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Some Generalizations of Weak Contractions

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Abstract

The concept of weak contraction appeared in [6], and its extension appeared in [7]. In this paper we introduce weak contractions in the sense of [7] that advances the Kannan, Reich, Chatterjea and Hardy-Rogers contractions. Some results related to the fixed point of these new contractions are proved with illustrative examples.

1 Preliminaries

Definition 1.1 ([1, 2])**.** A self mapping T for $\alpha \in [0, \frac{1}{2})$ defined on a metric space (X, d) such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ is called a *Kannan contraction.*

Definition 1.2 ([3])**.** A self mapping T for $\alpha \in [0, \frac{1}{3})$ defined on a metric space (X, d) such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty) + d(x, y)]$$

for all $x, y \in X$ is called a *Reich contraction.*

Definition 1.3 ([4])**.** A self mapping T for $\alpha \in [0, \frac{1}{2})$ defined on a metric space (X, d) such that

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ is called a *Chatterjea contraction.*

Definition 1.4 ([5])**.** A self mapping T for $\beta \in [0, \frac{1}{5})$ defined on a metric space (X, d) such that

$$d(Tx, Ty) \leq \beta[d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ is called a *Hardy-Rogers contraction.*

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Definition 1.5 ([6])**.** A self mapping T for $\delta \in (0,1)$ and $L > 0$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx)$$

for all $x,y \in X$ is called a (δ, L) *weak contraction.*

Definition 1.6 ([7])**.** A self mapping T for $\delta \in (0,1)$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta d(x,y) + (1-\delta)d(y,Tx)$$

for all $x,y \in X$ is called a $(\delta, 1-\delta)$ *weak contraction.*

Novelty: The novelty of the generalized weak contractions introduced in this paper lies in their ability to guarantee unique fixed points in metric spaces without requiring the mapping to be a strict contraction. This allows for broader application in nonlinear analysis

2 Main Results

Definition 2.1. A self mapping T for $\delta \in (0, \frac{1}{2})$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta[d(x,Tx) + d(y,Ty)] + (1-2\delta)d(y,Tx)$$

for all $x,y \in X$ is called $(\delta, 1-2\delta)$ *weak contraction of type I.*

Definition 2.2. A self mapping T for $\delta \in (0, \frac{1}{2})$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta[d(x,Ty) + d(y,Tx)] + (1-2\delta)d(y,Tx)$$

for all $x,y \in X$ is called $(\delta, 1-2\delta)$ *weak contraction of type II.*

Definition 2.3. A self mapping T for $\delta \in (0, \frac{1}{3})$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta[d(x,y) + d(x,Tx) + d(y,Ty)] + (1-3\delta)d(y,Tx)$$

for all $x,y \in X$ is called $(\delta, 1-3\delta)$ *weak contraction.*

Definition 2.4. A self mapping T for $\delta \in (0, \frac{1}{5})$ defined on a metric space (X,d) such that

$$d(Tx,Ty) \leq \delta[d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)] + (1-5\delta)d(y,Tx)$$

for all $x,y \in X$ is called $(\delta, 1-5\delta)$ *weak contraction.*

Theorem 2.5. *Assume that T be a self mapping on a complete metric space (X,d) such that T is $(\delta, 1-2\delta)$ weak contraction of type I. Then T admits a unique fixed point.*

Proof. Define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n$ for $n = 1, 2, \cdots$, then

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) && (1)\\
&\leq \delta[d(Tx_n, x_n) + d(Tx_{n+1}, x_{n+1})] + (1-2\delta)d(Tx_n, x_{n+1}) && (2)\\
&\leq \delta[d(x_{n+1}, x_n) + d(x_{n+2}, x_{n+1})] + (1-2\delta)d(x_{n+1}, x_{n+1}) && (3)\\
(1-\delta)d(x_{n+1}, x_{n+2}) &\leq \delta d(x_n, x_{n+1}) && (4)\\
d(x_{n+1}, x_{n+2}) &\leq \frac{\delta}{(1-\delta)} d(x_n, x_{n+1}) && (5)\\
d(x_{n+1}, x_{n+2}) &\leq \gamma d(x_n, x_{n+1}) && (6)
\end{aligned}$$

where $\gamma = \frac{\delta}{1-\delta} < 1$. By induction we get

$$d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1).$$

Now with the help of triangle inequality and the above inequality 1, for all $n \geq 0$ and $m \geq 1$ we get that

