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Properties of Integrals Involving Ratios of the Modified Gamma Function

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Abstract

In this article, we study the Modified Gamma function and more precisely we focus on properties of some integrals involving ratios of the Modified Gamma function. The properties studied involve estimates of square norms and Sobolev norms, using the definitions of L_2 and Sobolev functional spaces. Additionally, estimates are derived where the integrand is the product of two or more functions involving particular ratios of the Modified Gamma function. Lastly, continuous entropy is computed for a particular function defined as ratio of the Modified Gamma function, and the corresponding continuous entropy is calculated for the derivative of the negative ratio of the previously mentioned function.

1 Introduction

It is well known that the Gamma function is ubiquitous in many branches of applied and pure mathematics, and there is plenty of information in the mathematical literature [1], [2], [3], [4], [11]. Primarily, one significant application in the study of numbers (Number theory) is the association of the Gamma function with the factorial, which these two entities are intimately connected. There are many identities and inequalities obtained involving the Gamma function available in the literature. In this article we focus on two key elements:1. is to obtain Square and Sobolev norm estimates of ratios of the Modified Gamma Function and estimates for particular integrals involving quotients of the Modified Gamma Function, 2.Obtain formulas for the continuous differential entropies on certain types of functions involving quotients of the Modified Gamma Function . The modified Gamma function is an adjustment of the original Gamma function by changing the exponential function which is located inside the integrand, by an exponential function with arbitrary base a which is strictly greater than one. This idea comes from the articles which are available in the mathematical literature [5], [6], [7], [8], [9], [10], [12], [13] . In these articles, the authors have modified the kernel function. Consequently, this idea is employed here for deriving all the corresponding results.

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2 Preliminary Notes

Before proceeding with the theorems and the associated proofs, we provide some essential background information. The Gamma function is defined below

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t)dt,\ x \in]0, +\infty[,\ t \in]0, +\infty[. \tag{1}$$

The modified Gamma function, by making an adjustment inside the integrand of (1), is

$$\Gamma_a(x) = \int_0^{+\infty} t^{x-1} a^{-t} dt,\ x \in]0, +\infty[,\ t \in]0, +\infty[,\ a \in]1, +\infty[. \tag{2}$$

3 Main Results

The main results of this article are presented in this section. For the derivation of all the results below, the function prescribed by (2) is employed.

Theorem 3.1. *The following identity holds.*

$$\Gamma_a(x+1) = \frac{x}{\ln(a)} \Gamma_a(x), \forall x > 0. \tag{3}$$

Proof. A direct computation gives

$$\begin{aligned}
\Gamma_a(x+1) &= \int_0^{+\infty} t^{x+1-1} a^{-t} dt \\
&= \int_0^{+\infty} t^x a^{-t} dt \\
&= \int_0^{+\infty} t^x \frac{d}{dt}\left(\frac{-a^{-t}}{\ln(a)}\right) dt \\
&= \lim_{t \to +\infty} \left[t^x \left(\frac{-a^{-t}}{\ln(a)}\right)\right] - \lim_{t \to 0} \left[t^x \left(\frac{-a^{-t}}{\ln(a)}\right)\right] + \int_0^{+\infty} \frac{d}{dt}(t^x) \frac{a^{-t}}{\ln(a)} dt \\
&= \frac{x}{\ln(a)} \int_0^{+\infty} t^{x-1} a^{-t} dt \\
&= \frac{x}{\ln(a)} \Gamma_a(x), \forall x > 0
\end{aligned}$$

by using integration by parts. And this result is identical to (3). □

Theorem 3.2. *The following relationships hold*

$$\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|_{L_2((M_1,M_2))} = \ln(a)\sqrt{\frac{1}{M_1+k} - \frac{1}{M_2+k}}, \tag{4}$$

$$\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{L_2((M_1,M_2))} = \ln(a)\sqrt{\frac{1}{M_1+k+1} - \frac{1}{M_2+k+1}}, \tag{5}$$

$$\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|_{L_2((M_1,M_2))} > \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{L_2((M_1,M_2))}, \forall x > 0, k \in \mathbb{Z}^+, 0 < M_1 < M_2. \tag{6}$$

