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Differential Subordination Results of Multivalent Analytic Functions Defined by Borel Distribution Series

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Abstract

In this article, we introduce and study a certain family of functions which are analytic and multivalent in the open unit disk defined by the Borel distribution series. We determine some results related to inclusion relationship, argument estimate, integral representation and subordination property.

1. Introduction

Suppose that $\mathcal{A}_p$ be the family of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k , \qquad p \in \mathbb{N} = \{1,2,\dots\}, \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_1 = \mathcal{A}$.

Given two functions f and g which are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g\big(w(z)\big), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

For functions f given by (1.1) and $g \in \mathcal{A}_p$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k ,$$

the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k \, b_k \, z^k = (g * f)(z).$$

Elementary distributions such as the Poisson, Pascal, logarithmic, binomial, and beta negative binomial

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distributions have been partially investigated within Geometric Function Theory from a theoretical perspective (see for example [1,2,7,8,9,12]).

Very recently, Wanas and Khuttar [13] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=r)=\frac{(\lambda r)^{r-1}e^{-\lambda r}}{r!}, \quad r=1,2,3,\ldots$$

Wanas and Khuttar [13] introduced a series whose coefficients are probabilities of the Borel distribution (BD)

$$N_p(\lambda;z)=z^p+\sum_{k=p+1}^{\infty}\frac{(\lambda(k-p))^{k-p-1}e^{-\lambda(k-p)}}{(k-p)!}z^k,$$

where $0<\lambda\leq 1$.

Now, we define a linear operator $D(p,\lambda)f:\mathcal{A}_p\to\mathcal{A}_p$ defined by the convolution

$$D(p,\lambda)f(z)=N_p(\lambda;z)*f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{\left(\lambda(k-p)\right)^{k-p-1}e^{-\lambda(k-p)}}{(k-p)!}a_kz^k, \tag{1.2}$$

where $a_k\geq 0$, $0<\lambda\leq 1$ and $z\in U$. It is easy to deduce from (1.2) that

$$z\left(D(p,\lambda)f(z)\right)'=\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}D(p,\lambda+1)f(z)$$

$$+\left(p-\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\right)D(p,\lambda)f(z), \tag{1.3}$$

Let T be the family of functions h of the form:

$$h(z)=1+\sum_{k=1}^{\infty}h_kz^k,$$

which are analytic and convex univalent in U and satisfy the condition:

$$Re\{h(z)\}>0, \quad (z\in U).$$

We will require the following lemmas to prove our main results.

Lemma 1.1 [4]. *Let $u,v\in\mathbb{C}$ and suppose that ψ is convex and univalent in U with $\psi(0)=1$ and $Re\{u\psi(z)+v\}>0,(z\in U)$. If q is analytic in U with $q(0)=1$, then the subordination*

$$q(z)+\frac{zq'(z)}{uq(z)+v}\prec\psi(z)$$

implies that $q(z)\prec\psi(z)$.

Lemma 1.2 [5]. *Let h be convex univalent in U and $\mathcal{T}$ be analytic in U with $Re\{\mathcal{T}(z)\}\geq 0$, $(z\in U)$. If q is analytic in U and $q(0)=h(0)$, then the subordination*

$$q(z)+\mathcal{T}(z)zq'(z)\prec h(z)$$

implies that $q(z)\prec h(z)$.

Lemma 1.3 [3]**.** *Let q be analytic in U with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$. If there exists two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2}b_1 = arg\big(q(z_1)\big) < arg\big(q(z)\big) < arg\big(q(z_2)\big) = \frac{\pi}{2}b_2\,,$$

for some b_1 and b_2 $(b_1>0, b_2>0)$ and for all $z(|z|<|z_1|=|z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1+b_2}{2}\right)m \qquad and \qquad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1+b_2}{2}\right)m\,,$$

where

$$m \geq \frac{1-|\varepsilon|}{1+|\varepsilon|} \quad and \quad \varepsilon = i\,tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Lemma 1.4 [10]**.** *The function*

$$(1-z)^{\eta} \equiv exp(log(1-z)), \ \ (\eta\neq 0)$$

is univalent if and only if η is either in the closed disk $|\eta-1|\leq 1$ or in the closed disk $|\eta+1|\leq 1$.

Lemma 1.5 [6]**.** *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi\big(q(z)\big)$ and $h(z)=\theta\big(q(z)\big)+Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in U,*

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for $z\in U$.*

If G is analytic in U, with $G(0)=q(0), G(U)\subset D$ and

$$\theta\big(G(z)\big)+zG'(z)\phi\big(G(z)\big) \prec \theta\big(q(z)\big)+zq'(z)\phi\big(q(z)\big),$$

then $G\prec q$ and q is the best dominant.