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})\\
&\leq (\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+m-1})d(x_0, x_1)\\
&= \gamma^n d(x_0, x_1)\frac{1-\gamma^m}{1-\gamma}\\
&\leq d(x_0, x_1)\frac{\gamma^n}{1-\gamma}.
\end{aligned}$$

Thus for $n, m \to \infty$, $d(x_n, x_{n+m}) \to 0$. This shows that the sequence $\{x_n\} \subseteq X$ is Cauchy. Now, (X, d) being the complete metric space ensures the convergence of $\{x_n\}$ to a point $x^* \in X$. Now we claim that x^* is a fixed point of T. Observe that

$$\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*)\\
&\leq d(x^*, x_{n+1}) + \delta[d(x_n, Tx_n) + d(x^*, Tx^*)] + (1-2\delta)d(x^*, Tx_n)\\
&= d(x^*, x_{n+1}) + \delta[d(x_n, x_{n+1}) + d(x^*, Tx^*)] + (1-2\delta)d(x^*, x_{n+1}).
\end{aligned}$$

For $n \to \infty$, we have

$$d(x^*, Tx^*) \leq \delta d(x^*, Tx^*)$$

since $1 - \delta \neq 0$, then $d(x^*, Tx^*) = 0$, thus, $x^* = Tx^*$.

To prove uniqueness, suppose a and b are two distinct fixed points of T, then $d(a, b) > 0$. Now from the contractive condition of the theorem, we have

$$\begin{aligned}
d(a, b) &= d(Ta, Tb)\\
&\leq \delta[d(a, Ta) + d(b, Tb)] + (1-2\delta)d(b, Ta)\\
&= (1-2\delta)d(a, b)
\end{aligned}$$

δ being strictly less than $\frac{1}{2}$, it is a contradiction, thus $a = b$. Hence, we are done with the proof. □

Example 2.6. Let T be a self mapping on $([1,2], d)$ with $d(x,y) = |x-y|$ such that $T(x) = \frac{x+3}{4}$ $\forall$ $x \in X$. This map holds all the criteria of above theorem and admits a unique fixed point, i.e. $x = 1$. Additionally, for $\delta = \frac{1}{3}$ the following is the graph of $d(Tx,Ty) \leq \delta[d(x,Tx) + d(y,Ty)] + (1-2\delta)d(y,Tx)$. Observe that the left-hand side of the above inequality is at the bottom (RustTones) and the right-hand side of the above inequality is at the top (BlueGreenYellow).

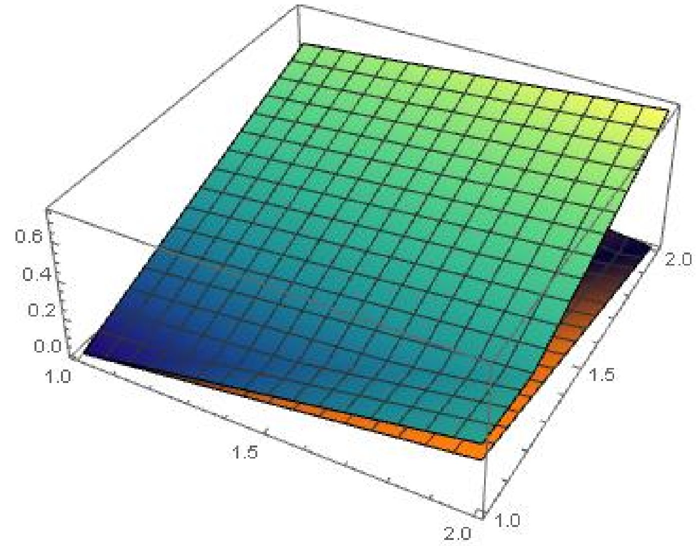

Figure 1: The graph of $T(x) = \dfrac{x+3}{4}$.

