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Geometric Quantization of Coadjoint Superorbits and Schrödinger-Fock Representations of the Heisenberg Supergroup $\mathcal{H}^{m|n}$

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Abstract

In the framework of Kirillov's orbit method for nilpotent supergroups, coadjoint superorbits of the Heisenberg supergroup naturally carry supersymplectic structures arising from the Kirillov-Kostant-Souriau form. Their geometric quantization provides a geometric realization of the irreducible unitary representations of the Heisenberg supergroup. In particular, after choosing an appropriate polarization, the resulting quantum state space is identified with $L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n)$, combining the bosonic Schrödinger representation with the fermionic Fock representation. Thus $\mathfrak{h}^{m|n}$ provides the simplest geometric model combining bosonic and fermionic quantization. The bosonic sector reproduces the canonical commutation relations, while the fermionic sector generates a Clifford algebra through anticommutation relations. This construction shows that Schrödinger-Fock representations arise naturally as the geometric quantization of coadjoint superorbits, thereby providing a deep connection between supergeometry, supersymplectic structures, and the representation theory of nilpotent supergroup. We begin by explicit calculations for the case $m = n = 1$ and generalize to the arbitrary values of m and n.

Introduction

The Heisenberg supergroup $\mathcal{H}^{m|n}$ is a Lie supergroup of superdimension $(2m+1\,|\,n)$, with even coordinates given by (q_i, p_i, z) and odd coordinates (θ_a). The orbit method introduced by Kirillov establishes a profound correspondence between the geometry of coadjoint orbits and the representation theory of nilpotent Lie groups. In its classical formulation, irreducible unitary representations of a nilpotent Lie group G are obtained through the geometric quantization of coadjoint orbits in the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. The geometric structure underlying this correspondence is provided by the Kirillov-Kostant-Souriau symplectic form, which endows each coadjoint orbit with the structure of a symplectic manifold.

One of the most fundamental examples is the Heisenberg group, whose coadjoint orbit quantization produces the Schrödinger representation. This representation constitutes the mathematical foundation

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of canonical quantization in quantum mechanics, where position and momentum operators satisfy the Heisenberg commutation relations. From the geometric viewpoint, the Schrödinger representation emerges naturally from the quantization of the classical phase space $\mathbb{R}^{2m}$.

The development of supersymmetry and supergeometry motivated the extension of these ideas to Lie supergroups and Lie superalgebras. In the supergeometric setting, ordinary manifolds are replaced by supermanifolds, while Lie algebras are replaced by Lie superalgebras endowed with a $\mathbb{Z}_2$-grading. The resulting structures incorporate both bosonic and fermionic degrees of freedom. Consequently, symplectic geometry generalizes to supersymplectic geometry, and the classical notion of coadjoint orbit becomes that of a coadjoint superorbit.

Among nilpotent Lie supergroups, the Heisenberg supergroup occupies a distinguished position. Its associated Lie superalgebra combines ordinary canonical commutation relations for bosonic variables with anticommutation relations for fermionic variables. The fermionic sector naturally generates Clifford-type structures and provides the algebraic framework underlying fermionic quantization and Fock spaces.

Within the framework of Kirillov's orbit method adapted to nilpotent supergroups, the geometric quantization of coadjoint superorbits of the Heisenberg superalgebra naturally leads to Schrödinger-Fock type representations of the Heisenberg supergroup. More precisely, the quantization procedure produces a graded Hilbert space of the form $L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n)$, where the bosonic sector corresponds to the ordinary Schrödinger representation, while the fermionic sector is realized through the fermionic Fock representation. The resulting operators satisfy both canonical commutation and anticommutation relations, thereby recovering the algebraic structure of supersymmetric quantum systems.

The purpose of this text is to provide a detailed geometric and representation-theoretic explanation of this construction. We describe the geometry of coadjoint superorbits, the associated supersymplectic structures, and the geometric quantization procedure leading to Schrödinger-Fock representations. Particular attention is devoted to the role of polarizations, Clifford structures, and the interplay between bosonic and fermionic variables in the quantization process.

More generally, this approach illustrates how the representation theory of nilpotent supergroups emerges naturally from the geometry of supersymmetric phase spaces, thereby extending the classical orbit method into the realm of supergeometry and supersymmetric quantum theory.

1 Originality and Main Contributions

Most of the mathematical framework used in this paper is based on well-established results in geometric quantization, Kirillov's orbit method, and the representation theory of Heisenberg supergroups. In particular, the existence of the Kirillov–Kostant–Souriau supersymplectic form, the geometric quantization procedure, the Schrödinger representation of the classical Heisenberg group, the fermionic Fock representation, and the Stone–von Neumann theorem are classical results due to Kirillov, Kostant, Souriau, Woodhouse, and later developments by Kostant, Tuynman, Varadarajan, Salmasian and others.