2. Main Results

Definition 2.1. A function $f\in\mathcal{A}_p$ is said to be in the family $W(\lambda,\gamma,p;h)$ if it satisfies the following differential subordination condition:

$$\frac{1}{p-\gamma}\left(\frac{z\big(D(p,\lambda)f(z)\big)'}{D(p,\lambda)f(z)}-\gamma\right) \prec h(z),$$

where $0<\lambda\leq 1, p\in\mathbb{N}, 0\leq\gamma<p$ and $h\in T$.

Theorem 2.1. *Let* $Re\left\{(p-\gamma)h(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right\}>0$. *Then*

$$W(\lambda+1,\gamma,p;h)\subset W(\lambda,\gamma,p;h).$$

Proof. Let $f\in W(\lambda+1,\gamma,p;h)$ and put

$$q(z)=\frac{1}{p-\gamma}\left(\frac{z\big(D(p,\lambda)f(z)\big)'}{D(p,\lambda)f(z)}-\gamma\right). \tag{2.1}$$

Then q is analytic in U with $q(0)=1$. According to (2.1) and using the relation (1.4), we obtain

$$\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\frac{D(p,\lambda+1)f(z)}{D(p,\lambda)f(z)}=(p-\gamma)q(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p. \qquad (2.2)$$

By logarithmically differentiating both sides of (2.2) with respect to z and multiplying by z, we get

$$q(z)+\frac{zq'(z)}{(p-\gamma)q(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p}=\frac{1}{p-\gamma}\left(\frac{z\left(D(p,\lambda+1)f(z)\right)'}{D(p,\lambda+1)f(z)}-\gamma\right)\prec h(z). \qquad (2.3)$$

Since $Re\left\{(p-\gamma)h(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right\}>0$, applying Lemma 1.1 to the subordination (2.3), yields $q(z)\prec h(z)$, which implies $f\in W(\lambda,\gamma,p;h)$.

Theorem 2.2. *Let* $f\in\mathcal{A}_p$, $0<a_1,a_2\leq 1$ *and* $0\leq\gamma<p$. *If*

$$-\frac{\pi}{2}a_1<arg\left(\frac{z\left(D(p,\lambda+1)f(z)\right)'}{D(p,\lambda+1)g(z)}-\gamma\right)<\frac{\pi}{2}a_2,$$

for some $g\in W\left(\lambda+1,\gamma,p;\frac{1+AZ}{1+Bz}\right)$, $(-1\leq B<A\leq 1)$, *then*

$$-\frac{\pi}{2}b_1<arg\left(\frac{z\left(D(p,\lambda)f(z)\right)'}{D(p,\lambda)g(z)}-\gamma\right)<\frac{\pi}{2}b_2,$$

where b_1 *and* b_2 $(0<b_1,b_2\leq 1)$ *are the solutions of the equations:*

$$a_1=\begin{cases}b_1+\frac{2}{\pi}tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B\neq-1\\ b_1, & B=-1\end{cases} \qquad (2.4)$$

and

$$a_2=\begin{cases}b_2+\frac{2}{\pi}tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B\neq-1\\ b_2, & B=-1\end{cases} \qquad (2.5)$$

with

$$\varepsilon=i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right)\quad and\quad t=\frac{2}{\pi}sin^{-1}\left(\frac{(A-B)(p-\gamma)}{\left(\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)(1-B^2)+(p-\gamma)(1-AB)}\right). \qquad (2.6)$$

Proof. Define the function G by

$$G(z)=\frac{1}{p-\tau}\left(\frac{z\left(D(p,\lambda)f(z)\right)'}{D(p,\lambda)g(z)}-\tau\right), \qquad (2.7)$$

where $g\in W\left(\lambda+1,\gamma,p;\frac{1+AZ}{1+Bz}\right)$, $(-1\leq B<A\leq 1)$ and $0\leq\tau<p$.

Then G is analytic in U with $G(0)=1$. Therefore, by making use of (1.3) and (2.7), we obtain

$$\big((p-\tau)G(z)+\tau\big)D(p,\lambda)g(z) = \frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}D(p,\lambda+1)f(z)+\left(p-\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\right)D(p,\lambda)f(z).$$

Differentiating above relation with respect to z and multiplying by z, we get

$$\begin{aligned}\big((p-\tau)G(z)+\tau\big)z\big(D(p,\lambda)g(z)\big)'+(p-\tau)zG'(z)D(p,\lambda)g(z) &= \frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\, z\big(D(p,\lambda+1)f(z)\big)' \\ &\quad+\left(p-\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\right)z\big(D(p,\lambda)f(z)\big)'. \end{aligned} \tag{2.8}$$

Suppose that

$$H(z)=\frac{1}{p-\gamma}\left(\frac{z\big(D(p,\lambda)g(z)\big)'}{D(p,\lambda)g(z)}-\gamma\right).$$