Theorem 2.7. *Assume that T be a self mapping on a complete metric space (X,d) such that T is $(\delta, 1-2\delta)$ weak contraction of type II. Then T admits a fixed point.*

Proof. Define a sequence $\{x_n\} \subseteq X$ by $x_{n+1} = Tx_n$ for $n = 1, 2, \cdots$, then

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \delta[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] + (1-2\delta)d(x_{n+1}, Tx_n) \\
&= \delta[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})] + (1-2\delta)d(x_{n+1}, x_{n+1}) \\
&= \delta d(x_n, x_{n+2}) \\
&\leq \delta[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
(1-\delta)d(x_{n+1}, x_{n+2}) &\leq \delta d(x_n, x_{n+1}) \\
d(x_{n+1}, x_{n+2}) &\leq \frac{\delta}{(1-\delta)} d(x_n, x_{n+1})
\end{aligned}$$

where $\gamma = \frac{\delta}{1-\delta} < 1$. By induction we get

$$d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1).$$

Similar to the proof of the above theorem, the sequence $\{x_n\}$ converges to a point $x^* \in X$. Now we show that x^* is a fixed point of T. Observe that

$$\begin{aligned} d(x^*,Tx^*) &\leq d(x^*,x_{n+1})+d(Tx_n,Tx^*) \\ &\leq d(x^*,x_{n+1})+\delta[d(x_n,Tx^*)+d(x^*,Tx_n)]+(1-2\delta)d(x^*,Tx_n) \\ &= d(x^*,x_{n+1})+\delta[d(x_n,Tx^*)+d(x^*,x_{n+1})]+(1-2\delta)d(x^*,x_{n+1}). \end{aligned}$$

For $n\to\infty$, we have $d(x^*,Tx^*)\leq \delta d(x^*,Tx^*)$. Since $1-\delta\neq 0$, then $d(x^*,Tx^*)=0$. So x^* is a fixed point of T. Thus, we are done with the proof. ☐

Remark 2.8. The above theorem does not guarantee the uniqueness of the fixed point. Suppose a and b are fixed points of T where a and b are distinct, then $d(a,b)>0$. So the contractive condition deduce that

$$\begin{aligned} d(a,b) &= d(Ta,Tb) \\ &\leq \delta[d(a,Tb)+d(b,Ta)]+(1-2\delta)d(b,Ta) \\ &= \delta[d(a,b)+d(a,b)]+(1-2\delta)d(a,b) \\ &= d(a,b). \end{aligned}$$

So no contradiction occurs implying that the fixed point may not be unique.

Example 2.9. Let T be a self mapping on $([2,3],d)$ with $d(x,y)=|x-y|$ such that $T(x)=\frac{x+6}{4}\ \forall\ x\in X$. This map holds all the criteria of above theorem and admits a fixed point, i.e. $x=2$. Additionally, for $\delta=\frac{1}{3}$ the following is the graph of $d(Tx,Ty)\leq\delta[d(x,Ty)+d(y,Tx)]+(1-2\delta)d(y,Tx)$. Observe that the left-hand side of the above inequality is at the bottom (RustTones) and the right-hand side of the above inequality is at the top (BlueGreenYellow).