Proof. By the definition of the L_2 functional norms and employing (3), we obtain

$$\begin{aligned}\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{L_2((M_1,M_2))} &= \left(\int_{M_1}^{M_2}\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)^2 dx\right)^{\frac{1}{2}}\\ &= \left(\int_{M_1}^{M_2}\left(\frac{\ln(a)}{x+k}\right)^2 dx\right)^{\frac{1}{2}} = \ln(a)\left(\int_{M_1}^{M_2}\frac{1}{(x+k)^2}dx\right)^{\frac{1}{2}}\\ &= \ln(a)\sqrt{\frac{1}{M_1+k}-\frac{1}{M_2+k}},\end{aligned}$$

$$\begin{aligned}\Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{L_2((M_1,M_2))} &= \left(\int_{M_1}^{M_2}\left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right)^2 dx\right)^{\frac{1}{2}}\\ &= \left(\int_{M_1}^{M_2}\left(\frac{ln(a)}{x+k+1}\right)^2 dx\right)^{\frac{1}{2}} = \ln(a)\left(\int_{M_1}^{M_2}\frac{1}{(x+k+1)^2}dx\right)^{\frac{1}{2}}\\ &= \ln(a)\sqrt{\frac{1}{M_1+k+1}-\frac{1}{M_2+k+1}}\end{aligned}$$

which are the identical to (4), (5). To prove the inequality (6) we compute the quantity

$$\mathcal{N} = \Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|^2_{L_2((M_1,M_2))} - \Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|^2_{L_2((M_1,M_2))}$$

and this yields

$$\begin{aligned}\mathcal{N} &= \Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|^2_{L_2((M_1,M_2))} - \Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|^2_{L_2((M_1,M_2))}\\ &= \ln^2(a)\left(\frac{1}{M_1+k}-\frac{1}{M_2+k}\right) - \ln^2(a)\left(\frac{1}{M_1+k+1}-\frac{1}{M_2+k+1}\right)\\ &= \ln^2(a)\ \frac{(M_2-M_1)}{(M_2\ +k)(M_1+k)} - \ln^2(a)\frac{(M_2-M_1)}{(M_2\ +k+1)(M_1+k+1)}\\ &= \ln^2(a)(M_2-M_1)\left(\frac{1}{(M_2\ +k)(M_1+k)} - \frac{1}{(M_2\ +k+1)(M_1+k+1)}\right)\\ &= \ln^2(a)(M_2-M_1)\ \frac{(M_1+M_2+2k+1)}{(M_2\ +k)(M_1+k)(M_2\ +k+1)(M_1+k+1)} > 0.\end{aligned}$$

Consequently, since this is a strictly positive quantity, we obtain the desired estimate

$$\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{L_2((M_1,M_2))} > \Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{L_2((M_1,M_2))}.$$

as prescribed by (6). ☐

Theorem 3.3. *The relationships below hold*

$$\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{H^1((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k)^2}}\right)\ \Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{L_2((M_1,+\infty))}, \tag{7}$$

$$\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{H^1((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k+1)^2}}\right)\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{L_2((M_1,+\infty))}, \tag{8}$$

$$\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|_{H^1((M_1,+\infty))} > \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{H^1((M_1,+\infty))}, \forall x>0, k\in\mathbb{Z}^+, M_1>0. \tag{9}$$

Proof. By using the definition of Sobolev norm, we obtain

$$\begin{aligned}
\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{H^1((M_1,+\infty))} &= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} + \left|\left|\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)'\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} + \left(\ln(a)\frac{\sqrt{3}}{3}(M_1+k)^{\frac{-3}{2}}\right)^2 \\
&= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} + \left(\ln(a)\frac{\sqrt{3}}{3}\,\frac{1}{\sqrt{M_1+k}}(M_1+k)^{-1}\right)^2 \\
&= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} + \frac{1}{3(M_1+k)^2}\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \left(1+\frac{1}{3(M_1+k)^2}\right)\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))},
\end{aligned}$$