Using (1.3) again, we have

$$\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}\frac{D(p,\lambda+1)g(z)}{D(p,\lambda)g(z)}=(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p. \tag{2.9}$$

From (2.8) and (2.9), we easily get

$$G(z)+\frac{zG'(z)}{(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p}=\frac{1}{p-\tau}\left(\frac{z\big(D(p,\lambda+1)f(z)\big)'}{D(p,\lambda)g(z)}-\tau\right). \tag{2.10}$$

Notice that from Theorem 2.1, $g\in W\left(\lambda+1,\gamma,p;\frac{1+AZ}{1+Bz}\right)$ implies $g\in W\left(\lambda,\gamma,p;\frac{1+AZ}{1+Bz}\right)$. Thus,

$$H(z)\prec\frac{1+AZ}{1+Bz}\quad(-1\le B<A\le 1).$$

By using the result of Silverman and Silvia [11], we have

$$\left|H(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}\quad(B\neq-1,\ z\in U) \tag{2.11}$$

and

$$Re\{H(z)\}>\frac{1-A}{2}\quad(B=-1,\ z\in U). \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$\left|(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p-\frac{\left(\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)(1-B^2)+(p-\gamma)(1-AB)}{1-B^2}\right| <\frac{(A-B)(p-\gamma)}{1-B^2},$$

when $B\neq-1,\ z\in U$

and

$$Re\left\{(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right\}$$
$$>\frac{(1-A)(p-\gamma)}{2}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\,,$$

when $B=-1,\ \ z\in U$.

Putting

$$(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p=\rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(p-\gamma)}{\left(\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)(1-B^2)+(p-\gamma)(1-AB)}<\phi$$
$$<\frac{(A-B)(p-\gamma)}{\left(\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)(1-B^2)+(p-\gamma)(1-AB)},\qquad (B\neq -1)$$

and $-1<\phi<1,\ (B=-1)$, then

$$\frac{(1-A)(p-\gamma)}{1-B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p<\rho$$
$$<\frac{(1+A)(p-\gamma)}{1+B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p,\quad (B\neq -1)$$

and

$$\frac{(1-A)(p-\gamma)}{1-B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p<\rho<\infty,\ \ (B=-1).$$

An application of Lemma 1.2 with $\mathcal{T}(z)=\frac{1}{(p-\gamma)H(z)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p}$, yields $G(z)\prec h(z)$.

If there exist two points $z_1, z_2\in U$ such that

$$-\frac{\pi}{2}b_1=arg\big(G(z_1)\big)<arg\big(G(z)\big)<arg\big(G(z_2)\big)=\frac{\pi}{2}b_2\,,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)}=-\frac{mi}{2}(b_1+b_2)\qquad\text{and}\qquad \frac{z_2G'(z_2)}{G(z_2)}=\frac{mi}{2}(b_1+b_2)\,,$$

where

$$m\geq\frac{1-|\varepsilon|}{1+|\varepsilon|}\quad\text{and}\quad \varepsilon=i\tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B\neq -1$, we obtain

$$\begin{aligned} arg\left(\frac{1}{p-\tau}\left(\frac{z_1\left(D(p,\lambda+1)f(z_1)\right)'}{D(p,\lambda+1)g(z_1)}-\tau\right)\right) &= arg\left(G(z_1)+\frac{z_1G'(z_1)}{(p-\gamma)H(z_1)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p}\right) \\ &= arg\left(G(z_1)\right)+arg\left(1+\frac{z_1G'(z_1)}{\left[(p-\gamma)H(z_1)+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right]G(z_1)}\right) \\ &= -\frac{\pi}{2}b_1+arg\left(1-\frac{mi}{2\rho}(b_1+b_2)e^{-i\frac{\pi}{2}\phi}\right) \\ &= -\frac{\pi}{2}b_1+arg\left(1-\frac{m}{2\rho}(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)+\frac{mi}{2\rho}(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)\right) \\ &\le -\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{m(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho+m(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)}\right) \\ &\le -\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right) \\ &= -\frac{\pi}{2}a_1, \end{aligned}$$

where a_1 and t are given by (2.4) and (2.6), respectively.

Also,

$$\begin{aligned} & arg\left(\frac{1}{p-\tau}\left(\frac{z_2\left(D(p,\lambda+1)f(z_2)\right)'}{D(p,\lambda+1)g(z_2)}-\tau\right)\right) \\ &\ge \frac{\pi}{2}b_2+\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-\gamma)}{1+B}+\frac{p-k}{1-(k-p)^{k-p-1}e^{p-k}}+\gamma-p\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right) \\ &= \frac{\pi}{2}a_2, \end{aligned}$$

where a_2 and t are given by (2.5) and (2.6), respectively.