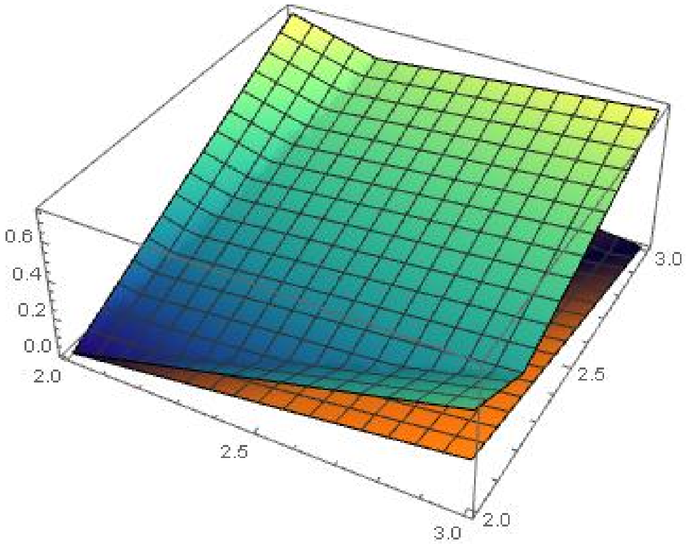

Figure 2: The graph of $T(x)=\dfrac{x+6}{4}$.

Theorem 2.10. *Assume that T be a self mapping on a complete metric space (X,d) such that T is $(\delta,1-3\delta)$ weak contraction. Then T admits a unique fixed point.*

Proof. Define a sequence $\{x_n\} \subseteq X$ as in Theorem 2.5 by $x_{n+1} = Tx_n$ for $n = 1, 2, \cdots$, then

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\le \delta[d(x_n, x_{n+1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] + (1 - 3\delta)d(x_{n+1}, Tx_n) \\
&= \delta[d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + (1 - 3\delta)d(x_{n+1}, x_{n+1}) \\
(1 - \delta)d(x_{n+1}, x_{n+2}) &\le 2\delta d(x_n, x_{n+1}) \\
d(x_{n+1}, x_{n+2}) &\le \frac{2\delta}{(1 - \delta)} d(x_n, x_{n+1})
\end{aligned}$$

where $\gamma = \frac{2\delta}{1-\delta} < 1$. By induction we get $d(x_n, x_{n+1}) \le \gamma^n d(x_0, x_1)$. Similar to the proof of Theorem 2.5, the sequence $\{x_n\}$ converges to a point $a \in X$. Now we show that a is a fixed point of T. Observe that

$$\begin{aligned}
d(a, Ta) &\le d(a, x_{n+1}) + d(Tx_n, Ta) \\
&\le d(a, x_{n+1}) + \delta[d(x_n, a) + d(x_n, Tx_n) + d(a, Ta)] + (1 - 3\delta)d(a, Tx_n) \\
&= d(a, x_{n+1}) + \delta[d(x_n, a) + d(x_n, x_{n+1}) + d(a, Ta)] + (1 - 3\delta)d(a, x_{n+1}).
\end{aligned}$$

For $n \to \infty$, we have $d(a, Ta) \le \delta d(a, Ta)$. Since $1 - \delta \neq 0$, then $d(a, Ta) = 0$. So a is a fixed point of T. For uniqueness let T has two distinct fixed points c and d, then $d(c, d) > 0$. From the contractive condition we have

$$\begin{aligned}
d(c, d) &= d(Tc, Td) \\
&\le \delta[d(c, d) + d(c, Tc) + d(d, Td)] + (1 - 3\delta)d(d, Tc) \\
&= (1 - 2\delta)d(c, d)
\end{aligned}$$

δ being strictly less than $\frac{1}{3}$, it is a contradiction, thus $a = b$. Hence, we are done with the proof. □

Example 2.11. Let T be a self mapping on $([0, 1], d)$ with $d(x, y) = |x - y|$ such that $T(x) = \frac{x^2}{3}$ $\forall\, x \in X$. This map holds all the criteria of above theorem and admits a unique fixed point, i.e. $x = 0$. Additionally, for $\delta = \frac{1}{4}$ the following is the graph of $d(Tx, Ty) \le \delta[d(x, y) + d(x, Tx) + d(y, Ty)] + (1 - 3\delta)d(y, Tx)$. Observe that the left-hand side of the above inequality is at the bottom (LakeColors) and the right-hand side of the above inequality is at the top (BlueGreenYellow).

