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# Some New Young-type Integral Inequalities

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**Abstract**

In this article, we present new forms of Young-type integral inequalities. Some of these extend the classical Young integral inequality by including a function and its inverse, while others introduce a function and its ratio transformation or derivative. The proofs of the main results are based on the standard Young product inequality and well-calibrated analytical tools, such as the Chasles integral formula and the Cauchy-Schwarz integral inequality, as well as changes of variables. Several inequalities also address the case where the main function is decreasing, thus expanding the traditional framework, which assumes an increasing function. In particular, the decreasing case leads to elegant formulations. Some theoretical aspects are illustrated with examples.

## 1 Introduction

Integral inequalities are fundamental mathematical tools. They are used to solve both pure and applied problems, particularly those involving complex integral frameworks. The most notable examples can be found in the books [1–5]. For the purposes of this article, we emphasize the Young integral inequality, which was originally established by W.H. Young in [6]. This inequality provides a simple lower bound on the sum of the integral of a continuous (strictly) increasing function and the integral of its inverse. A formal statement of this inequality is given below. Let $c > 0$, $f : [0, c] \to [0, +\infty)$ be a continuous increasing function with $f(0) = 0$, $f^{-1}$ be its inverse, i.e., such that $f^{-1}(f(u)) = f(f^{-1}(u)) = u$ for any $u \in [0, c]$, $a \in [0, c]$ and $b \in [0, f(c)]$. Then the following holds:

$$\int_0^a f(u)du + \int_0^b f^{-1}(u)du \geq ab. \tag{1}$$

This result has found extensive applications across various fields of mathematics. In particular, it is useful in real and functional analysis, optimization theory, probability, signal processing, and the study of differential and integral equations. Over the years, numerous generalizations, refinements and extensions of this inequality have been developed to cover broader classes of functions and more complex settings. Notable contributions and developments in this direction can be found in [7–31]. Of the many

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developments, the results presented in [28] are of particular interest due to their originality in two key areas: the introduction of a tunable parameter $p$ and the use of the ratio (or reciprocal) function $1/f$ instead of the inverse function $f^{-1}$. One of the main contributions is stated in [28, Theorem 7], as formally given below. Let $p > 1$, $q = p/(p-1)$, $c > 0$, $f : [0, c] \to [0, +\infty)$ be a continuous increasing function with $f(0) = 0$, $f^{-1}$ be its inverse, $a \in [0, c]$ and $b \in [0, f(c)]$. Then the following holds:

$$\frac{b}{p}\int_0^a f^p(u)du + \frac{a}{q}\int_0^b [f^{-1}(u)]^q du \geq \left[\int_0^a f(u)du\right]\left[\int_0^a [b - f(u)]du\right]. \tag{2}$$

The introduction of the parameter $p$, together with the use of a product-integral form for the lower bound, makes this inequality particularly innovative. Another important result is [28, Theorem 8]. It deals with the ratio function $1/f$ instead of the classical inverse $f^{-1}$. A formal statement of this theorem is given below. Let $p > 1$, $q = p/(p-1)$, $a, b \geq 0$ with $b > a$, and $f : [0, b] \to [0, +\infty)$ be a function such that $\int_0^b f^{p-1}(u)du < +\infty$ and $\int_0^a [1/f(u)]^{q-1} du < +\infty$. Then the following holds:

$$\left[\frac{1}{p}\int_0^a f(u)du + \frac{1}{q}\int_0^b \frac{1}{f(u)}du\right]\left[\int_0^b f^{p-1}(u)du\right]^{1/p}\left[\int_0^a \frac{1}{f^{q-1}(u)}du\right]^{1/q} \geq ab. \tag{3}$$

In particular, for the case $p = 2$, this inequality reduces to

$$\left[\int_0^a f(u)du + \int_0^b \frac{1}{f(u)}du\right]\sqrt{\left[\int_0^b f(u)du\right]\left[\int_0^a \frac{1}{f(u)}du\right]} \geq 2ab. \tag{4}$$

In addition to the introduction of $1/f$, we highlight the simplicity of this inequality, offering inspiration for further exploration.