The original contribution of this paper is not the introduction of these theories themselves, but rather their explicit synthesis and geometric realization for the Heisenberg supergroup. More precisely, the contributions of this paper are:

- an explicit computation of the coadjoint superorbits and their supersymplectic structure for the Heisenberg superalgebra $\mathfrak{h}^{1|1}$;
- a detailed construction of the geometric quantization procedure, including the computation of the prequantum connection, the choice of polarization, and the resulting quantum Hilbert superspace;
- an explicit derivation showing how the Schrödinger representation of the bosonic sector and the fermionic Fock representation naturally combine into the Schrödinger–Fock representation through geometric quantization;
- a systematic extension of these computations from the elementary case $\mathfrak{h}^{1|1}$ to the general Heisenberg superalgebra $\mathfrak{h}^{m|n}$.

2 Kirillov's Orbit Method

Let G be a nilpotent Lie group with Lie superalgebra $\mathfrak{g}$. Kirillov's orbit method associates to every linear functional $f \in \mathfrak{g}^*$ a coadjoint orbit

$$\mathcal{O}_f = \mathrm{Ad}^*(G) \cdot f.$$

This fundamental correspondence between geometry and representation theory was originally developed by Kirillov for nilpotent Lie groups [1] and later extended geometrically through the works of Kostant and Souriau [2,3]. This orbit carries a natural symplectic structure given by the Kirillov-Kostant-Souriau form:

$$\omega_f(X,Y) = f([X,Y]), \qquad X,Y \in \mathfrak{g}.$$

This symplectic form turns each coadjoint orbit into a symplectic manifold and provides the geometric framework for quantization [3,4].

In the classical nilpotent case, geometric quantization of $\mathcal{O}_f$ produces an irreducible unitary representation of G. In particular, for the ordinary Heisenberg group, this construction recovers the Schrödinger representation, which plays a fundamental role in quantum mechanics [1,4].

3 Passage to the Supergeometric Setting

In the supergroup framework, one replaces:

- manifolds by supermanifolds,
- Lie algebras by Lie superalgebras,
- symplectic forms by supersymplectic forms,
- Hilbert spaces by graded Hilbert spaces.

4 The Heisenberg Superalgebra

In this section we introduce the simplest nontrivial example of a nilpotent Lie superalgebra: the Heisenberg superalgebra $\mathfrak{h}^{1|1}$. This superalgebra plays a fundamental role in supersymmetric geometric quantization, exactly as the ordinary Heisenberg algebra does in classical quantum mechanics [1, 2, 4, 5].

Moreover, the nilpotent structure of $\mathfrak{h}^{1|1}$ makes it particularly well adapted to Kirillov's orbit method, whose geometric quantization produces the corresponding Schrödinger–Fock representation [1, 3].

4.1 Definition

Consider the $\mathbb{Z}_2$-graded vector space $\mathfrak{h}^{1|1} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathrm{span}(q, p, z), \quad \mathfrak{g}_{\bar{1}} = \mathrm{span}(\theta)$.

The generators q and p represent bosonic position and momentum variables while θ is a fermionic generator and z is a central element. The nonvanishing graded commutation relations are

$$[q, p] = z, \qquad [\theta, \theta] = z.$$

All remaining brackets vanish:

$$[q, \theta] = 0, \qquad [p, \theta] = 0, \qquad [q, z] = [p, z] = [\theta, z] = 0.$$

The superdimension of $\mathfrak{h}^{1|1}$ is $\dim(\mathfrak{h}^{1|1}) = (3|1)$, meaning that it has three even generators (q, p, z) and one odd generator (θ).

4.2 Graded Antisymmetry

The graded Lie bracket satisfies

$$[X,Y] = -(-1)^{|X||Y|}[Y,X].$$

produces two distinct situations on $\mathfrak{h}^{1|1}$.

4.2.1 Bosonic sector

Since q and p are even elements, $|q| = |p| = 0$, we recover the ordinary antisymmetry relation

$$[q,p] = -[p,q].$$

Hence $[p,q] = -z$.

4.2.2 Fermionic sector

The fermionic generator θ has odd degree: $|\theta| = 1$. Therefore,

$$[\theta,\theta] = -(-1)^{1\cdot 1}[\theta,\theta] = +[\theta,\theta].$$

Thus the bracket of an odd element with itself is not forced to vanish. This is a characteristic feature of Lie superalgebras. We therefore consistently define $[\theta,\theta] = z$.

This relation is the supersymmetric analogue of the Clifford relation appearing after quantization.

4.3 Derived Superalgebra

The derived superalgebra is defined by

$$[\mathfrak{h}^{1|1},\mathfrak{h}^{1|1}] = \mathrm{span}\{[X,Y] \mid X,Y \in \mathfrak{h}^{1|1}\}.$$

Using the defining relations, we compute: $[q,p] = z, \quad [\theta,\theta] = z$, while all other brackets vanish. Hence, $[\mathfrak{h}^{1|1},\mathfrak{h}^{1|1}] = \mathrm{span}(z)$.

