarantees that the image distances are uniformly dominated by the contraction constant k. Consequently, rational contractions combine the shrinking behavior of classical contractions with the stabilizing effect of a bounded nonlinear control term.

Remark 3.1. *If*

$$d_h(x_1, x_2, \ldots, x_{m+1}) \to 0,$$

then

$$\frac{d_h(x_1, x_2, \ldots, x_{m+1})}{1 + d_h(x_1, x_2, \ldots, x_{m+1})} \sim d_h(x_1, x_2, \ldots, x_{m+1}),$$

and the rational contraction behaves asymptotically like a Banach-type contraction. Hence the proposed condition may be viewed as a nonlinear extension of the classical contraction principle in hemi-metric spaces.

Example 3.1. *Let $H = \mathbb{R}$ be equipped with the pairwise-sum hemi-metric*

$$d_h(h_1, \ldots, h_{m+1}) = \sum_{1 \leq i < j \leq m+1} |h_i - h_j|.$$

Define

$$T : H \to H, \qquad T(x) = 0,$$

for all $x \in H$, and let $k \in [0, 1)$.

Then T is a rational contraction.

Proof. *For arbitrary $x_1, \ldots, x_{m+1} \in H$,*

$$Tx_1 = \cdots = Tx_{m+1} = 0,$$

and hence

$$d_h(Tx_1, \ldots, Tx_{m+1}) = d_h(0, \ldots, 0) = 0.$$

Since

$$k\frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})} \geq 0,$$

it follows immediately that

$$d_h(Tx_1, \ldots, Tx_{m+1}) = 0 \leq k\frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})}.$$

Therefore T is a rational contraction. □

Example 3.2. *Let $H = \mathbb{R}$ be equipped with the pairwise-sum hemi-metric*

$$d_h(h_1, \ldots, h_{m+1}) = \sum_{1 \leq i < j \leq m+1} |h_i - h_j|.$$

Define

$$T(x) = \lambda x, \qquad 0 \leq \lambda < 1.$$

Assume that there exists $k \in [0, 1)$ such that

$$\lambda \leq \frac{k}{1 + d_h(x_1, \ldots, x_{m+1})}$$

for all $x_1, \ldots, x_{m+1} \in H$.

Then T is a rational contraction.

Proof. *For arbitrary $x_1, \ldots, x_{m+1} \in H$,*

$$\begin{aligned} d_h(Tx_1, \ldots, Tx_{m+1}) &= \sum_{1 \leq i < j \leq m+1} |\lambda x_i - \lambda x_j| \\ &= \lambda \sum_{1 \leq i < j \leq m+1} |x_i - x_j| \\ &= \lambda\, d_h(x_1, \ldots, x_{m+1}). \end{aligned}$$

Using the assumption on λ, we obtain

$$d_h(Tx_1, \ldots, Tx_{m+1}) = \lambda\, d_h(x_1, \ldots, x_{m+1}) \leq k\frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})}.$$

Hence T is a rational contraction. □

Example 3.3. *Let X be a nonempty set and let $H = C_b(X)$ be equipped with the supremum-diameter hemi-metric*

$$d_h(f_1, \ldots, f_{m+1}) = \sup_{x \in X} \max_{1 \leq i < j \leq m+1} |f_i(x) - f_j(x)|.$$

Fix $g \in C_b(X)$ and define

$$T : H \to H, \qquad T(f) = g,$$

for every $f \in H$.

Then T is a rational contraction for every $k \in [0,1)$.

Proof. *For arbitrary $f_1, \ldots, f_{m+1} \in H$,*

$$Tf_1 = \cdots = Tf_{m+1} = g.$$

Therefore,

$$d_h(Tf_1, \ldots, Tf_{m+1}) = d_h(g, \ldots, g) = 0.$$

Since

$$k\frac{d_h(f_1, \ldots, f_{m+1})}{1 + d_h(f_1, \ldots, f_{m+1})} \geq 0,$$

we have

$$d_h(Tf_1, \ldots, Tf_{m+1}) = 0 \leq k\frac{d_h(f_1, \ldots, f_{m+1})}{1 + d_h(f_1, \ldots, f_{m+1})}.$$

Thus T is a rational contraction. □

4 Main Result

In this section, we establish an existence and uniqueness theorem for rational contractions in h-complete m-hemi-metric spaces. We first record a standard estimate obtained by repeated applications of the m-simplex inequality.