$$\begin{aligned}
\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{H^1((M_1,+\infty))} &= \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} + \left|\left|\left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right)'\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} + \left(\ln(a)\frac{\sqrt{3}}{3}(M_1+k+1)^{\frac{-3}{2}}\right)^2 \\
&= \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} \\
&+ \left(\ln(a)\frac{\sqrt{3}}{3}\,\frac{1}{\sqrt{M_1+k+1}}(M_1+k+1)^{-1}\right)^2 \\
&= \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} \\
&+ \frac{1}{3(M_1+k+1)^2}\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \left(1+\frac{1}{3(M_1+k+1)^2}\right)\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))}
\end{aligned}$$

by employing the formulas (4), (5) where the upper bound of integration tends to $+\infty$. Taking the square root on the previous relationships, this yields

$$\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|_{H^1((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k)^2}}\right)\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|_{L_2((M_1,+\infty))},$$

$$\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{H^1((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k+1)^2}}\right)\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|_{L_2((M_1,+\infty))}$$

identical to (7), (8) To prove the inequality, we take the squares of Sobolev norms and we compute the difference, and this yields

$$\begin{aligned}
\mathcal{P} &= \Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|^2_{H^1((M_1,+\infty))} - \Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|^2_{H^1((M_1,+\infty))} \\
&= \left(\left(\sqrt{1+\frac{1}{3(M_1+k)^2}}\right)\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{L_2((M_1,+\infty))}\right)^2 \\
&- \left(\left(\sqrt{1+\frac{1}{3(M_1+k+1)^2}}\right)\Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{L_2((M_1,+\infty))}\right)^2 \\
&= \ln^2(a)\left(\frac{1}{M_1+k}+\frac{1}{3(M_1+k)^3}-\frac{1}{M_1+k+1}-\frac{1}{3(M_1+k+1)^3}\right) \\
&= \ln^2(a)\left(\frac{1}{t}+\frac{1}{3t^3}-\frac{1}{t+1}-\frac{1}{3(t+1)^3}\right) \\
&= \frac{\ln^2(a)\left(3t^2(t+1)^2+(t+1)^3-t^3\right)}{3t^3(t+1)^3} \\
&= \frac{\ln^2(a)\left(3t^4+6t^3+6t^2+3t+1\right)}{3t^3(t+1)^3} > 0,
\end{aligned}$$

where $t = t(k) = M_1 + k$. Since $\mathcal{P}$ is a strictly positive quantity, the desired inequality follows (9). □

Theorem 3.4. *The following relationships hold*

$$\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{H^2((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k)^2}+\frac{4}{5(M_1+k)^4}}\right)\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{L_2((M_1,+\infty))}, \tag{10}$$

$$\Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{H^2((M_1,+\infty))} = \left(\sqrt{1+\frac{1}{3(M_1+k+1)^2}+\frac{4}{5(M_1+k+1)^4}}\right) \tag{11}$$

$$\times\Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{L_2((M_1,+\infty))}, \tag{12}$$

$$\Big|\Big|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\Big|\Big|_{H^2((M_1,+\infty))} > \Big|\Big|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\Big|\Big|_{H^2((M_1,+\infty))}, \forall x>0, k\in\mathbb{Z}^+, M_1>0. \tag{13}$$

Proof. By the definition of H^2 norm we obtain

$$\begin{aligned}
\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{H^2((M_1,+\infty))} &= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))} + \left|\left|\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)'\right|\right|^2_{L_2((M_1,+\infty))} \\
&+ \left|\left|\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)''\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \frac{\ln^2(a)}{M_1+k} + \frac{\ln^2(a)}{3(M_1+k)^3} + \frac{4\ln^2(a)}{5(M_1+k)^5} \\
&= \frac{\ln^2(a)}{(M_1+k)}\left(1+\frac{1}{3(M_1+k)^2}+\frac{4}{5(M_1+k)^4}\right) \\
&= \left(1+\frac{1}{3(M_1+k)^2}+\frac{4}{5(M_1+k)^4}\right)\left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{L_2((M_1,+\infty))}, \\
\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{H^2((M_1,+\infty))} &= \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))} + \left|\left|\left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right)'\right|\right|^2_{L_2((M_1,+\infty))} \\
&+ \left|\left|\left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right)''\right|\right|^2_{L_2((M_1,+\infty))} \\
&= \frac{\ln^2(a)}{M_1+k+1} + \frac{\ln^2(a)}{3(M_1+k+1)^3} + \frac{4\ln^2(a)}{5(M_1+k+1)^5} \\
&= \frac{\ln^2(a)}{(M_1+k+1)}\left(1+\frac{1}{3(M_1+k+1)^2}+\frac{4}{5(M_1+k+1)^4}\right) \\
&= \left(1+\frac{1}{3(M_1+k+1)^2}+\frac{4}{5(M_1+k+1)^4}\right)\left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{L_2((M_1,+\infty))}.
\end{aligned}$$