Building on this foundation, in this article, we propose new forms of Young-type integral inequalities, some of which are defined with $1/f$ and some with the derivative of $f$, i.e., $f'$. These are derived using classical techniques, such as the Young product inequality, the Chasles integral formula, the Cauchy-Schwarz integral inequality and changes of variables. Special attention is given to the case where $f$ is decreasing, departing from the standard increasing assumption of the classical Young integral inequality. This leads to elegant inequalities under monotonicity assumptions. To support this claim, we briefly state an example of one of our notable results dealing with the ratio function $1/f$ below. Let $a, b \geq 0$ with $b > a$, $p > 1$, $q = p/(p-1)$ and $f : [0, b] \to [0, +\infty)$ be a decreasing function such that $\int_0^a f^p(u)du < +\infty$ and $\int_0^b [1/f^q(u)]du < +\infty$. Then the following holds:

$$\int_0^a f^p(u)du + \int_0^b \frac{1}{f^q(u)}du \geq (ap)^{1/p}(bq)^{1/q}.$$

Compared to [28, Theorem 8], i.e., Equations (3) and (4), a simplified form is considered and with a more manageable lower bound, but under assumption that $f$ is decreasing. Other results of this type are demonstrated, also using $f'$. All the proofs are presented in detail, making the article self-contained and accessible. In addition, examples are provided to illustrate the theory where this is of interest.

The rest of the article is divided into three main sections: Section 2 presents two general theorems on integral inequalities involving two functions. These can be considered as being of independent interest.

Section 3 uses these theorems to establish the new forms of Young-type integral inequalities, such as the one presented briefly above. A conclusion is given in Section 4.

## 2 Two General Integral Inequalities

The theorem below presents a general integral inequality that mirrors the structure of the classical Young integral inequality, but incorporates two functions, $f$ and $g$, rather than just one. A notable feature of this result, beyond its originality, is that it is derived using the standard Young product inequality.

**Theorem 2.1.** *Let $a, b \geq 0$, $p > 1$, $q = p/(p-1)$, and $f : [0, a] \to [0, +\infty)$ and $g : [0, b] \to [0, +\infty)$ be two functions such that $\int_0^a f^p(u)du < +\infty$ and $\int_0^b g^q(u)du < +\infty$. Then the following holds:*

$$\int_0^a f^p(u)du + \int_0^b g^q(u)du \geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f(u)du\right] \left[\int_0^b g(u)du\right].$$

**Proof of Theorem 2.1.** The first steps of the proof involve using the Young product inequality and introducing a new variable, $v$. To begin with, the Young product inequality states that for any $x, y \geq 0$,

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy.$$

Applying this to the following special configuration: $x = (p/b)^{1/p} f(u)$ and $y = (q/a)^{1/q} g(v)$, for any $u \in [0, a]$ and $v \in [0, b]$, we get

$$\frac{1}{p}\left[\left(\frac{p}{b}\right)^{1/p} f(u)\right]^p + \frac{1}{q}\left[\left(\frac{q}{a}\right)^{1/q} g(v)\right]^q \geq \left(\frac{p}{b}\right)^{1/p} f(u) \left(\frac{q}{a}\right)^{1/q} g(v).$$

This can be rewritten as

$$\frac{1}{b}f^p(u) + \frac{1}{a}g^q(v) \geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} f(u)g(v).$$

Integrating both sides with respect to $(u, v) \in [0, a] \times [0, b]$, we obtain

$$\int_0^a \int_0^b \left[\frac{1}{b}f^p(u) + \frac{1}{a}g^q(v)\right] dudv \geq \int_0^a \int_0^b \left[\left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} f(u)g(v)\right] dudv,$$

with

$$\begin{aligned}\int_0^a \int_0^b \left[\frac{1}{b}f^p(u) + \frac{1}{a}g^q(v)\right] dvdu &= \frac{1}{b}\int_0^a \int_0^b f^p(u)dvdu + \frac{1}{a}\int_0^a \int_0^b g^q(v)dvdu \\ &= \left[\int_0^a f^p(u)du\right] \frac{1}{b}\left(\int_0^b dv\right) + \left[\int_0^b g^q(v)dv\right] \frac{1}{a}\left(\int_0^a du\right) \\ &= \int_0^a f^p(u)du + \int_0^b g^q(v)dv = \int_0^a f^p(u)du + \int_0^b g^q(u)du\end{aligned}$$

and

$$\int_0^a \int_0^b \left[\left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q} f(u)g(v)\right] dvdu = \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b g(v)dv\right]$$
$$= \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b g(u)du\right].$$

So we have

$$\int_0^a f^p(u)du + \int_0^b g^q(u)du \geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b g(u)du\right].$$