Lemma 4.1. *Let (H, d_h) be an m-hemi-metric space and let $\{x_n\} \subseteq H$. Put*

$$D_n = d_h(x_n, x_{n-1}, \ldots, x_{n-m}), \qquad n \geq m.$$

Then, for all integers

$$n_0 > n_1 > \cdots > n_m \geq N \geq m,$$

repeated applications of the m-simplex inequality give

$$d_h(x_{n_0}, x_{n_1}, \ldots, x_{n_m}) \leq m \sum_{r=N}^{n_0} D_r.$$

Consequently, if

$$\sum_{r=m}^{\infty} D_r < \infty,$$

then $\{x_n\}$ is an h-Cauchy sequence.

Proof. Beginning with

$$d_h(x_{n_0}, x_{n_1}, \ldots, x_{n_m}),$$

apply the m-simplex inequality successively with the intermediate points

$$x_{n_0-1}, x_{n_0-2}, \ldots, x_{n_1},$$

and repeat the same procedure for the remaining coordinates. At each step, the resulting consecutive $(m+1)$-tuple is bounded by some D_r. Since each D_r occurs at most m times, we obtain

$$d_h(x_{n_0}, x_{n_1}, \ldots, x_{n_m}) \leq m \sum_{r=N}^{n_0} D_r.$$

If $\sum D_r$ converges, its tails tend to zero, and therefore

$$\lim_{n_0,\ldots,n_m \to \infty} d_h(x_{n_0}, \ldots, x_{n_m}) = 0.$$

Hence $\{x_n\}$ is h-Cauchy. □

Theorem 4.1. *Let (H, d_h) be an h-complete m-hemi-metric space. Suppose that $T : H \to H$ satisfies*

$$d_h(Tx_1, \ldots, Tx_{m+1}) \leq k \frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})}$$

for all $x_1, \ldots, x_{m+1} \in H$, where $k \in [0, 1)$. Then T has a unique fixed point in H.

Proof. Choose $x_0 \in H$ and define the Picard sequence

$$x_{n+1} = Tx_n, \qquad n \geq 0.$$

For $n \geq m$, put

$$D_n = d_h(x_n, x_{n-1}, \ldots, x_{n-m}).$$

Using the contractive condition, we obtain

$$\begin{aligned} D_{n+1} &= d_h(x_{n+1}, x_n, \ldots, x_{n+1-m}) \\ &= d_h(Tx_n, Tx_{n-1}, \ldots, Tx_{n-m}) \\ &\leq k \frac{D_n}{1 + D_n} \\ &\leq kD_n. \end{aligned} \tag{4.1}$$

By induction,

$$D_n \leq k^{n-m} D_m, \qquad n \geq m. \tag{4.2}$$

Hence,

$$\sum_{n=m}^{\infty} D_n \leq D_m \sum_{n=m}^{\infty} k^{n-m} = \frac{D_m}{1-k} < \infty. \tag{4.3}$$

Let $\varepsilon > 0$. By the convergence of the series, there exists $N \geq m$ such that

$$m \sum_{r=N}^{\infty} D_r < \varepsilon. \tag{4.4}$$

For

$$n_0 > n_1 > \cdots > n_m \geq N,$$

Lemma 4.1 gives

$$\begin{aligned} d_h(x_{n_0}, x_{n_1}, \ldots, x_{n_m}) &\leq m \sum_{r=N}^{n_0} D_r \\ &\leq m \sum_{r=N}^{\infty} D_r \\ &< \varepsilon. \end{aligned} \tag{4.5}$$

Thus,

$$\lim_{n_0,\ldots,n_m \to \infty} d_h(x_{n_0}, \ldots, x_{n_m}) = 0,$$

so $\{x_n\}$ is an h-Cauchy sequence.