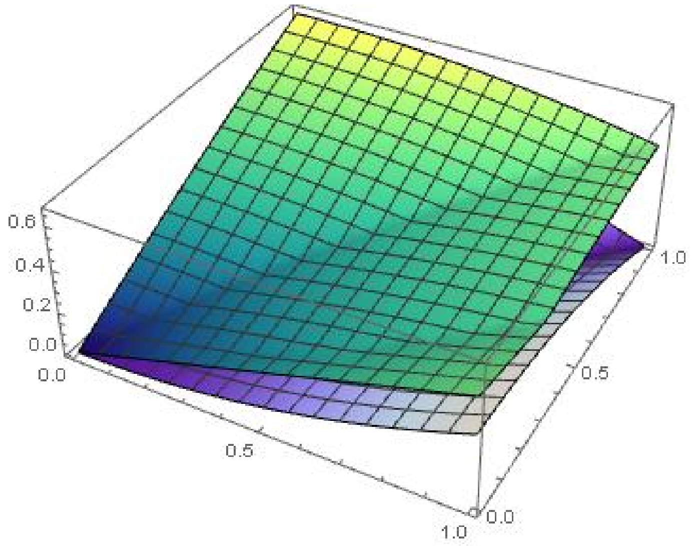

Figure 3: The graph of $T(x) = \dfrac{x^2}{3}$.

Theorem 2.12. *Assume that T be a self mapping on a complete metric space (X, d) such that T is $(\delta, 1-5\delta)$ weak contraction. Then T admits a unique fixed point.*

Proof. Define a sequence $\{x_n\} \subseteq X$ as in Theorem 2.5 by $x_{n+1} = Tx_n$ for $n = 1, 2, \cdots$, then

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \delta[d(x_n, x_{n+1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] \\
&\quad + (1-5\delta) d(x_{n+1}, Tx_n) \\
&= \delta[d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})] \\
&\quad + (1-5\delta) d(x_{n+1}, x_{n+1}) \\
&\leq \delta[2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\
&= \delta[3d(x_n, x_{n+1}) + 2d(x_{n+1}, x_{n+2})] \\
(1-2\delta) d(x_{n+1}, x_{n+2}) &\leq 3\delta d(x_n, x_{n+1}) \\
d(x_{n+1}, x_{n+2}) &\leq \frac{3\delta}{(1-2\delta)} d(x_n, x_{n+1})
\end{aligned}$$

where $\gamma = \frac{3\delta}{1-2\delta} < 1$. By induction we get $d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1)$. Similar to the proof of Theorem 2.5, the sequence $\{x_n\}$ converges to a point $a \in X$. Now we show that a is a fixed point of T. Observe that

$$\begin{aligned}
d(a, Ta) &\leq d(a, x_{n+1}) + d(Tx_n, Ta) \\
&\leq d(a, x_{n+1}) + \delta[d(x_n, a) + d(x_n, Tx_n) + d(a, Ta) + d(x_n, Ta) + d(a, Tx_n)] + (1-5\delta) d(a, Tx_n) \\
&\leq d(a, x_{n+1}) + \delta[d(x_n, a) + d(x_n, x_{n+1}) + d(a, Ta) + d(x_n, Ta) + d(a, x_{n+1})] + (1-5\delta) d(a, x_{n+1}).
\end{aligned}$$

For $n \to \infty$, we have $d(a, Ta) \leq 2\delta d(a, Ta)$. Since $1 - 2\delta \neq 0$, then $d(a, Ta) = 0$. So a is a fixed point of T. For uniqueness let T has two distinct fixed points c and d, then $d(c, d) > 0$. From the contractive condition, we have

$$\begin{aligned} d(c,d) &= d(Tc,Td) \\ &\le \delta[d(c,d)+d(c,Tc)+d(d,Td)+d(c,Td)+d(d,Tc)]+(1-5\delta)d(d,Tc) \\ &= (1-2\delta)d(c,d) \end{aligned}$$