This concludes the proof of Theorem 2.1. □

As an additional point, note that the constant factor can also be expressed without $q$, as follows:

$$\left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q} = \frac{p}{a(p-1)}\left(\frac{a(p-1)}{b}\right)^{1/p}.$$

Thanks to its generality and the mild integrability assumptions required for the functions involved, Theorem 2.1 can be applied to various mathematical problems. However, we do not guarantee its sharpness in all situations. In the next section, we will mainly use it to derive new forms of Young-type integral inequalities. As a basic example of its application, when $g = f$, we get

$$\int_0^a f^p(u)du + \int_0^b f^q(u)du \geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b f(u)du\right].$$

This can be considered a combination of the Minkowski and Hölder integral inequalities. To the best of our knowledge, it is not mentioned in the literature. It is of particular interest when either $\int_0^a f^p(u)du$ or $\int_0^b f^q(u)du$ has no closed form expressions, but $\int_0^a f^p(u)du$ does. For example, if we set $a = b = 1$ and $f(u) = u + e^u$, $u \in [0, 1]$, and $g = f$, then we have

$$\int_0^1 (u+e^u)^p du + \int_0^1 (u+e^u)^q du = \int_0^a f^p(u)du + \int_0^b f^q(u)du$$
$$\geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b f(u)du\right] = p^{1/p}q^{1/q}\left[\int_0^1 (u+e^u)du\right]\left[\int_0^1 (u+e^u)du\right]$$
$$= p^{1/p}q^{1/q}\left[\int_0^1 udu + \int_0^1 e^u du\right]^2 = p^{1/p}q^{1/q}\left(e - \frac{1}{2}\right)^2 \approx 2.21828 \times p^{1/p}q^{1/q}.$$

As another example, if we set $a = b = 1$ and $f(u) = \sin((\pi/2)u)$, $u \in [0, 1]$, and $g = f$, then we have

$$\int_0^1 \left[\sin\left(\frac{\pi}{2}u\right)\right]^p du + \int_0^1 \left[\sin\left(\frac{\pi}{2}u\right)\right]^q du = \int_0^a f^p(u)du + \int_0^b f^q(u)du$$
$$\geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b f(u)du\right] = p^{1/p}q^{1/q}\left[\int_0^1 \sin\left(\frac{\pi}{2}u\right)du\right]\left[\int_0^1 \sin\left(\frac{\pi}{2}u\right)du\right]$$
$$= p^{1/p}q^{1/q}\left[\int_0^1 \sin\left(\frac{\pi}{2}u\right)du\right]^2 = p^{1/p}q^{1/q}\left(\frac{2}{\pi}\right)^2 = \frac{4}{\pi^2}p^{1/p}q^{1/q}.$$

An inequality involving an integral and trigonometric and power functions can be used to solve problems in applied mathematics, particularly in Fourier analysis. Of course, the case $f \neq g$, which is covered by Theorem 2.1, offers another dimension of application that needs to be explored in more detail.

The theorem below supplements Theorem 2.1 by providing a lower bound for the product of two integrals, each of which involves a specific function. It is important to note that these functions are assumed to be decreasing, i.e., a monotonicity assumption is imposed on them. The proof is based on a special decomposition of an integral, followed by the application of the Chasles integral formula and the Cauchy-Schwarz integral inequality.

**Theorem 2.2.** *Let $a, b \geq 0$ with $b > a$, $f : [0, a] \to [0, +\infty)$ and $g : [0, b] \to [0, +\infty)$ be two decreasing functions such that $\int_0^a f(u)du < +\infty$ and $\int_0^b [1/g(u)]du < +\infty$, and there exists a constant $\theta > 0$ satisfying, for any $u \in [0, a]$,*

$$\frac{f(u)}{g(u)} \geq \theta. \tag{5}$$

*Then the following holds:*

$$\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{g(u)}du\right] \geq \theta ab.$$

**Proof of Theorem 2.2.** Since $b > a$, the Chasles integral formula at the threshold value $u = a$ gives

$$\begin{aligned}\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{g(u)}du\right] &= \left[\int_0^a f(u)du\right]\left[\int_0^a \frac{1}{g(u)}du + \int_a^b \frac{1}{g(u)}du\right] \\ &= \left[\int_0^a f(u)du\right]\left[\int_0^a \frac{1}{g(u)}du\right] + \left[\int_0^a f(u)du\right]\left[\int_a^b \frac{1}{g(u)}du\right].\end{aligned}$$