Since (H, d_h) is h-complete, there exists $x^* \in H$ such that

$$x_n \xrightarrow{h} x^*. \tag{4.6}$$

We now prove that x^* is a fixed point of T. By the rational contractive condition,

$$\begin{aligned} &d_h(Tx^*, Tx_{n_1}, \ldots, Tx_{n_m}) \\ &\qquad \leq k \, \frac{d_h(x^*, x_{n_1}, \ldots, x_{n_m})}{1 + d_h(x^*, x_{n_1}, \ldots, x_{n_m})}. \end{aligned} \tag{4.7}$$

Since $x_n \xrightarrow{h} x^*$, the right-hand side tends to zero as $n_1, \ldots, n_m \to \infty$. Therefore,

$$d_h(Tx^*, x_{n_1+1}, \ldots, x_{n_m+1}) \longrightarrow 0. \tag{4.8}$$

On the other hand, the shifted sequence $\{x_{n+1}\}$ also h-converges to x^*. Hence, by the uniqueness of an h-limit, (4.8) implies

$$Tx^* = x^*.$$

Thus, x^* is a fixed point of T.

To prove uniqueness, suppose that $y^* \in H$ is another fixed point. Set

$$D = d_h(x^*, y^*, \ldots, y^*).$$

Then

$$\begin{aligned} D &= d_h(Tx^*, Ty^*, \ldots, Ty^*) \\ &\leq k \frac{D}{1 + D}. \end{aligned}$$

If $D > 0$, division by D gives

$$1 \leq \frac{k}{1+D} < 1,$$

which is impossible. Therefore, $D = 0$, and the identity property yields

$$x^* = y^*.$$

Hence T has a unique fixed point in H. □

Corollary 4.1. *Under the hypotheses of Theorem 4.1, the Picard sequence*

$$x_{n+1} = Tx_n$$

h-converges to the unique fixed point x^* of T. Moreover,

$$d_h(x_n, x_{n-1}, \ldots, x_{n-m}) \leq k^{n-m} d_h(x_m, x_{m-1}, \ldots, x_0), \qquad n \geq m.$$

Proof. The convergence of the Picard sequence and the stated estimate follow from (4.2)–(4.6) in the proof of Theorem 4.1. □

Remark 4.1. *Theorem 4.1 may be viewed as an analogue of the Banach fixed point theorem [3] in the setting of m-hemi-metric spaces. The contractive condition is motivated by the rational contractions introduced by Dass and Gupta [9] and Jaggi [18]. Owing to the multi-point nature of the distance, the proof relies on the m-simplex inequality rather than the classical triangle inequality.*

5 Examples

In this section, we present examples illustrating the applicability of Theorem 4.1. In particular, the following example provides a nonconstant rational contraction.

Example 5.1. *Let*

$$H = [0, 1]$$

and define

$$d_h(x_1, \ldots, x_{m+1}) = \sum_{1 \leq i < j \leq m+1} |x_i - x_j|.$$

Then (H, d_h) is an h-complete m-hemi-metric space.

Put

$$M = \binom{m+1}{2}$$

and choose

$$0 < \alpha < \frac{1}{1+M}.$$

Define $T : H \to H$ *by*

$$T(x) = \alpha x.$$

Then T *is nonconstant and maps* H *into itself.*

For arbitrary $x_1, \ldots, x_{m+1} \in H$*, let*

$$D = d_h(x_1, \ldots, x_{m+1}).$$

Since $x_i \in [0, 1]$*, we have*

$$0 \leq D \leq M.$$

Moreover,

$$\begin{aligned} d_h(Tx_1, \ldots, Tx_{m+1}) &= \sum_{1 \leq i < j \leq m+1} |\alpha x_i - \alpha x_j| \\ &= \alpha \sum_{1 \leq i < j \leq m+1} |x_i - x_j| \\ &= \alpha D. \end{aligned}$$

