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An Exponential-logarithmic Maximum Hardy-Hilbert-type Integral Inequality

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Abstract

This paper introduces a new variant of the Hardy-Hilbert integral inequality, referred to as an exponential-logarithmic maximum Hardy-Hilbert-type integral inequality. Using this result, we establish an additional related inequality. For completeness and accessibility, the paper is self-contained, and all proofs are provided in detail.

1 Introduction

We begin by recalling the classical Hardy-Hilbert integral inequality, which serves as the foundation for the results developed in this paper. Let $p, q > 1$ such that $1/p + 1/q = 1$, and $f, g : (0, \infty) \to (0, \infty)$ be two measurable functions such that

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty.$$

Then the following inequality holds:

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{x+y} f(x)g(y)dxdy \le \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}.$$

A well-known maximum version of this inequality is given below. Let $p, q > 1$ such that $1/p + 1/q = 1$, and $f, g : (0, \infty) \to (0, \infty)$ be two measurable functions such that

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty.$$

Then the following inequality holds:

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{\max(x,y)} f(x)g(y)dxdy \le pq \left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \tag{1}$$

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In both cases, the corresponding upper bounds are known to be best possible. Moreover, the inequalities involve only the unweighted integral norms of f and g. For the foundational theory, we refer to [7,8,13]. The study of Hardy-Hilbert-type integral inequalities continues to be an active area of research. In particular, two-dimensional variants have attracted attention, leading to a substantial body of literature. Significant contributions and developments in this direction can be found in [1–6, 9–12].

In this paper, we propose to generalize the inequality in Equation (1) by developing an exponential-logarithmic maximum Hardy-Hilbert-type integral inequality. Its main feature is to incorporate an exponential-logarithmic term depending on an adjustable parameter, c. Equation (1) is recovered by taking $c = 0$. Then we use this inequality to obtain another Hardy-Hilbert-type integral inequality based on the following double integral:

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{\max(x,y)}\left(\frac{1-\exp\left(-k\left|\ln\frac{x}{y}\right|\right)}{\left|\ln\frac{x}{y}\right|}\right) f(x)g(y)dxdy,$$

where $k \geq 0$. To the best of our knowledge, all the inequalities presented here appear to be new to the literature. This paper is self-contained; all proofs are given in full detail to ensure clarity and completeness.

The rest of the paper is as follows: The two Hardy-Hilbert-type integral inequalities are presented and proved in the next section. A conclusion is given in Section 3.

2 Theorems

2.1 First theorem

Our exponential-logarithmic maximum Hardy-Hilbert-type integral inequality is formulated in the theorem below.

Theorem 2.1. *Let $p, q > 1$ such that $1/p + 1/q = 1$, $c > -\min(1/p, 1/q)$, and $f, g : (0,\infty) \to (0,\infty)$ be two measurable functions such that*

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty.$$

Then the following inequality holds:

$$\int_0^{\infty}\int_0^{\infty} \frac{\exp\left(-c\left|\ln\frac{x}{y}\right|\right)}{\max(x,y)} f(x)g(y)dxdy \leq \frac{2c+1}{c^2+c+\frac{1}{pq}}\left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}.$$

Proof. Let us set

$$I = \int_0^{\infty}\int_0^{\infty} K(x,y)f(x)g(y)dxdy,$$

where

$$K(x,y) = \frac{\exp\left(-c\left|\ln \frac{x}{y}\right|\right)}{\max(x,y)}.$$

Using $1/p + 1/q = 1$, we can write

$$I = \int_0^{\infty}\int_0^{\infty} \left(K(x,y)^{\frac{1}{p}} \left(\frac{x}{y}\right)^{\frac{1}{pq}} f(x)\right)\left(K(x,y)^{\frac{1}{q}} \left(\frac{y}{x}\right)^{\frac{1}{pq}} g(y)\right) dxdy.$$