It follows from the Cauchy-Schwarz integral inequality and the assumption in Equation (5) that

$$\left[\int_0^a f(u)du\right]\left[\int_0^a \frac{1}{g(u)}du\right] \geq \left[\int_0^a \sqrt{\frac{f(u)}{g(u)}}du\right]^2 \geq \left[\sqrt{\theta}\int_0^a du\right]^2 = \theta a^2.$$

Since $f$ and $g$ are decreasing, using again Equation (5) at $u = a$, we get

$$\begin{aligned}\left[\int_0^a f(u)du\right]\left[\int_a^b \frac{1}{g(u)}du\right] &\geq f(a)\left(\int_0^a du\right)\frac{1}{g(a)}\left(\int_a^b du\right) = \frac{f(a)}{g(a)}a(b-a) \\ &\geq \theta a(b-a).\end{aligned}$$

We therefore have

$$\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{g(u)}du\right] \geq \theta a^2 + \theta a(b-a) = \theta ab.$$

This ends the proof of Theorem 2.2. $\square$

Analogous to Theorem 2.1, this general result can be applied to various mathematical problems. In the next section, we will mainly use it to derive new forms of Young-type integral inequalities. As a basic example of application, when $g = f$, we obviously have $\theta = 1$ and the inequality obtained simplifies to

$$\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{f(u)}du\right] \geq ab.$$

This can be considered a combination of the Chebyshev and Cauchy-Schwarz integral inequalities, taking into account the monotonicity assumption for the Chebyshev integral inequality. To the best of our knowledge, this has not been mentioned in the literature before. For example, if we set $a = b = 1$ and $f(u) = 1/(1+u)^\alpha$, $u \in [0,1]$, with $\alpha > 0$, and $g = f$, then we have

$$\begin{aligned}\frac{2^{-\alpha}(2^{2\alpha+1}+2-5\times 2^\alpha)}{\alpha^2-1} &= \left[\int_0^a \frac{1}{(1+u)^\alpha}du\right]\left[\int_0^b (1+u)^\alpha du\right] \\ &= \left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{f(u)}du\right] \geq ab = 1.\end{aligned}$$

Some numerical work shows the sharpness of this inequality. To support this claim, Figure 1 plots the curve of the difference function

$$h(\alpha) = \frac{2^{-\alpha}(2^{2\alpha+1}+2-5\times 2^\alpha)}{\alpha^2-1} - 1,$$

with $\alpha \in [0,1]$ to highlight the small values of $\alpha$.

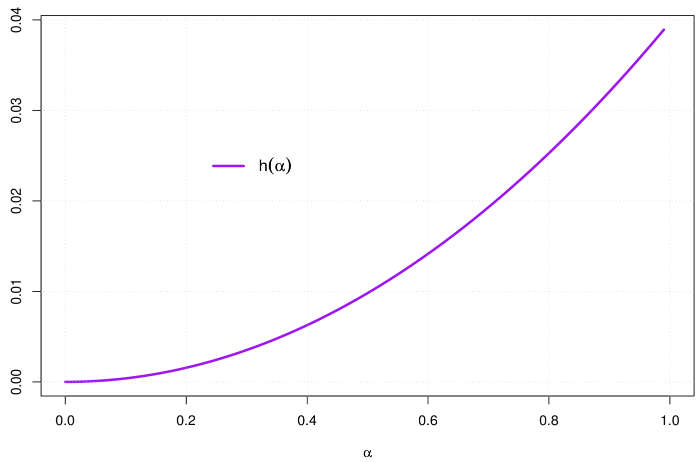


Figure 1: Curve of the function $h(\alpha)$ with $\alpha \in [0,1]$.

We can see that the maximum error is approximately 0.04 for $\alpha = 1$, which is relatively small. Therefore, Theorem 2.2 can be applied to determine new analytical inequalities that go beyond the scope of Young-type integral inequalities.

# 3 New Forms of the Young-type Integral Inequalities

This section is organized into three independent subsections. The first is dedicated to a result involving the inverse function, in the same vein as the classical Young integral inequality. The second focuses on results based on the ratio function and the third on inequalities involving the derivative of the function.

## 3.1 A result involving the inverse function

A new form of the classical Young integral inequality is presented in the proposition below. It is on the same basis as [28, Theorem 7], as recalled in Equation (2), but there are notable differences, which will be discussed later. The proof is based on Theorem 2.1 and the classical Young integral inequality.