Choose

$$k = \alpha(1 + M).$$

By the choice of α*, we have* $0 < k < 1$*. Since* $D \leq M$*,*

$$\alpha = \frac{k}{1+M} \leq \frac{k}{1+D}.$$

Therefore,

$$\alpha D \leq k \frac{D}{1+D}.$$

Hence,

$$d_h(Tx_1, \ldots, Tx_{m+1}) \leq k \, \frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})}.$$

Thus, T *is a nonconstant rational contraction.*

By Theorem 4.1, T *has a unique fixed point in* H*. Solving*

$$T(x) = x$$

gives

$$\alpha x = x.$$

Since $0 < \alpha < 1$*, it follows that*

$$x = 0.$$

Therefore, the unique fixed point of T *is*

$$x^* = 0.$$

Example 5.2. *Let X be a nonempty set and let $C_b(X)$ denote the space of all bounded real-valued functions on X. Consider the m-hemi-metric*

$$d_h(f_1, \ldots, f_{m+1}) = \sup_{x \in X} \max_{1 \le i < j \le m+1} |f_i(x) - f_j(x)|.$$

Choose a constant

$$0 < \alpha < 1$$

and define the operator

$$T : C_b(X) \to C_b(X), \qquad (Tf)(x) = \alpha f(x), \quad x \in X.$$

Then T is nonconstant and maps $C_b(X)$ into itself.

For arbitrary $f_1, \ldots, f_{m+1} \in C_b(X)$, we have

$$\begin{aligned} d_h(Tf_1, \ldots, Tf_{m+1}) &= \sup_{x \in X} \max_{1 \le i < j \le m+1} |\alpha f_i(x) - \alpha f_j(x)| \\ &= \alpha \sup_{x \in X} \max_{1 \le i < j \le m+1} |f_i(x) - f_j(x)| \\ &= \alpha\, d_h(f_1, \ldots, f_{m+1}). \end{aligned}$$

Since

$$\frac{d_h(f_1, \ldots, f_{m+1})}{1 + d_h(f_1, \ldots, f_{m+1})} \le d_h(f_1, \ldots, f_{m+1}),$$

choose

$$k = \alpha.$$

Then

$$d_h(Tf_1, \ldots, Tf_{m+1}) = \alpha d_h(f_1, \ldots, f_{m+1}) \le k\, \frac{d_h(f_1, \ldots, f_{m+1})}{1 + d_h(f_1, \ldots, f_{m+1})},$$

whenever

$$d_h(f_1, \ldots, f_{m+1}) \le \frac{k - \alpha}{\alpha}.$$

Hence T is a rational contraction on every bounded subset of $C_b(X)$ satisfying the above condition.

By Theorem 4.1, T has a unique fixed point. Indeed,

$$Tf = f$$

implies

$$\alpha f = f,$$

and therefore

$$f \equiv 0.$$

Thus the unique fixed point of T is the zero function.

Example 5.3. *Let*

$$H = \{0, 1, \ldots, m+1\}$$

and define

$$d_h(x_1, \ldots, x_{m+1}) = \sum_{1 \le i < j \le m+1} |x_i - x_j|.$$

Then (H, d_h) is an h-complete m-hemi-metric space.

Define $T : H \to H$ by

$$T(x) = 0, \qquad x \in H.$$

For arbitrary $x_1, \ldots, x_{m+1} \in H$, we have

$$Tx_1 = \cdots = Tx_{m+1} = 0,$$

and hence

$$d_h(Tx_1, \ldots, Tx_{m+1}) = d_h(0, \ldots, 0) = 0.$$

Therefore, for every $k \in [0, 1)$,

$$d_h(Tx_1, \ldots, Tx_{m+1}) \le k \, \frac{d_h(x_1, \ldots, x_{m+1})}{1 + d_h(x_1, \ldots, x_{m+1})}.$$

Thus, T is a rational contraction.

The fixed-point equation

$$T(x) = x$$

gives

$$x = 0.$$

Hence, by Theorem 4.1, the unique fixed point of T is

$$x^* = 0.$$

6 Applications

In this section, we illustrate an application of Theorem 4.1 to establish the existence and uniqueness of solutions of nonlinear Fredholm integral equations.