Taking the square root of the previous formulas, we obtain (10), (11). To prove the inequality (13) we calculate the difference between the H^2 norms as follows

$$\begin{aligned}
\mathcal{D} &= \left|\left|\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right|\right|^2_{H^2((M_1,+\infty))} - \left|\left|\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}\right|\right|^2_{H^2((M_1,+\infty))} \\
&= \frac{\ln^2(a)}{t(t+1)} + \frac{\ln^2(a)\left((t+1)^3-t^3\right)}{t^3(t+1)^3} + \frac{4\ln^2(a)\left((t+1)^5-t^5\right)}{t^5(t+1)^5} \\
&= \ln^2(a)\frac{15t^4(t+1)^4+5t^2(t+1)^2\left((t+1)^3-t^3\right)+12\left((t+1)^5-t^5\right)}{15t^5(t+1)^5} \\
&= \frac{\ln^2(a)}{15t^5(t+1)^5}\left(15t^8+60t^7+105t^6+105t^5+125t^4+145t^3+125t^2+60t+12\right) > 0,
\end{aligned}$$

where $t = t(k) = M_1 + k$. Since the quantity $\mathcal{D}$ is strictly positive, the inequality (13) follows. □

Theorem 3.5. *The estimate*

$$\int_{M_1}^{M_2} \sqrt{\frac{\ln(x+1)\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}}dx \leq \sqrt{\ln(a)} \left(\ln\left(\frac{(1+M_2)^{(1+M_2)}}{(1+M_1)^{(1+M_1)}}\right) + (M_1 - M_2) \right)^{\frac{1}{2}} \times \left(\ln\left(\frac{k+M_2+1}{k+M_1+1}\right)\right)^{\frac{1}{2}}$$

holds $\forall x > 0, 0 < M_1 < M_2, k \in \mathbb{Z}^+$. *The integrand function is strictly positive and it is assumed that the integral over the prescribed domain absolutely converges.*

Proof. By direct calculation, we obtain

$$\begin{aligned}
&\int_{M_1}^{M_2} \sqrt{\frac{\ln(x+1)\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}}dx = \int_{M_1}^{M_2} \sqrt{\ln(x+1)} \sqrt{\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}}\, dx \\
&\leq \left(\int_{M_1}^{M_2} \left(\sqrt{\ln(x+1)}\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \left(\sqrt{\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}}\right)^2 dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_{M_1}^{M_2} \ln(x+1)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_{M_1}^{M_2} \ln(x+1)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{\ln(a)}{x+k+1}dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{\ln(a)}\ (M_2\ \ln(M_2+1) - M_1\ ln(M_1+1) - M_2 + M_1 + \ln(M_2+1) - \ln(M_1+1))^{\frac{1}{2}} \\
&\times \left(\ln\left(\frac{k+M_2+1}{k+M_1+1}\right)\right)^{\frac{1}{2}} \\
&\leq \sqrt{\ln(a)} \left(\ln\left(\frac{(1+M_2)^{(1+M_2)}}{(1+M_1)^{(1+M_1)}}\right) + (M_1 - M_2) \right)^{\frac{1}{2}} \\
&\times \left(\ln\left(\frac{k+M_2+1}{k+M_1+1}\right)\right)^{\frac{1}{2}},
\end{aligned}$$

by employing the Schwarz-Cauchy inequality and integration by parts on the integral $\int_{M_1}^{M_2} \ln(x+1)dx$. □