**Proposition 3.1.** *Let $p > 1$, $q = p/(p-1)$, $c > 0$, $f : [0, c] \to [0, +\infty)$ be a continuous increasing function with $f(0) = 0$, $a \in [0, c]$ and $b \in [0, f(c))$. Then the following holds:*

$$\int_0^a f^p(u)du + \int_0^b [f^{-1}(u)]^q du \geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f(u)du\right] \left[\int_0^a [b - f(u)]du\right].$$

**Proof of Proposition 3.1.** Applying Theorem 2.1, followed by the classical Young integral inequality as recalled in Equation (1), we get

$$\begin{aligned}
\int_0^a f^p(u)du + \int_0^b [f^{-1}(u)]^q du &\geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f(u)du\right] \left[\int_0^b f^{-1}(u)du\right] \\
&\geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f(u)du\right] \left[ab - \int_0^a f(u)du\right] \\
&= \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f(u)du\right] \left[\int_0^a [b - f(u)]du\right].
\end{aligned}$$

This completes the proof of Proposition 3.1. □

In contrast to [28, Theorem 7], we see that the integrals are not weighted by specific constants; the main term is defined as the direct sum of the main integral. This is more in keeping with the classical Young integral inequality, which considers such a sum. In this sense, Proposition 3.1 and [28, Theorem 7] can be seen as complementary.

## 3.2 Results involving the ratio function

This subsection is devoted to new forms of Young-type integral inequalities involving the ratio function. It builds on the pioneering result presented in [28, Theorem 8], as recalled in Equations (3) and (4).

The result below is one such inequality and involves two functions, $f$ and $g$. Its proof relies on a basic power inequality combined with Theorem 2.2.

**Proposition 3.2.** *Let $a, b \geq 0$ with $b > a$, $f : [0, a] \to [0, +\infty)$ and $g : [0, b] \to [0, +\infty)$ be two decreasing functions such that $\int_0^a f(u)du < +\infty$ and $\int_0^b [1/g(u)]du < +\infty$, and there exists a constant $\theta > 0$ satisfying, for any $u \in [0, a]$,*

$$\frac{f(u)}{g(u)} \geq \theta.$$

*Then the following holds:*

$$\int_0^a f(u)du + \int_0^b \frac{1}{g(u)}du \geq 2\sqrt{\theta ab}.$$

**Proof of Proposition 3.2.** Applying the elementary inequality $x^2 + y^2 \geq 2xy$ for any $x, y \geq 0$, with $x = \sqrt{\int_0^a f(u)du}$ and $y = \sqrt{\int_0^b [1/g(u)]du}$, and Theorem 2.2, we get

$$\int_0^a f(u)du + \int_0^b \frac{1}{g(u)}du \geq 2\sqrt{\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{g(u)}du\right]} \geq 2\sqrt{\theta ab}.$$

This concludes the proof of Proposition 3.2. □

In particular, when $g = f$, we obviously have $\theta = 1$ and the inequality obtained reduces to

$$\int_0^a f(u)du + \int_0^b \frac{1}{f(u)}du \geq 2\sqrt{ab}.$$

A much simpler result is established compared to Equation (4), but with the monotonicity assumption on $f$. One of the advantages of Proposition 3.2 is that it deals with two independent functions, i.e., $f$ and $g$, rather than just one. Consequently, it can be adapted to various mathematical scenarios beyond the Young-type integral inequalities.

A variant of Proposition 3.2 is presented below, using an adjustable parameter $p$. This variant also complements [28, Theorem 8], as recalled in Equation (3). The proof incorporates key elements from two of our previously established theorems: Theorems 2.1 and 2.2.

**Proposition 3.3.** *Let $a, b \geq 0$ with $b > a$, $p > 1$, $q = p/(p-1)$, $f : [0, a] \to [0, +\infty)$ and $g : [0, b] \to [0, +\infty)$ be two decreasing functions such that $\int_0^a f^p(u)du < +\infty$ and $\int_0^b 1/g^q(u)du < +\infty$, and there exists a constant $\theta > 0$ satisfying, for any $u \in [0, a]$,*

$$\frac{f(u)}{g(u)} \geq \theta.$$

*Then the following holds:*

$$\int_0^a f^p(u)du + \int_0^b \frac{1}{g^q(u)}du \geq \theta(ap)^{1/p}(bq)^{1/q}.$$