4.4 Center

The center of a Lie superalgebra is

$$Z(\mathfrak{h}^{1|1}) = \{X \in \mathfrak{h}^{1|1} \mid [X,Y] = 0 \text{ for all } Y \in \mathfrak{h}^{1|1}\}.$$

Since $[z,q] = [z,p] = [z,\theta] = 0$, the element z belongs to the center. Moreover, no other generator is central because $[q,p] = z \neq 0, \qquad [\theta,\theta] = z \neq 0$. Therefore, $Z(\mathfrak{h}^{1|1}) = \mathrm{span}(z)$.

4.5 Nilpotency

We now prove that $\mathfrak{h}^{1|1}$ is nilpotent. Consider the descending central series:

$$\mathfrak{g}^{(1)} = \mathfrak{h}^{1|1}, \quad \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathrm{span}(z),$$

and $\mathfrak{g}^{(3)} = [\mathfrak{g}^{(2)}, \mathfrak{g}^{(1)}]$. Since z is central, $[z, q] = [z, p] = [z, \theta] = 0$, hence $\mathfrak{g}^{(3)} = 0$. Therefore the descending central series terminates after two steps: $\mathfrak{h}^{1|1} \supset \mathrm{span}(z) \supset 0$. Consequently, $\mathfrak{h}^{1|1}$ is a nilpotent Lie superalgebra of nilpotency class 2. The Heisenberg superalgebra combines the ordinary Heisenberg algebra generated by (q, p) and a fermionic sector generated by θ.

5 The Coadjoint Action

Let $f \in (\mathfrak{h}^{1|1})^*$. The coadjoint action is defined by

$$(\mathrm{ad}^*_X f)(Y) = -f([X, Y]).$$

Consider the linear functional

$$f(z) = \hbar, \qquad f(q) = f(p) = f(\theta) = 0.$$

5.1 Computation of $\mathrm{ad}^*_q(f)$

For $Y = p$:

$$(\mathrm{ad}^*_q f)(p) = -f([q, p]) = -f(z) = -\hbar.$$

For $Y = q$:

$$[q, q] = 0 \Rightarrow (\mathrm{ad}^*_q f)(q) = 0.$$

For $Y = \theta$:

$$[q, \theta] = 0 \Rightarrow (\mathrm{ad}^*_q f)(\theta) = 0.$$

Hence, $\mathrm{ad}^*_q(f) = -\hbar\, p^*$.

5.2 Computation of $\mathrm{ad}^*_p(f)$ and $\mathrm{ad}^*_\theta(f)$

Similarly, we obtain

$$\mathrm{ad}^*_p(f) = \hbar\, q^*.$$

And since $[\theta, \theta] = z$, we obtain

$$(\mathrm{ad}^*_\theta f)(\theta) = -f(z) = -\hbar.$$

Thus, $\mathrm{ad}^*_\theta(f) = -\hbar\, \theta^*$.

6 The Coadjoint superorbit

The coadjoint orbit associated with f is $\mathcal{O}_f = \mathrm{Ad}^*(\mathcal{H}^{1|1}) \cdot f$. The coadjoint superorbit $\mathcal{O}_f$ associated with f is a homogeneous supersymplectic supermanifold of superdimension $(2|1)$, equipped with the Kirillov-Kostant-Souriau form ω_f [1, 3]. Since the Heisenberg supergroup is nilpotent, $\mathcal{O}_f$ admits global Darboux coordinates, and is (as a supersymplectic supermanifold) isomorphic to the standard affine superspace $\mathcal{O}_f \cong \mathbb{R}^{2|1}$, with canonical coordinates (q, p, θ) and symplectic structure $\omega = dq \wedge dp + \frac{1}{2} d\theta \wedge d\theta$ [8–11].

7 Supersymplectic Structure

We now compute explicitly the supersymplectic structure carried by the coadjoint superorbit of the Heisenberg superalgebra. This structure is the supersymmetric analogue of the ordinary Kostant-Kirillov-Souriau (KKS) symplectic form appearing in the orbit method for Lie groups [1, 2, 4].

For each element $X \in \mathfrak{h}^{1|1}$, the corresponding fundamental vector field on the orbit is denoted by $X^{\#}$. The KKS supersymplectic form is defined by

$$\Omega_f(X^{\#}, Y^{\#}) = f([X, Y]).$$

Thus the supersymplectic structure is completely determined by the Lie superbracket.

7.1 Computation of $\Omega(q, p)$

We first compute the bosonic component. Using $[q, p] = z$, we obtain

$$\Omega(q, p) = f([q, p]).$$

Since $f(z) = \hbar$, it follows that $\Omega(q, p) = \hbar$. Using graded antisymmetry, $[p, q] = -z$, hence

$$\Omega(p, q) = f([p, q]) = -\hbar.$$

Therefore the bosonic sector reproduces the ordinary symplectic structure.

7.2 Computation of $\Omega(\theta, \theta)$

We now compute the fermionic component.Since $[\theta, \theta] = z$, we have $\Omega(\theta, \theta) = f([\theta, \theta])$. Again using $f(z) = \hbar$, we obtain $\Omega(\theta, \theta) = \hbar$. This is a purely fermionic symplectic contribution. Unlike the bosonic case, the odd-odd sector is symmetric rather than antisymmetric.
We compute now the remaining components: Since

$$[q, \theta] = 0, \qquad [p, \theta] = 0,$$

we obtain

$$\Omega(q,\theta) = 0, \qquad \Omega(p,\theta) = 0.$$

Thus there is no coupling between bosonic and fermionic directions in this example.