Theorem 3.6. *The inequality*

$$\int_{M_1}^{M_2} \sqrt{sinh_a(x)sin(x)\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+2)}}dx \leq \frac{\begin{array}{c}(\ln(a)(cosh_a(M_2)sin(M_2) - cosh_a(M_1)sin(M_1)) \\ -sinh_a(M_2)cos(M_2) + sinh_a(M_1)cos(M_1))^{\frac{1}{2}}\end{array}}{\left(\ln^2(a)+1\ \right)^{\frac{1}{2}}} \times \ln(a) \left(\ln\frac{(M_2+k)(M_1+k+1)}{(M_1+k)(M_2+k+1)}\right)^{\frac{1}{2}}$$

is valid $\forall x > 0, 0 < M_1 < M_2 < \pi, k \in \mathbb{Z}^+$, *where* $sinh_a(x) = \frac{a^x - a^{-x}}{2}, a > 1$. *The integrand function is strictly positive and it is assumed that the integral over the prescribed domain absolutely converges.*

Proof. A straightforward computation gives

$$\int_{M_1}^{M_2} \sqrt{sinh_a(x)sin(x)\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+2)}}dx = \int_{M_1}^{M_2} \sqrt{sinh_a(x)sin(x)}\ \sqrt{\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+2)}}\ dx$$
$$\leq \left(\int_{M_1}^{M_2} \left(\sqrt{sinh_a(x)sin(x)}\right)^2 dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \left(\sqrt{\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+2)}}\right)^2 dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{M_1}^{M_2} sinh_a(x)sin(x)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+2)}dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{M_1}^{M_2} sinh_a(x)sin(x)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\ \frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{M_1}^{M_2} sinh_a(x)sin(x)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{\ln(a)}{(x+k)}\ \frac{\ln(a)}{(x+k+1)}dx\right)^{\frac{1}{2}}$$
$$\leq \ln(a) \left(\int_{M_1}^{M_2} sinh_a(x)sin(x)dx\right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \frac{1}{(x+k)}\ \frac{1}{(x+k+1)}dx\right)^{\frac{1}{2}}$$

$$\leq \frac{\begin{array}{c}(\ln(a)(cosh_a(M_2)sin(M_2) - cosh_a(M_1)sin(M_1)) \\ -sinh_a(M_2)cos(M_2) + sinh_a(M_1)cos(M_1))^{\frac{1}{2}}\end{array}}{\left(\ln^2(a)+1\ \right)^{\frac{1}{2}}}$$
$$\times \ln(a)\ \left(ln\frac{(M_2+k)(M_1+k+1)}{(M_1+k)(M_2+k+1)}\right)^{\frac{1}{2}}$$

by using the Schwarz-Cauchy inequality, employing integration by parts on the integral

$$\int_{M_1}^{M_2} sinh_a(x)sin(x)dx$$

and using partial fraction analysis on the integral $\int_{M_1}^{M_2} \frac{1}{(x+k)}\ \frac{1}{(x+k+1)}dx$. □

Theorem 3.7. *The inequality*

$$\int_{M_1}^{M_2} \sqrt{\frac{x^2}{1+x^2}}\ \frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)}dx \leq \ln(a)\sqrt{\arctan(M_1) - \arctan(M_2) + M_2 - M_1}$$
$$\times \sqrt{\frac{1}{M_1+k+1} - \frac{1}{M_2+k+1}}$$

holds $\forall x > 0, 0 < M_1 < M_2, k \in \mathbb{Z}^+$. *The integrand function is strictly positive and it is assumed that the integral over the prescribed domain absolutely converges.*

Proof. A direct calculation yields

$$\int_{M_1}^{M_2} \sqrt{\frac{x^2}{1+x^2}} \frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)} dx \leq \left(\int_{M_1}^{M_2} \left(\sqrt{\frac{x^2}{1+x^2}} \right)^2 dx \right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)} \right)^2 dx \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{M_1}^{M_2} \frac{x^2}{1+x^2} dx \right)^{\frac{1}{2}} \left(\int_{M_1}^{M_2} \left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)} \right)^2 dx \right)^{\frac{1}{2}}$$

$$\leq \left(\int_{M_1}^{M_2} \frac{1+x^2-1}{1+x^2} dx \right)^{\frac{1}{2}} \left\| \left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)} \right) \right\|_{L_2((M_1,M_2))}$$