Applying the Hölder integral inequality over the double integral, we get

$$I \leq I_f^{\frac{1}{p}} I_g^{\frac{1}{q}},$$

where

$$I_f = \int_0^{\infty}\int_0^{\infty} K(x,y)\left(\frac{x}{y}\right)^{\frac{1}{q}} f^p(x)dydx$$

and

$$I_g = \int_0^{\infty}\int_0^{\infty} K(x,y)\left(\frac{y}{x}\right)^{\frac{1}{p}} g^q(y)dxdy.$$

Applying the Fubini-Tonelli integral theorem, we can write

$$I_f = \int_0^{\infty} \omega_1(x) f^p(x)dx,$$

where

$$\omega_1(x) = \int_0^{\infty} K(x,y)\left(\frac{x}{y}\right)^{\frac{1}{q}} dy.$$

Substituting $y = xt$, which implies that $dy = xdt$, noting that $xK(x,xt) = K(1,t)$, we obtain

$$\omega_1(x) = \int_0^{\infty} K(1,t)t^{-\frac{1}{q}}dt = \int_0^{\infty} \frac{\exp(-c|\ln t|)}{\max(1,t)} t^{-\frac{1}{q}}dt.$$

Splitting this integral at $t = 1$ to resolve the absolute value and the max function, and using $c > -\min(1/p, 1/q)$ and $1/p + 1/q = 1$, we have

$$\begin{aligned}
\omega_1(x) &= \int_0^1 \frac{e^{c\ln t}}{1} t^{-\frac{1}{q}}dt + \int_1^{\infty} \frac{e^{-c\ln t}}{t} t^{-\frac{1}{q}}dt \\
&= \int_0^1 t^{c-\frac{1}{q}}dt + \int_1^{\infty} t^{-c-1-\frac{1}{q}}dt \\
&= \left[\frac{t^{c+\frac{1}{p}}}{c+\frac{1}{p}}\right]_0^1 + \left[\frac{t^{-c-\frac{1}{q}}}{-c-\frac{1}{q}}\right]_1^{\infty} \\
&= \frac{1}{c+\frac{1}{p}} + \frac{1}{c+\frac{1}{q}} = \frac{c+\frac{1}{q}+c+\frac{1}{p}}{\left(c+\frac{1}{p}\right)\left(c+\frac{1}{q}\right)} \\
&= \frac{2c+1}{c^2 + c\left(\frac{1}{p}+\frac{1}{q}\right) + \frac{1}{pq}} = \frac{2c+1}{c^2+c+\frac{1}{pq}}.
\end{aligned}$$

Therefore, we have

$$I_f = \frac{2c+1}{c^2+c+\frac{1}{pq}} \int_0^{\infty} f^p(x)dx.$$

For I_g, we proceed similarly. Applying the Fubini-Tonelli integral theorem, we can write

$$I_g = \int_0^{\infty} \omega_2(y) g^q(y) dy,$$

where

$$\omega_2(y) = \int_0^{\infty} K(x,y) \left(\frac{y}{x}\right)^{\frac{1}{p}} dx.$$

Due to the symmetry of $K(x,y)$, i.e., $K(x,y) = K(y,x)$, we compute

$$\omega_2(y) = \omega_1(x) = \frac{2c+1}{c^2+c+\frac{1}{pq}}.$$

Therefore, we have

$$I_g = \frac{2c+1}{c^2+c+\frac{1}{pq}} \int_0^{\infty} g^q(y)dy.$$

Finally, using $1/p + 1/q = 1$, we obtain

$$\begin{aligned} I &\leq \left(\frac{2c+1}{c^2+c+\frac{1}{pq}} \int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\frac{2c+1}{c^2+c+\frac{1}{pq}} \int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}} \\ &= \frac{2c+1}{c^2+c+\frac{1}{pq}} \left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of the theorem. □

If we take $c = 0$ in Theorem 2.1, then we get Equation (1), i.e.,

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{\max(x,y)} f(x)g(y)dxdy \leq pq \left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}.$$

For another example, if we take $c = 1$, then we get

$$\int_0^{\infty}\int_0^{\infty} \frac{\exp\left(-\left|\ln \frac{x}{y}\right|\right)}{\max(x,y)} f(x)g(y)dxdy \leq \frac{3}{2+\frac{1}{pq}} \left(\int_0^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}.$$

2.2 Second theorem

Using Theorem 2.1, we obtain another Hardy-Hilbert-type integral inequality, stated below.