Similarly, for the case $B=-1$, we have

$$arg\left(\frac{1}{p-\tau}\left(\frac{z_1\left(D(p,\lambda+1)f(z_1)\right)'}{D(p,\lambda+1)g(z_1)}-\tau\right)\right)\le -\frac{\pi}{2}b_1$$

and

$$arg\left(\frac{1}{p-\tau}\left(\frac{z_2\left(D(p,\lambda+1)f(z_2)\right)'}{D(p,\lambda+1)g(z_2)}-\tau\right)\right)\ge \frac{\pi}{2}b_2.$$

The above two cases contradict the assumptions. Consequently, the proof of the theorem is complete.

In the following theorem, we find integral representation of the class $W(\lambda,\gamma,p;h)$.

Theorem 2.3. *Let* $f\in W(\lambda,\gamma,p;h)$. *Then*

$$D(p,\lambda)f(z) = z^p . exp\left[(p-\gamma)\int_0^z \frac{h\big(w(s)\big)-1}{s}ds\right],$$

where w *is analytic in* U *with* $w(0) = 0$ *and* $|w(z)| < 1$, $(z \in U)$.

Proof. Assume that $f \in W(\lambda,\gamma,p;h)$. It is easy to see that subordination condition (1.5) can be written as follows

$$\frac{z\big(D(p,\lambda)f(z)\big)'}{D(p,\lambda)f(z)} = (p-\gamma)h\big(w(z)\big)+\gamma, \tag{2.13}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$.

From (2.13), we find that

$$\frac{\big(D(p,\lambda)f(z)\big)'}{D(p,\lambda)f(z)} - \frac{p}{z} = (p-\gamma)\frac{h\big(w(z)\big)-1}{z}, \tag{2.14}$$

After integrating both sides of (2.14), we have

$$\log\left(\frac{D(p,\lambda)f(z)}{z^p}\right) = (p-\gamma)\int_0^z \frac{h\big(w(s)\big)-1}{s}ds. \tag{2.15}$$

Therefore, from (2.15), we obtain the required result.

Theorem 2.4. *Let* $1 < \beta < 2$ *and* $\eta \in \mathbb{R} \setminus \{0\}$ *such that either* $\left|\frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}+1\right| \le 1$ *or* $\left|\frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}-1\right| \le 1$. *If* $f \in \mathcal{A}_p$ *satisfies the condition*

$$Re\left\{1+\frac{D(p,\lambda+1)f(z)}{D(p,\lambda)f(z)}\right\} > 2-\beta+\frac{\big(1-(k-p)^{k-p-1}e^{p-k}\big)(1-p)}{p-k}, \tag{2.16}$$

then,

$$\big(zD(p,\lambda)f(z)\big)^{\eta} \prec (1-z)^{-\frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}}$$

and $(1-z)^{-\frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}}$ *is the best dominant.*

Proof. Define the function k by

$$k(z) = \big(zD(p,\lambda)f(z)\big)^{\eta}. \tag{2.17}$$

Differentiating (2.17) with respect to z logarithmically and using (1.3), we obtain

$$\frac{zk'(z)}{k(z)} = \frac{\eta(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}\frac{D(p,\lambda+1)f(z)}{D(p,\lambda)f(z)} - \frac{\eta(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}+p.$$

Now, in view of the condition (2.16), we have the following subordination

$$1+\frac{\big(1-(k-p)^{k-p-1}e^{p-k}\big)zk'(z)}{\eta(p-k)k(z)} \prec \frac{1+(2\beta-3)z}{1-z}.$$

Assume that

$$\theta(w) = 1\,, \quad \phi(w) = \frac{\left(1-(k-p)^{k-p-1}e^{p-k}\right)}{\eta(p-k)w}$$

and

$$q(z) = (1-z)^{-\frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}},$$

then by making use of Lemma 1.4, we know that q is univalent in U. It now follows that

$$Q(z) = zq'(z)\phi\big(q(z)\big) = \frac{2(\beta-1)z}{1-z}$$

and

$$h(z) = \theta\big(q(z)\big) + Q(z) = \frac{1+(2\beta-3)z}{1-z}.$$

If we define the domain D by

$$q(U) = \left\{w\colon \left|w^{\frac{1}{\sigma}}-1\right| < \left|w^{\frac{1}{\sigma}}\right|, \sigma = \frac{2\eta(\beta-1)(p-k)}{1-(k-p)^{k-p-1}e^{p-k}}\right\} \subset D,$$

then, it is easy to check that the conditions of Lemma 1.5 hold true. Therefore, we get the desired result.

Conclusion

The purpose of this paper is to obtain some results related to inclusion relationship, argument estimate, integral representation and subordination property for a certain family of functions which are analytic and multivalent in the open unit disk defined by Borel distribution. Moreover, it is remarked that we can apply the results presented here to various families of analytic and meromorphic functions.

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