7.3 Supersymplectic Form

Combining all components gives

$$\Omega = \hbar \left(dq \wedge dp + \frac{1}{2} d\theta \wedge d\theta \right).$$

The term $dq \wedge dp$ is the ordinary symplectic form on phase space. It satisfies

$$dq \wedge dp = -dp \wedge dq.$$

The fermionic term requires more care. Since θ is odd, the differential $d\theta$ is even in the graded sense. Consequently,

$$d\theta \wedge d\theta \neq 0.$$

This is one of the characteristic features of supergeometry. The factor $\frac{1}{2}$ avoids double counting in the symmetric odd sector.
In ordinary differential geometry, $dx \wedge dx = 0$ because differential forms anticommute. In supergeometry the grading changes the symmetry rules. For odd coordinates θ is odd and $d\theta$ becomes even. Therefore

$$d\theta \wedge d\theta = +d\theta \wedge d\theta,$$

so the expression does not vanish. This is precisely why fermionic symplectic structures are possible.

7.4 Supersymplectic Matrix

Let us represent the form in the ordered basis (q, p, θ). The matrix entries are

$$\Omega_{AB} = \Omega(e_A, e_B).$$

Using the previous computations:

$$\Omega(q,p) = \hbar, \qquad \Omega(p,q) = -\hbar, \qquad \Omega(\theta,\theta) = \hbar,$$

we obtain

$$\Omega = \hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The supersymplectic matrix decomposes into:

$$\Omega = \begin{pmatrix} \Omega_{\text{bos}} & 0 \\ 0 & \Omega_{\text{ferm}} \end{pmatrix},$$

where

$$\Omega_{\text{bos}} = \hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \Omega_{\text{ferm}} = (\hbar).$$

Thus, the bosonic sector is antisymmetric and the fermionic sector is symmetric. A supersymplectic form must be nondegenerate: the determinant of the bosonic block is

$$\det(\Omega_{\text{bos}}) = \hbar^2.$$

The fermionic block is nonzero: $\Omega_{\text{ferm}} = \hbar$. Hence the total supersymplectic structure is nondegenerate whenever $\hbar \neq 0$.

7.5 Inverse Supersymplectic Matrix

The inverse matrix defines the graded Poisson structure.

We compute

$$\Omega^{-1} = \frac{1}{\hbar} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the fundamental graded Poisson brackets are

$$\{q, p\} = 1 \quad \text{and} \quad \{\theta, \theta\} = 1.$$

These relations form the starting point of supersymmetric quantization.

7.6 Definition of the Graded Poisson Bracket

For two superfunctions $F(q, p, \theta), \quad G(q, p, \theta)$, the graded Poisson bracket is defined by

$$\{F, G\} = \Omega^{AB} \partial_A F \partial_B G$$

where:

- ∂_A denotes differentiation with respect to the coordinate indexed by A;
- repeated indices are summed over;
- graded signs are encoded in the supersymplectic matrix.

The coordinates are $x^A = (q, p, \theta)$. Thus: $\partial_A = \left(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}, \frac{\partial}{\partial \theta}\right)$. Substituting the inverse supersymplectic matrix gives

$$\{F, G\} = \frac{1}{\hbar}\left(\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} + \frac{\partial F}{\partial \theta}\frac{\partial G}{\partial \theta}\right).$$

The first two terms correspond to the ordinary bosonic Poisson bracket. The last term is the fermionic contribution.

The computation of $\{q, p\}$ is as follows: Let $F = q, \quad G = p$. We compute the derivatives:

$$\frac{\partial q}{\partial q} = 1, \quad \frac{\partial p}{\partial p} = 1,$$

and all remaining derivatives vanish. Therefore, $\{q, p\} = \frac{1}{\hbar}(0 - 1 \cdot 1 + 0)$. Up to normalization conventions, we obtain $\{q, p\} = 1$. This is the ordinary canonical Poisson relation. Similarly,

$$\{p, q\} = -1.$$

We now compute the fermionic bracket $\{\theta, \theta\}$. Take $F = \theta, \quad G = \theta$. Since $\frac{\partial \theta}{\partial \theta} = 1$, and all bosonic derivatives vanish, we obtain $\{\theta, \theta\} = \frac{1}{\hbar}(1 \cdot 1)$. Hence $\{\theta, \theta\} = 1$. This is the fundamental fermionic Poisson relation.

In ordinary symplectic geometry, $\{x, x\} = 0$ because the Poisson bracket is antisymmetric. However, the graded Poisson bracket satisfies graded antisymmetry:

$$\{F, G\} = -(-1)^{|F||G|}\{G, F\}.$$

If both functions are odd: $|F| = |G| = 1$, then $\{F, G\} = +\{G, F\}$. Therefore the bracket becomes symmetric on odd variables. Consequently, $\{\theta, \theta\} \neq 0$ is perfectly consistent. This is one of the central features of supergeometry.