Let

$$H = C([0, 1], \mathbb{R})$$

be the Banach space of all real-valued continuous functions on $[0, 1]$, equipped with the supremum norm

$$\|u\|_\infty = \sup_{t \in [0,1]} |u(t)|.$$

For $u_1, \ldots, u_{m+1} \in H$, define

$$d_h(u_1, \ldots, u_{m+1}) = \max_{1 \le i < j \le m+1} \|u_i - u_j\|_\infty.$$

By Example 2.3, d_h is an m-hemi-metric on H.

Consider the nonlinear Fredholm integral equation

$$u(t) = \int_0^1 J(t, s, u(s))\, ds, \qquad t \in [0, 1]. \tag{1}$$

Assume that the kernel

$$J : [0, 1] \times [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$$

satisfies the following conditions.

(i) For every $u \in H$, the mapping

$$t \longmapsto \int_0^1 J(t, s, u(s))\, ds$$

is continuous on $[0, 1]$.

(ii) There exists a constant $0 \le \gamma < 1$ such that

$$|J(t, s, x) - J(t, s, y)| \le \gamma \frac{|x - y|}{1 + |x - y|}$$

for all $t, s \in [0, 1]$ and $x, y \in \mathbb{R}$.

Theorem 6.1. *Let J satisfy assumptions* (i) *and* (ii)*. Then the nonlinear Fredholm integral equation* (1) *admits a unique solution in $C([0, 1], \mathbb{R})$.*

Proof. Define the operator

$$T : H \longrightarrow H$$

by

$$(Tu)(t) = \int_0^1 J(t, s, u(s))\, ds, \qquad t \in [0, 1].$$

By assumption (i), $T(H) \subseteq H$.

Let $u_i, u_j \in H$. Then, for every $t \in [0, 1]$,

$$\begin{aligned} |(Tu_i)(t) - (Tu_j)(t)| &\le \int_0^1 |J(t, s, u_i(s)) - J(t, s, u_j(s))|\, ds \\ &\le \gamma \int_0^1 \frac{|u_i(s) - u_j(s)|}{1 + |u_i(s) - u_j(s)|}\, ds. \end{aligned}$$

Since the function

$$\phi(r) = \frac{r}{1+r}, \qquad r \geq 0,$$

is increasing,

$$\frac{|u_i(s) - u_j(s)|}{1 + |u_i(s) - u_j(s)|} \leq \frac{\|u_i - u_j\|_\infty}{1 + \|u_i - u_j\|_\infty}.$$

Hence,

$$\|Tu_i - Tu_j\|_\infty \leq \gamma \frac{\|u_i - u_j\|_\infty}{1 + \|u_i - u_j\|_\infty}.$$

Therefore, for arbitrary $u_1, \dots, u_{m+1} \in H$,

$$\begin{aligned} d_h(Tu_1, \dots, Tu_{m+1}) &= \max_{1 \leq i < j \leq m+1} \|Tu_i - Tu_j\|_\infty \\ &\leq \gamma \max_{1 \leq i < j \leq m+1} \frac{\|u_i - u_j\|_\infty}{1 + \|u_i - u_j\|_\infty} \\ &= \gamma \frac{\max_{1 \leq i < j \leq m+1} \|u_i - u_j\|_\infty}{1 + \max_{1 \leq i < j \leq m+1} \|u_i - u_j\|_\infty} \\ &= \gamma \frac{d_h(u_1, \dots, u_{m+1})}{1 + d_h(u_1, \dots, u_{m+1})}, \end{aligned}$$

where the third equality follows from the monotonicity of $\phi(r) = r/(1+r)$.

Thus T satisfies the rational contractive condition of Theorem 4.1.

Furthermore,

$$d_h(\underbrace{u, \dots, u}_{m \text{ times}}, v) = \|u - v\|_\infty, \qquad u, v \in H.$$

Hence h-convergence and the h-Cauchy property are equivalent to the usual convergence and Cauchy property in the supremum norm. Since

$$\big(C([0,1], \mathbb{R}), \|\cdot\|_\infty\big)$$

is a Banach space, (H, d_h) is h-complete.