$$\leq \left(\int_{M_1}^{M_2} dx - \int_{M_1}^{M_2} \frac{1}{1+x^2} dx \right)^{\frac{1}{2}} \left\| \left(\frac{\Gamma_a(x+k+1)}{\Gamma_a(x+k+2)} \right) \right\|_{L_2((M_1,M_2))}$$

$$\leq \ln(a) \sqrt{\arctan(M_1) - \arctan(M_2) + M_2 - M_1}$$
$$\times \sqrt{\frac{1}{M_1+k+1} - \frac{1}{M_2+k+1}}$$

by using the Schwarz-Cauchy inequality and employing relationship (5). □

Theorem 3.8. *The following estimate holds*

$$\int_{M_1}^{M_2} \sqrt{\sqrt{\sqrt{\ln^2(x+1)}}} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)} dx \leq 3^{\frac{3}{4}} \ln(a) \left(\ln\left(\frac{(1+M_2)^{(1+M_2)}}{(1+M_1)^{(1+M_1)}} \right) + (M_1 - M_2\ ) \right)^{\frac{1}{4}}$$
$$\times \left((M_1+k)^{\frac{-1}{3}} - (M_2+k)^{\frac{-1}{3}}\ \right)^{\frac{3}{4}}$$

$\forall x > 0, 0 < M_1 < M_2, k \in \mathbb{Z}^+$. *The integrand function is strictly positive and it is assumed that the integral over the prescribed domain absolutely converges.*

Proof.

$$\int_{M_1}^{M_2} \sqrt{\sqrt{\sqrt{\ln^2(x+1)}}} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)} dx = \int_{M_1}^{M_2} \left(\left(\left(\ln^2(x+1) \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)} dx$$

$$= \int_{M_1}^{M_2} \left(\ln^2(x+1) \right)^{\frac{1}{8}} \frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)} dx$$

$$\leq \left(\int_{M_1}^{M_2} \left(\left(\ln^2(x+1) \right)^{\frac{1}{8}} \right)^4 dx \right)^{\frac{1}{4}} \left(\int_{M_1}^{M_2} \left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)} \right)^{\frac{4}{3}} dx \right)^{\frac{3}{4}}$$

$$\leq \ln(a) \ \left(\int_{M_1}^{M_2} \ln(x+1) dx \right)^{\frac{1}{4}} \left(\int_{M_1}^{M_2} (x+k)^{-\frac{4}{3}} dx \right)^{\frac{3}{4}}$$

$$\leq 3^{\frac{3}{4}} \ln(a) \left(\ln\left(\frac{(1+M_2)^{(1+M_2)}}{(1+M_1)^{(1+M_1)}} \right) + (M_1 - M_2\ ) \right)^{\frac{1}{4}}$$

$$\times \left((M_1+k)^{\frac{-1}{3}} - (M_2+k)^{\frac{-1}{3}}\ \right)^{\frac{3}{4}}$$

by employing Hölder's inequality and integration by parts on the integral $\int_{M_1}^{M_2} ln(x+1)dx$. □

Theorem 3.9. *The continuous differential entropy over the finite interval* (M_1, M_2) *where* $0 < M_1 < M_2$ *of the functions*

$$\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}, \ \left(-\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)'$$

is provided in closed form by the following formulas

$$\begin{aligned}
\mathcal{H}_1((M_1, M_2)) &= -\int_{M_1}^{M_2} log_a\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)dx \\
&= \ln(a)\ (\ln(M_2+k)\ log_a(M_2+k) - \ln(M_1+k)\ log_a(M_1+k)) \\
&- \frac{1}{2}\left(\ln^2(M_2+k) - \ln^2(M_1+k)\right) \\
&- \ln(a)\ log_a(\ln(a))\ \ln\left(\frac{M_2+k}{M_1+k}\right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_2((M_1, M_2)) &= -\int_{M_1}^{M_2} log_a\left(\left(-\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)'\right)\left(-\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)' dx \\
&= 2\ \ln(a)\left(\frac{log_a(M_1+k)}{(M_1+k)} - \frac{log_a(M_2+k)}{(M_2+k)}\right) \\
&+ 2\left(\frac{1}{M_1+k} - \frac{1}{M_2+k}\right) - \ln(a)\ log_a(\ln(a))\ \left(\frac{1}{M_1+k} - \frac{1}{M_2+k}\right)
\end{aligned}$$