**Proof of Proposition 3.3.** Applying Theorem 2.1 to the functions $f$ and $1/g$ instead of $g$, and Theorem 2.2, and using $1/p + 1/q = 1$, we get

$$\int_0^a f^p(u)du + \int_0^b \frac{1}{g^q(u)}du \geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f(u)du\right]\left[\int_0^b \frac{1}{g(u)}du\right]$$
$$\geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\theta ab = \theta p^{1/p}a^{1-1/q}q^{1/q}b^{1-1/p} = \theta(ap)^{1/p}(bq)^{1/q}.$$

This ends the proof of Proposition 3.3. □

In particular, when $g = f$, we obviously have $\theta = 1$ and the inequality obtained simplifies to

$$\int_0^a f^p(u)du + \int_0^b \frac{1}{f^q(u)}du \geq (ap)^{1/p}(bq)^{1/q}.$$

Therefore, under the assumption of monotonicity on $f$, the result [28, Theorem 8] can be significantly simplified. The generality of Proposition 3.3 induced by the presence of $g$ is also advantageous for various mathematical applications involving sums of integrals.

### 3.3 Results involving the derivative

An interesting approach to new forms of Young-type integral inequalities involves using the derivative of the main function $f$. The first such results are presented in this subsection.

The proposition below demonstrates the effective use of Theorem 3.6 to derive a new "derivative form" of Young integral inequalities.

**Proposition 3.4.** *Let $a, b \geq 0$, $p > 1$, $q = p/(p-1)$ and $f : [0, a] \to [0, +\infty)$, and $g : [0, b] \to [0, +\infty)$ be two differentiable functions such that $\int_0^a [f'(u)]^p du < +\infty$ and $\int_0^b [g'(u)]^q du < +\infty$. Then the following holds:*

$$\int_0^a [f'(u)]^p du + \int_0^b [g'(u)]^q du \geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}[f(a) - f(0)]\,[g(b) - g(0)]\,.$$

**Proof of Proposition 3.4.** Applying Theorem 2.1 to the functions "$f = f'$" and "$g = g'$", we immediately get

$$\int_0^a [f'(u)]^p du + \int_0^b [g'(u)]^q du \geq \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}\left[\int_0^a f'(u)du\right]\left[\int_0^b g'(u)du\right]$$
$$= \left(\frac{p}{b}\right)^{1/p}\left(\frac{q}{a}\right)^{1/q}[f(a) - f(0)]\,[g(b) - g(0)]\,.$$

This completes the proof of Proposition 3.4. □

It is interesting to see how the values of $f$ and $g$ at the limit points 0, $a$ and $b$, together with $p$, completely determine the lower bound.

**Proposition 3.5.** *Let $p > 1$, $q = p/(p-1)$, $c > 0$, $f : [0, c] \to [0, +\infty)$ be a differentiable convex function with $f'(0) = 0$, $a \in [0, c]$ and $b \in [0, f'(c)]$. Then the following holds:*

$$\int_0^a [f'(u)]^p du + \int_0^b [f'^{,-1}(u)]^q du \geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} [f(a) - f(0)] [ab - f(a) + f(0)],$$

*where $f'^{,-1}$ is the inverse function of the derivative $f'$ (to the best of our knowledge, it has no general and analytical expression).*

**Proof of Proposition 3.5.** Since $f$ is a differentiable convex function, $f'$ is increasing. We also have $f'(0) = 0$. It follows from Proposition 3.1 applied to the function "$f = f'$" that

$$\begin{aligned}\int_0^a [f'(u)]^p du + \int_0^b [f'^{,-1}(u)]^q du &\geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} \left[\int_0^a f'(u) du\right] \left[\int_0^a [b - f'(u)] du\right] \\ &= \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} [f(a) - f(0)] [ab - f(a) + f(0)].\end{aligned}$$

This ends the proof of Proposition 3.5. □

In particular, if $f(0) = 0$, which is not imposed since we deal with $f'$ in the proposition and not $f$, then the inequality obtained simplifies to

$$\int_0^a [f'(u)]^p du + \int_0^b [f'^{,-1}(u)]^q du \geq \left(\frac{p}{b}\right)^{1/p} \left(\frac{q}{a}\right)^{1/q} f(a) [ab - f(a)].$$

To the best of our knowledge, this is a new form of Young-type integral inequalities with original derivative and lower bound properties.

In the proposition below, we build on this work by adopting a different approach: we still use the derivative of $f$, but there is a more direct connection to the classical Young product inequality.