δ being strictly less than $\frac{1}{5}$, it is a contradiction, thus $a=b$. Hence, we are done with the proof. □

Example 2.13. Let T be a self mapping on $([0,1],d)$ with $d(x,y)=|x-y|$ such that $T(x)=\frac{x^2}{5}\ \forall\, x\in X$. This map holds all the criteria of the above theorem and admits a unique fixed point, i.e. $x=0$. Additionally, for $\delta=\frac{1}{6}$ the following is the graph of $d(Tx,Ty)\le\delta[d(x,y)+d(x,Tx)+d(y,Ty)+d(x,Ty)+d(y,Tx)]+(1-5\delta)d(y,Tx)$. Observe that the left-hand side of the above inequality is at the bottom (LakeColors) and the right-hand side of the above inequality is at the top (BlueGreenYellow).

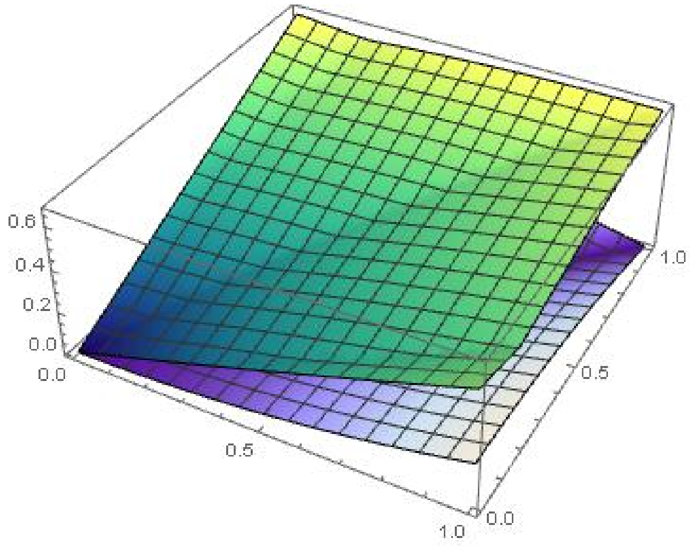

Figure 4: The graph of $T(x)=\dfrac{x^2}{5}$.

3 Conclusion

In this paper, we introduced several new classes of weak contractions that generalize the Kannan, Reich, Chatterjea, and Hardy-Rogers contractions within the framework of [7]. For each new contractive condition, we established fixed point existence and uniqueness theorems in complete metric spaces. Illustrative examples confirmed the validity of the obtained results. The proposed contractions offer greater flexibility in applications where standard contractive parameters fail. Future work may extend these notions to more general settings such as b-metric spaces, partial metric spaces, modular metric spaces, or fuzzy metric spaces, thereby broadening the applicability of weak contraction principles.

References

[1] Kannan, R. (1968). Some results on fixed points. *Bulletin of the Calcutta Mathematical Society*, *60*, 71–76. https://doi.org/10.2307/2316437

[2] Kannan, R. (1969). Some results on fixed points II. *The American Mathematical Monthly*, *76*(4), 405–408. https://doi.org/10.1080/00029890.1969.12000228

[3] Reich, S. (1971). Some results concerning contraction mappings. *Canadian Mathematical Bulletin*, *14*(1), 121–124. https://doi.org/10.4153/CMB-1971-024-9

[4] Chatterjea, S. K. (1972). Fixed point theorems. *Comptes Rendus de l'Académie Bulgare des Sciences*, *25*, 727–730.

[5] Hardy, G. E., & Rogers, T. D. (1973). A generalization of a fixed point theorem of Reich. *Canadian Mathematical Bulletin*, *16*(2), 201–206. https://doi.org/10.4153/CMB-1973-036-0

[6] Berinde, V. (2004). Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Analysis Forum*, *9*(1), 43–53.

[7] Ampadu, C. B. (2019). An almost contraction mapping theorem in metric spaces with unique fixed point. *Fundamental Journal of Mathematics and Mathematical Sciences*, *11*(2), 47–50.