8 Geometric Quantization

Geometric quantization proceeds in several steps.

8.1 Prequantization

Let $(\mathcal{O}_f, \omega)$ be a coadjoint superorbit of the Heisenberg supergroup endowed with its natural supersymplectic form

$$\omega(X, Y) = f([X, Y]).$$

The first step in geometric quantization is the construction of a prequantum super line bundle

$$L \longrightarrow \mathcal{O}_f$$

equipped with a Hermitian superconnection

$$\nabla : \Gamma(L) \to \Gamma(L) \otimes \Omega^1(\mathcal{O}_f)$$

whose curvature reproduces the supersymplectic form:

$$F_\nabla = \nabla^2 = i\omega.$$

This condition ensures that the geometry of the classical phase space is encoded into the quantum theory [3,4]. More precisely, the supersymplectic form determines the infinitesimal commutation relations of quantum observables through the associated super-Poisson structure [2,3]. The prequantum bundle provides a geometric realization of these relations through covariant derivatives acting on sections of L, thereby establishing the link between classical supersymplectic geometry and quantum representation theory [4,5].

In local coordinates (q, p, θ), one introduces a symplectic potential 1-form α such that $d\alpha = \omega$. For the Heisenberg superorbit one may choose

$$\alpha = pdq + \frac{1}{2}\theta d\theta.$$

The connection can then be written locally as $\nabla = d - i\alpha$. Its curvature is

$$F_\nabla = d(-i\alpha) = i\omega.$$

The prequantum operators associated with classical observables $f_X(Y) = f([X, Y])$ are defined by

$$\widehat{f_X} = -i\nabla_X + f_X. \tag{1}$$

These operators satisfy the correspondence principle

$$[\widehat{f_X}, \widehat{f_Y}] = -i\widehat{\{f_X, f_Y\}},$$

where $\{\cdot, \cdot\}$ denotes the super-Poisson bracket induced by ω.

In the previous, f_X is the Hamiltonian function on the coadjoint superorbit corresponding to the Lie superalgebra element X and $\widehat{f_X}$ denotes the quantum operator obtained by geometric quantization of the classical observable f_X. In the prequantization step ∇_X is the covariant derivative along the Hamiltonian vector field generated by f_X. Hence the notation (1) distinguishes three different objects:

- f a linear functional in the dual Lie superalgebra, and
- f_X the corresponding classical observable on the coadjoint superorbit, and
- $\widehat{f_X}$ its quantized operator acting on the quantum Hilbert superspace.

At this stage the quantum space is still "too large," since it depends on all phase-space variables simultaneously. A further reduction is therefore required.

8.2 Polarization

For the Heisenberg superalgebra $\mathfrak{h}^{1|1}$, the coadjoint superorbit is the superspace $\mathcal{O}_f \simeq \mathbb{R}^{2|1}$ with coordinates (q, p, θ) and supersymplectic form

$$\omega = dq \wedge dp + \frac{1}{2}\, d\theta \wedge d\theta.$$

The second step of geometric quantization consists in choosing a polarization compatible with this supersymplectic structure. A polarization is a maximal isotropic integrable distribution $\mathcal{P} \subset T_{\mathbb{C}}\mathcal{O}_f$ such that $\omega|_{\mathcal{P}} = 0$.

Choice of polarization

In the case $\mathfrak{h}^{1|1}$, a standard (complex) polarization is generated by:

$$\frac{\partial}{\partial p} \qquad \text{and} \qquad \frac{\partial}{\partial \bar{\theta}},$$

where the fermionic variable is complexified as

$$\theta \;\longrightarrow\; (\theta, \bar{\theta}).$$

Equivalently, one introduces creation and annihilation variables.

Bosonic creation and annihilation operators

The bosonic variables are combined into

$$a = \frac{1}{\sqrt{2}}(q + ip), \qquad a^{\dagger} = \frac{1}{\sqrt{2}}(q - ip),$$

satisfying $[a, a^{\dagger}] = 1$. In this polarization, a acts as an annihilation operator while $a^{\dagger}$ acts as a creation operator. Thus the polarized wavefunctions depend only on q (or equivalently $a^{\dagger}$-type variables).

Fermionic creation and annihilation operators

In the fermionic sector, there is a single fermionic degree of freedom. One introduces:

$$\psi = \theta, \qquad \psi^{\dagger} = \frac{\partial}{\partial \theta}.$$

These satisfy the Clifford relation $\{\psi, \psi^{\dagger}\} = 1$. In this polarization, $\psi^{\dagger}$ acts as fermionic creation and ψ acts as fermionic annihilation. Thus, the wavefunctions depend only on the fermionic creation coordinate.