**Proposition 3.6.** *Let $a, b, c \geq 0$ with $b > a$ $p > 1$, $q = p/(p-1)$, $f : [a, b] \to [0, +\infty)$ be a differentiable increasing function such that $\int_a^b [f'(u)]^q du < +\infty$, and $g : [0, +\infty) \to [0, +\infty)$ be a function such that $\int_a^b g^p(f(u)) du < +\infty$ and $\int_0^c g(u) du < +\infty$. Then the following holds:*

$$\frac{1}{p} \int_a^b g^p(f(u)) du + \frac{1}{q} \int_a^b [f'(u)]^q du \geq \int_0^c g(u) du.$$

**Proof of Proposition 3.6.** Introducing the inverse function $f^{-1}$ and making the change of variables $u = f(v)$, we have

$$\int_0^c g(u) du = \int_0^c g(f(f^{-1}(u))) du = \int_{f^{-1}(0)}^{f^{-1}(c)} g(f(v)) f'(v) dv, \tag{6}$$

where $f^{-1}(0) = \lim_{u \to 0} f^{-1}(u)$ and $f^{-1}(c) = \lim_{u \to c} f^{-1}(u)$. Since $f$ is differentiable and increasing, we have $f'(v) \geq 0$ for any $v \in [a, b]$. This combined with the positivity of $g$ gives

$$\int_{f^{-1}(0)}^{f^{-1}(c)} g(f(v)) f'(v) dv \leq \int_a^b g(f(v)) f'(v) dv. \tag{7}$$

On the other hand, the Young product inequality ensures that, for any $x, y \geq 0$,

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy.$$

Applying this to $x = g(f(v))$ and $y = f'(v)$, for any $v \in [a, b]$, we get

$$\frac{1}{p}g^p(f(v)) + \frac{1}{q}[f'(v)]^q \geq g(f(v))f'(v).$$

Integrating both sides with respect to $v$ with $v \in [a, b]$, we obtain

$$\int_a^b \left[\frac{1}{p}g^p(f(v)) + \frac{1}{q}[f'(v)]^q\right] dv \geq \int_a^b g(f(v))f'(v)dv, \tag{8}$$

with

$$\begin{aligned}\int_a^b \left[\frac{1}{p}g^p(f(v)) + \frac{1}{q}[f'(v)]^q\right] dv &= \frac{1}{p}\int_a^b g^p(f(v))dv + \frac{1}{q}\int_a^b [f'(v)]^q dv \\ &= \frac{1}{p}\int_a^b g^p(f(u))du + \frac{1}{q}\int_a^b [f'(u)]^q du.\end{aligned} \tag{9}$$

Combining Equations (6), (7), (8) and (9), we have

$$\frac{1}{p}\int_a^b g^p(f(u))du + \frac{1}{q}\int_a^b [f'(u)]^q du \geq \int_0^c g(u)du.$$

This concludes the proof of Proposition 3.6. □

In particular, when $g = f$, after an adjustment of the interval of definition, we get

$$\frac{1}{p}\int_a^b f^p(f(u))du + \frac{1}{q}\int_a^b [f'(u)]^q du \geq \int_0^c f(u)du.$$

To the best of our knowledge, this is an innovative Young-type integral inequality, which has the property of involving various transformations of $f$.

The proposition below completes Proposition 3.6 using similar methods but examining the inequality from a new perspective. More precisely, it considers $g(f^{-1})$ rather than $g(f)$ as in Proposition 3.6.

**Proposition 3.7.** *Let $a, b, c \geq 0$ with $b > a$ and $c \in [a, b]$, $p > 1$, $q = p/(p-1)$, $f : [a, b] \to [0, +\infty)$ be a differentiable increasing function such that $\int_0^c [1/[f'(u)]^{q-1}]du < +\infty$, and $g : [0, +\infty) \to [0, +\infty)$ be a function such that $\int_0^{+\infty} g^p(f^{-1}(u))du < +\infty$ and $\int_0^c g(u)du < +\infty$. Then the following holds:*

$$\frac{1}{p}\int_0^{+\infty} g^p(f^{-1}(u))du + \frac{1}{q}\int_0^c \frac{1}{[f'(u)]^{q-1}}du \geq \int_0^c g(u)du.$$

**Proof of Proposition 3.7.** Making the change of variables $u = f^{-1}(v)$, we have

$$\int_0^c g(u)du = \int_0^c g(f^{-1}(f(u)))du = \int_{f(0)}^{f(c)} g(f^{-1}(v))\frac{1}{f'(f^{-1}(v))}dv, \tag{10}$$

where $f(0) = \lim_{u\to 0} f(u)$ and $f(c) = \lim_{u\to c} f(u)$.