Theorem 2.2. *Let $p, q > 1$ such that $1/p+1/q = 1$, $k \geq 0$, and $f, g : (0,\infty) \to (0,\infty)$ be two measurable functions such that*

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty.$$

Then the following inequality holds:

$$\int_0^\infty \int_0^\infty \frac{1}{\max(x,y)} \left(\frac{1-\exp\left(-k\left|\ln\frac{x}{y}\right|\right)}{\left|\ln\frac{x}{y}\right|} \right) f(x)g(y)dxdy$$
$$\leq \ln\left(pq\left(k^2+k\right)+1\right)\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}.$$

Proof. Considering the inequality in Theorem 2.1, we integrate both sides with respect to c over the interval $(0,k)$, which yields

$$\int_0^k \int_0^\infty \int_0^\infty \frac{\exp\left(-c\left|\ln\frac{x}{y}\right|\right)}{\max(x,y)} f(x)g(y)dxdydc$$
$$\leq \int_0^k \frac{2c+1}{c^2+c+\frac{1}{pq}} \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}} dc.$$

By the Fubini-Tonelli integral theorem (for the left side only), we have

$$\int_0^\infty \int_0^\infty \left(\int_0^k \exp\left(-c\left|\ln\frac{x}{y}\right|\right)dc\right) \frac{1}{\max(x,y)} f(x)g(y)dxdy$$
$$\leq \left(\int_0^k \frac{2c+1}{c^2+c+\frac{1}{pq}}dc\right) \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}.$$

We derive

$$\int_0^k \exp\left(-c\left|\ln\frac{x}{y}\right|\right)dc = \left[\frac{\exp\left(-c\left|\ln\frac{x}{y}\right|\right)}{-\left|\ln\frac{x}{y}\right|}\right]_0^k = \frac{1-\exp\left(-k\left|\ln\frac{x}{y}\right|\right)}{\left|\ln\frac{x}{y}\right|}$$

and

$$\int_0^k \frac{2c+1}{c^2+c+\frac{1}{pq}}dc = \left[\ln\left(c^2+c+\frac{1}{pq}\right)\right]_0^k$$
$$= \ln\left(k^2+k+\frac{1}{pq}\right) - \ln\left(\frac{1}{pq}\right)$$
$$= \ln\left(pq\left(k^2+k\right)+1\right).$$

Finally, we obtain

$$\int_0^\infty \int_0^\infty \frac{1}{\max(x,y)} \left(\frac{1-\exp\left(-k\left|\ln\frac{x}{y}\right|\right)}{\left|\ln\frac{x}{y}\right|} \right) f(x)g(y)dxdy$$
$$\leq \ln\left(pq\left(k^2+k\right)+1\right)\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}.$$

This completes the proof of the theorem. □

In particular, if we take $k = 1$, then we have

$$\int_0^{\infty} \int_0^{\infty} \frac{1}{\max(x,y)} \left(\frac{1 - \exp\left(-\left|\ln \frac{x}{y}\right|\right)}{\left|\ln \frac{x}{y}\right|} \right) f(x)g(y)dxdy$$
$$\leq \ln(2pq+1) \left(\int_0^{\infty} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy \right)^{\frac{1}{q}}.$$

3 Conclusion

In this paper, we introduce a new exponential-logarithmic maximum Hardy-Hilbert-type integral inequality. We then apply this result to derive another Hardy-Hilbert-type integral inequality. Possible directions for future research include developing further extensions and refinements of these inequalities, particularly those associated with the double integral

$$\int_0^{\infty} \int_0^{\infty} \frac{\exp\left(-c\left|\ln \frac{x}{y}\right|\right)}{(\max(x,y))^{\alpha}} f(x)g(y)dxdy,$$

where $\alpha \geq 0$. Future work may also explore applications of these results to related problems in analysis.

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