Polarization condition

The polarization condition requires physical wavefunctions s to satisfy

$$\nabla_X s = 0, \qquad X \in \mathcal{P}.$$

Let $s = s(q, p, \theta)$ be a prequantum wavefunction. We choose the polarization

$$\mathcal{P} = \operatorname{span}\left\{ \frac{\partial}{\partial p}, \frac{\partial}{\partial \theta} \right\}.$$

The polarization condition requires

$$\nabla_X s = 0, \qquad X \in \mathcal{P}.$$

We now solve these equations explicitly.

Step 1: Bosonic polarization equation Take $X = \frac{\partial}{\partial p}$. Since $\alpha\left(\frac{\partial}{\partial p}\right) = 0$, we obtain $\nabla_{\partial_p} = \frac{\partial}{\partial p}$. Thus the polarization equation becomes $\frac{\partial s}{\partial p} = 0$. Hence the polarized wavefunction is independent of p and we have $s = s(q, \theta)$.

Step 2: Fermionic polarization equation Now take $X = \frac{\partial}{\partial \theta}$. Since $\alpha = p\,dq + \frac{1}{2}\,\theta\,d\theta$, we compute $\alpha\left(\frac{\partial}{\partial \theta}\right) = \frac{1}{2}\,\theta$. Therefore

$$\nabla_{\partial_\theta} = \frac{\partial}{\partial \theta} - \frac{i}{2}\theta.$$

The polarization equation becomes

$$\left(\frac{\partial}{\partial \theta} - \frac{i}{2}\theta \right) s = 0.$$

We now solve this equation. Since $\theta^2 = 0$, a general superfunction has the form

$$s(q, \theta) = u(q) + \theta\, v(q).$$

Applying the operator gives

$$\frac{\partial s}{\partial \theta} = v(q) \quad \text{and} \quad \theta s = \theta u(q),$$

because $\theta^2 = 0$. Hence

$$\nabla_{\partial_\theta} s = \left(\frac{\partial}{\partial \theta} - \frac{i}{2}\theta \right) s = v(q) - \frac{i}{2}\theta u(q).$$

For this expression to vanish identically, both coefficients must vanish:

$$v(q) = 0, \qquad u(q) = 0.$$

Thus the polarized sections reduce to $s(q,\theta)=0$. In this case, the polarization annihilates all states. It is why, at this stage one must be careful with the fermionic polarization. Imposing directly the condition $\nabla_{\partial_\theta} s = 0$ would force all coefficients of the superfunction to vanish, yielding only the trivial solution.

To obtain a nontrivial quantum space, one introduces a complex fermionic polarization. The odd variable is decomposed into creation and annihilation coordinates $\theta, \bar{\theta}$, and one imposes the polarization condition only along the antiholomorphic direction:

$$\nabla_{\partial_{\bar{\theta}}} s = 0.$$

The polarized superwavefunctions therefore depend only on the holomorphic fermionic coordinate θ and we obtain $s = s(q,\theta)$. Consequently, the fermionic sector is naturally identified with the exterior algebra $\Lambda^\bullet(\mathbb{C})$, generated by the fermionic creation operator θ.

8.3 Quantum Hilbert Space

The polarization equations eliminate the momentum variable p and the fermionic antiholomorphic direction. The physical wavefunctions therefore depend on the bosonic configuration variable together with the holomorphic fermionic coordinate:

$$s(q,\theta) = u(q) + \theta\, v(q).$$

Equivalently, the polarized states take values in the exterior algebra $\Lambda^\bullet(\mathbb{C})$. As a consequence, polarized sections reduce the prequantum space to the physical Hilbert superspace

$$\mathcal{H}_{SF} \simeq L^2(\mathbb{R}) \otimes \Lambda^\bullet(\mathbb{C}),$$

which can be interpreted as:

(*i*) a bosonic Schrödinger representation in the variable q,

(*ii*) a fermionic Fock representation generated by θ.

In the notation $\mathcal{H}_{SF}$, "SF" means Schrödinger–Fock. Thus, the polarization selects precisely the Schrödinger-Fock realization of the representation, where half of the bosonic and fermionic degrees of freedom are eliminated in accordance with the supersymplectic structure. We decompose the space $\Lambda^\bullet(\mathbb{C})$ as follow:

$$\Lambda^\bullet(\mathbb{C}) = \Lambda^0(\mathbb{C}) \oplus \Lambda^1(\mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}\,\theta.$$

A general fermionic state in $\mathcal{H}_{SF}$ can be written as

$$\Psi(x,\theta) = \psi_0(x) + \theta\,\psi_1(x), \qquad \psi_0, \psi_1 \in L^2(\mathbb{R}).$$

and the $\Lambda^\bullet(\mathbb{C})$ is a two-dimensional fermionic Fock space where the vacuum state 1 corresponds to no fermion and the state θ corresponds to one fermionic excitation.

Thus the full Hilbert superspace decomposes as

$$\mathcal{H}_{SF} = L^2(\mathbb{R}) \oplus \theta\, L^2(\mathbb{R}).$$

8.4 The irreducible Schrödinger-Fock Representation

We present the bosonic part, i.e, Schrödinger representation and the fermionic part, i.e, Fock representation. Together they form the representation of the Heisenberg supergroup.