On the other hand, the Young product inequality ensures that, for any $x, y \geq 0$,

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy.$$

Applying this to $x = g(f^{-1}(v))$ and $y = 1/f'(f^{-1}(v))$, which is non-negative because $f$ is differentiable and increasing, for any $v \in [f(0), f(c)]$, we get

$$\frac{1}{p}g^p(f^{-1}(v)) + \frac{1}{q}\frac{1}{[f'(f^{-1}(v))]^q} \geq g(f^{-1}(v))\frac{1}{f'(f^{-1}(v))}.$$

Integrating both sides with respect to $v$ with $v \in [f(0), f(c)]$, we obtain

$$\int_{f(0)}^{f(c)} \left[\frac{1}{p}g^p(f^{-1}(v)) + \frac{1}{q}\frac{1}{[f'(f^{-1}(v))]^q}\right] dv \geq \int_{f(0)}^{f(c)} g(f^{-1}(v))\frac{1}{f'(f^{-1}(v))}dv, \tag{11}$$

with, by the change of variables $w = f^{-1}(v)$,

$$\begin{aligned}
&\int_{f(0)}^{f(c)} \left[\frac{1}{p}g^p(f^{-1}(v)) + \frac{1}{q}\frac{1}{[f'(f^{-1}(v))]^q}\right] dv \\
&= \frac{1}{p}\int_{f(0)}^{f(c)} g^p(f^{-1}(v))dv + \frac{1}{q}\int_{f(0)}^{f(c)} \frac{1}{[f'(f^{-1}(v))]^q}dv \\
&= \frac{1}{p}\int_{f(0)}^{f(c)} g^p(f^{-1}(v))dv + \frac{1}{q}\int_0^c \frac{1}{[f'(w)]^q}f'(w)dw \\
&= \frac{1}{p}\int_{f(0)}^{f(c)} g^p(f^{-1}(u))du + \frac{1}{q}\int_0^c \frac{1}{[f'(u)]^{q-1}}du. 
\end{aligned}\tag{12}$$

By the positivity of the integrand, we can bound the first integral as

$$\int_{f(0)}^{f(c)} g^p(f^{-1}(u))du \leq \int_0^{+\infty} g^p(f^{-1}(u))du. \tag{13}$$

Combining Equations (10), (11), (12) and (13), we have

$$\frac{1}{p}\int_0^{+\infty} g^p(f^{-1}(u))du + \frac{1}{q}\int_0^c \frac{1}{[f'(u)]^{q-1}}du \geq \int_0^c g(u)du.$$

This completes the proof of Proposition 3.7. □

In particular, when $g = f$, after an adjustment of the interval of definition, we get

$$\frac{1}{p}\int_0^{+\infty} f^p(f^{-1}(u))du + \frac{1}{q}\int_0^c \frac{1}{[f'(u)]^{q-1}}du \geq \int_0^c f(u)du.$$

To the best of our knowledge, in full generality, it is one of the few integral inequalities that deals with two functions, including their inverse and derivative. In the combined setting of Propositions 3.6 and 3.7, we therefore derive the following original upper bound:

$$\int_0^c g(u)du \leq \min\left[\frac{1}{p}\int_a^b g^p(f(u))du + \frac{1}{q}\int_a^b [f'(u)]^q du, \frac{1}{p}\int_0^{+\infty} g^p(f^{-1}(u))du + \frac{1}{q}\int_0^c \frac{1}{[f'(u)]^{q-1}}du\right].$$

In a sense, this opens a theoretical door to new "derivative forms" of Young-type integral inequalities.

## 4 Conclusion

In conclusion, our contribution to mathematical analysis has been to introduce new forms of Young-type integral inequalities. Some of these innovations include functions and their inverses, while others present innovative approaches using ratios or derivatives. Additionally, we have delved into the lesser-studied case of decreasing functions, offering elegant formulations. Direct examples and discussions complement the theory.

Future work could involve extending these inequalities to convex or concave functions to further generalize the results. It would be interesting to combine the new techniques with those in [12]. Additionally, adapting the developed inequalities to functions of several variables could contribute to multivariate analysis and its applications.

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