Therefore, all the hypotheses of Theorem 4.1 are satisfied. Consequently, T possesses a unique fixed point $u^* \in H$, that is,

$$Tu^* = u^*.$$

Equivalently,

$$u^*(t) = \int_0^1 J(t, s, u^*(s))\, ds, \qquad t \in [0,1].$$

Hence the nonlinear Fredholm integral equation (1) admits a unique continuous solution. □

Example 6.1. *Consider the nonlinear Fredholm integral equation*

$$u(t) = \int_0^1 \frac{1}{8} \frac{u(s)}{1 + u(s)}\, ds, \qquad t \in [0,1],$$

where

$$u \in C([0,1],[0,\infty)).$$

The corresponding kernel is

$$J(t,s,x) = \frac{1}{8}\frac{x}{1+x}, \qquad (t,s,x) \in [0,1]\times[0,1]\times[0,\infty).$$

Clearly, J is continuous. Moreover, for every $x, y \geq 0$,

$$\left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \leq \frac{|x-y|}{1+|x-y|},$$

since

$$(1+x)(1+y) = 1+x+y+xy \geq 1+|x-y|.$$

Hence

$$|J(t,s,x) - J(t,s,y)| \leq \frac{1}{8}\frac{|x-y|}{1+|x-y|},$$

so assumption (ii) *holds with*

$$\gamma = \frac{1}{8} < 1.$$

Furthermore, for every $u \in C([0,1],[0,\infty))$, the mapping

$$t \longmapsto \int_0^1 \frac{1}{8}\frac{u(s)}{1+u(s)}\, ds$$

is constant and therefore continuous. Thus assumption (i) *is also satisfied.*

Consequently, all the hypotheses of Theorem 6.1 are fulfilled. Hence the above nonlinear Fredholm integral equation admits a unique nonnegative continuous solution.

Finally,

$$(T0)(t) = \int_0^1 \frac{1}{8}\frac{0}{1+0}\, ds = 0,$$

so the zero function is a fixed point of T. By uniqueness,

$$u(t) \equiv 0, \qquad t \in [0,1],$$

is the unique continuous solution.

Remark 6.1. *The above example illustrates that Theorem 4.1 is directly applicable to nonlinear Fredholm integral equations whose kernels satisfy a rational contractive condition with respect to the m-hemi-metric*

$$d_h(u_1,\ldots,u_{m+1}) = \max_{1\leq i<j\leq m+1} \|u_i - u_j\|_\infty.$$

Similar arguments may be employed for other classes of nonlinear integral equations, such as Hammerstein and Volterra integral equations.

7 Conclusion

In this paper, we introduced a new class of rational contractions in the framework of m-hemi-metric spaces and established a Banach-type fixed point theorem for such mappings in h-complete m-hemi-metric spaces. The obtained theorem guarantees the existence and uniqueness of fixed points under a nonlinear rational contractive condition and extends the classical Banach contraction principle from ordinary metric spaces to the more general setting of multi-point distance spaces. Several illustrative examples were presented to demonstrate the applicability of the proposed theory, including both finite-dimensional and function-space settings. Furthermore, the developed fixed point theorem was successfully applied to prove the existence and uniqueness of solutions for a class of nonlinear Fredholm integral equations, thereby illustrating its usefulness in the analysis of nonlinear integral equations.

The results presented in this paper enrich the theory of generalized metric spaces and provide a unified framework for studying rational contractive mappings in m-hemi-metric spaces. The techniques developed here may also be useful in investigating other classes of nonlinear operators. Future research may focus on extending these results to common and coupled fixed points, multivalued mappings, cyclic and interpolative contractions, ordered m-hemi-metric spaces, fuzzy and probabilistic m-hemi-metric spaces, and applications to differential equations, integral equations, fractional differential equations, optimization problems, and other nonlinear mathematical models.

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