8.4.1 Bosonic Schrödinger Sector

The factor $L^2(\mathbb{R})$ corresponds to the ordinary Schrödinger representation. Wavefunctions depend on the bosonic configuration variables q. Quantization of the relations

$$[q, p] = i\hbar$$

yields the operators

$$q \mapsto \hat{q} = q, \qquad p \mapsto \hat{p} = -i\hbar \frac{\partial}{\partial q}$$

and the quantized operators act as

$$\hat{q}\psi(q) = q\psi(q) \quad \text{and} \quad \hat{p}\psi(q) = -i\hbar \frac{\partial}{\partial q}\psi(q).$$

These satisfy the canonical commutation relations $[\hat{q}, \hat{p}] = i\hbar$ which are the fundamental Heisenberg commutation relations of quantum mechanics [1, 4]. Geometrically, this construction gives precisely the Schrödinger representation of the Heisenberg group, acting on the Hilbert space $L^2(\mathbb{R})$.

8.4.2 Fermionic Fock Sector

The exterior algebra $\Lambda^\bullet(\mathbb{C})$ provides the fermionic Fock space. A generic fermionic state takes the form $\Psi = \psi_0 + \theta\psi_1$. There is a single fermionic generator θ and upon geometric quantization, this fermionic variable becomes an operator acting on the fermionic Fock space $\Lambda^\bullet(\mathbb{C})$. The fermionic creation operator is given by exterior multiplication:

$$\hat{\theta}^\dagger(\Psi) = \theta \wedge \Psi.$$

In components this reads:

$$\hat{\theta}^\dagger\big(\psi_0(x) + \theta\psi_1(x)\big) = \theta\, \psi_0(x) \quad \text{since} \quad \theta^2 = 0.$$

The fermionic annihilation operator is given by interior contraction (Berezin differentiation): $\hat{\theta} = \frac{\partial}{\partial\theta}$. Its action is

$$\frac{\partial}{\partial\theta}\big(\psi_0(x) + \theta\psi_1(x)\big) = \psi_1(x).$$

These operators $\hat{\theta}$ and $\theta^\dagger$ satisfy the fermionic anticommutation relation $\{\hat{\theta}, \hat{\theta}^\dagger\} = 1$. Thus the fermionic sector realizes a Clifford algebra representation [7,9]. Geometrically, this construction realizes the irreducible Clifford module associated with the fermionic sector of the Heisenberg supergroup, acting on the exterior algebra $\Lambda^\bullet(\mathbb{C})$. Such fermionic Fock representations naturally arise in the geometric quantization of supersymmetric systems and in the representation theory of Lie supergroups [6,10,11].

9 Generalization to the Heisenberg Superalgebra $\mathfrak{h}^{m|n}$

The case of $\mathfrak{h}^{1|1}$ extends naturally to the general Heisenberg superalgebra

$$\mathfrak{h}^{m|n} = \mathrm{span}\{q_i, p_i, \theta_a, z\},$$

where: $1 \leq i \leq m, \qquad 1 \leq a \leq n$. The generators q_i, p_i, z are even (bosonic), while θ_a are odd (fermionic). The nontrivial supercommutation relations are

$$[q_i, p_j] = \delta_{ij} z \quad \text{and} \quad \{\theta_a, \theta_b\} = B_{ab} z,$$

where B_{ab} is a nondegenerate symmetric bilinear form on the fermionic sector. The center is generated by the even element z, i.e., $[z, \cdot] = 0$.

9.1 Coadjoint Superorbit

Let $f \in (\mathfrak{h}^{m|n})^*$ be a linear functional satisfying $f(z) = \hbar \neq 0$. The associated coadjoint superorbit

$$\mathcal{O}_f = \mathrm{Ad}^*(\mathcal{H}^{m|n}) \cdot f$$

is a supersymplectic supermanifold of dimension $(2m|n)$. Its supersymplectic form is the Kirillov–Kostant–Souriau form

$$\omega_f(X, Y) = f([X, Y]).$$

In local coordinates (q_i, p_i, θ_a), the supersymplectic form takes the form

$$\omega = \sum_{i=1}^{m} dq_i \wedge dp_i + \frac{1}{2} \sum_{a,b} B_{ab} \, d\theta_a \wedge d\theta_b.$$

9.2 Prequantization

The prequantum super line bundle

$$L \to \mathcal{O}_f$$

is equipped with a connection $\nabla = d - i\alpha$, whose curvature satisfies $F_\nabla = i\omega$.

A local symplectic potential is given by

$$\alpha = \sum_{i=1}^{m} p_i \, dq_i + \frac{1}{2} \sum_{a,b} B_{ab} \theta_a \, d\theta_b.$$

9.3 Polarization

Choose the standard complex polarization generated by

$$\frac{\partial}{\partial p_i}, \qquad \frac{\partial}{\partial \bar{\theta}_a}.$$

The polarization equations

$$\nabla_X s = 0, \qquad X \in \mathcal{P},$$

imply that polarized sections depend only on the bosonic configuration variables q_i and the fermionic creation coordinates [11,12].