$\forall x > 0, 0 < M_1 < M_2, k \in \mathbb{Z}^+$. *The entropy integrals over the prescribed domain, absolutely converge and they are finite.*

Proof. By the definition of the continuous differential entropy restricted on the interval (M_1, M_2) and some algebraic manipulations we obtain

$$\begin{aligned}
\mathcal{H}_1((M_1, M_2)) &= -\int_{M_1}^{M_2} log_a\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)\left(\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)dx \\
&= -\int_{M_1}^{M_2} log_a\left(\frac{\ln(a)}{x+k}\right)\left(\frac{\ln(a)}{x+k}\right)\ dx \\
&= \ln(a)\ \int_{M_1}^{M_2} \frac{log_a(x+k)}{x+k}\ dx - \ln(a)\ log_a(\ln(a)) \int_{M_1}^{M_2} \frac{1}{x+k}dx \\
&= \ \ln(a)\ \int_{M_1}^{M_2} \frac{log_a(x+k)}{x+k}\ dx\ - \ln(a)\ log_a(\ln(a))\ \ln\left(\frac{M_2+k}{M_1+k}\right) \\
&= \ln(a)\ (\ln(M_2+k)\ log_a(M_2+k) - \ln(M_1+k)\ log_a(M_1+k)) \\
&- \frac{1}{2}\left(\ln^2(M_2+k) - \ln^2(M_1+k)\right) \\
&- \ln(a)\ log_a(\ln(a))\ \ln\left(\frac{M_2+k}{M_1+k}\right)
\end{aligned}$$

employing integration by parts on the integral $\int_{M_1}^{M_2} \frac{log_a(x+k)}{x+k}\, dx$ and using the result $\int_{M_1}^{M_2} \frac{\ln(x+k)}{(x+k)} dx = \frac{1}{2}\left(\ln^2(M_2+k) - \ln^2(M_1+k)\right)$. Similarly, for the second continuous differential entropy,we obtain

$$\begin{aligned}
\mathcal{H}_2((M_1, M_2)) &= -\int_{M_1}^{M_2} log_a\left(\left(-\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)'\right)\left(-\frac{\Gamma_a(x+k)}{\Gamma_a(x+k+1)}\right)' dx \\
&= -\int_{M_1}^{M_2} log_a\left(\ln(a)(x+k)^{-2}\right)\ \ln(a)\ (x+k)^{-2} dx \\
&= -\ln(a)\int_{M_1}^{M_2} \left(log_a(\ln(a))\ (x+k)^{-2} - 2log_a(x+k)\ (x+k)^{-2}\right) dx \\
&= 2\ln(a)\int_{M_1}^{M_2} log_a(x+k)\ (x+k)^{-2} dx - \ln(a) log_a(\ln(a))\int_{M_1}^{M_2} (x+k)^{-2} dx \\
&= 2\ \ln(a)\left(\frac{log_a(M_1+k)}{(M_1+k)} - \frac{log_a(M_2+k)}{(M_2+k)}\right) \\
&+ 2\left(\frac{1}{M_1+k} - \frac{1}{M_2+k}\right) - \ln(a)\ log_a(\ln(a))\ \left(\frac{1}{M_1+k} - \frac{1}{M_2+k}\right)
\end{aligned}$$

using integration by parts on the integral $\int_{M_1}^{M_2} log_a(x+k)\ (x+k)^{-2} dx$. □

4 Conclusions

In this article, we have obtained estimates of integrals involving ratios of the modified Gamma function, and more precisely inequalities involving L_2, H^1, H^2 norms during the first part of the article. Additionally, we obtained inequalities where the ratio of the Modified Gamma function is multiplied by another type of function on the positive interval (M_1, M_2). Lastly we obtained closed formulas for the continuous differential entropy of the ratio of the Modified Gamma function over the positive one dimensional interval (M_1, M_2).

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