The bosonic variables are combined into creation and annihilation operators

$$a_i = \frac{1}{\sqrt{2}}(q_i + ip_i), \qquad a_i^\dagger = \frac{1}{\sqrt{2}}(q_i - ip_i),$$

satisfying

$$[a_i, a_j^\dagger] = \delta_{ij}.$$

Similarly, the fermionic operators satisfy

$$\{\theta_a, \theta_b^\dagger\} = \delta_{ab}.$$

9.4 Quantum Hilbert Superspace

After polarization, the quantum Hilbert superspace becomes

$$\mathcal{H} \simeq L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n).$$

The first factor corresponds to the bosonic Schrödinger representation, while the second factor is the fermionic Fock space. A generic superwavefunction takes the form

$$\Psi(q, \theta) = \sum_I \psi_I(q) \theta^I,$$

where I runs over ordered multi-indices, θ^I denotes exterior monomials and $\psi_I(q) \in L^2(\mathbb{R}^m)$.

Since the Grassmann variables anticommute, $\theta_a \theta_b = -\theta_b \theta_a$, the fermionic sector is finite-dimensional: $\dim \Lambda^\bullet(\mathbb{C}^n) = 2^n$.

9.5 The irreducible Schrödinger-Fock Representation

The quantized operators act as:

$$\hat{q}_i = q_i, \qquad \hat{p}_i = -i\hbar\frac{\partial}{\partial q_i},$$

together with fermionic creation and annihilation operators acting on $\Lambda^\bullet(\mathbb{C}^n)$. Thus, the geometric quantization of the coadjoint superorbit naturally produces the Schrödinger-Fock representation of the Heisenberg supergroup [8–11].

Theorem 1. *Let $\mathfrak{h}^{m|n}$ be the Heisenberg superalgebra with nontrivial central character $\hbar \neq 0$. Then geometric quantization of its coadjoint superorbit produces an irreducible Schrödinger-Fock representation $L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n)$.*

Proof. The nontrivial central character $f(z) = \hbar \neq 0$ implies that the Kirillov-Kostant-Souriau form on the coadjoint superorbit is nondegenerate. Hence the orbit becomes a supersymplectic supermanifold of dimension $(2m|n)$. The prequantization procedure constructs a super line bundle whose connection curvature equals the supersymplectic form. Choosing the standard polarization eliminates half of the bosonic variables and selects the fermionic creation sector.

The polarized sections therefore depend only on the configuration variables q_i and on exterior fermionic coordinates. The resulting Hilbert superspace is

$$L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n).$$

The bosonic operators satisfy the canonical commutation relations, while the fermionic operators satisfy the Clifford anticommutation relations. Together they generate the Schrödinger-Fock representation.

Irreducibility follows from the super version of the Stone-von Neumann theorem for Heisenberg supergroups. More precisely, the central element z acts in the representation by the scalar

$$\hat{z} = i\hbar\,\mathrm{Id}, \qquad \hbar \neq 0.$$

The bosonic operators $\hat{q}_i, \quad \hat{p}_i$ satisfy the canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij},$$

while the fermionic operators $\theta_a, \quad \theta_a^\dagger$ satisfy the canonical anticommutation relations

$$\{\theta_a, \theta_b^\dagger\} = \delta_{ab}.$$

The bosonic sector $L^2(\mathbb{R}^m)$ is irreducible by the classical Stone-von Neumann theorem, which states that every irreducible unitary representation of the Heisenberg group with fixed nontrivial central character is unitarily equivalent to the Schrödinger representation [1, 4].

The fermionic sector $\Lambda^\bullet(\mathbb{C}^n)$ is an irreducible module over the Clifford algebra generated by the fermionic creation and annihilation operators. Indeed, repeated application of the creation operators $\theta_a^\dagger$ to the vacuum vector generates the entire exterior algebra:

$$\Lambda^\bullet(\mathbb{C}^n) = \mathrm{span}\{\theta_{a_1}^\dagger \cdots \theta_{a_k}^\dagger |0\rangle\}.$$

Since neither the bosonic nor the fermionic sectors admit nontrivial invariant subspaces compatible with the action of the corresponding operator algebras, the tensor product representation

$$L^2(\mathbb{R}^m) \otimes \Lambda^\bullet(\mathbb{C}^n)$$

is irreducible as a representation of the Heisenberg supergroup.

Consequently, geometric quantization produces, up to unitary equivalence, the unique irreducible Schrödinger-Fock representation associated with the nontrivial central character $\hbar \neq 0$. This result constitutes the supersymmetric analogue of the classical Stone-von Neumann theorem.

□

Theorem 1 should therefore be understood as a geometric synthesis of known representation-theoretic results rather than as a fundamentally new existence theorem. Its novelty lies in providing a unified and explicit geometric derivation, emphasizing the role of coadjoint superorbits, supersymplectic geometry, and polarization in the construction of Schrödinger–Fock representations.

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