

# Landau's Theorem for Biharmonic Mappings Constructed from Holomorphic Functions

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## Abstract

This article establishes the existence of a Landau-type constant for biharmonic mappings derived from holomorphic functions in the unit disk  $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ , provides the insights into the geometric properties and behavior of these mappings.

## 1 Introduction

Biharmonic functions constitute an important class of solutions to fourth-order elliptic partial differential equations and naturally extend the theory of harmonic functions. Besides their theoretical significance in complex analysis, biharmonic mappings arise in a variety of applications, including elasticity theory, fluid mechanics, plate bending problems, and mathematical physics. In geometric function theory, they provide a broader framework for investigating geometric properties such as univalence, convexity, starlikeness, distortion and covering phenomena. Owing to these diverse applications, the study of biharmonic mappings has attracted considerable attention over the past two decades.

A complex-valued function  $F$  defined on a domain  $D \subset \mathbb{C}$  is called biharmonic if it satisfies the fourth-order partial differential equation

$$\Delta^2 F = 0,$$

where  $\Delta$  denotes the Laplacian operator. Equivalently, the Laplacian of  $F$  is harmonic in  $D$ . Every biharmonic mapping on a simply connected domain can be expressed as

$$F(\zeta) = |\zeta|^2 G(\zeta) + H(\zeta),$$

where  $G$  and  $H$  are harmonic mappings. This decomposition plays a central role in the analysis of biharmonic mappings and provides an effective tool for extending many classical results from holomorphic and harmonic function theory to the biharmonic setting.

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One of the fundamental problems in geometric function theory is the investigation of univalence and geometric behaviour of mappings satisfying suitable normalization and boundedness conditions. For biharmonic mappings, the Jacobian determinant serves as the principal criterion for local univalence and orientation preservation. In particular, a biharmonic mapping is sense-preserving whenever its Jacobian remains positive throughout the domain of definition. Consequently, estimates involving the Jacobian and its associated quantities have become indispensable in deriving coefficient estimates, distortion inequalities and covering results for biharmonic mappings.

Landau's theorem is one of the classical cornerstones of geometric function theory. It provides an explicit lower bound for the radius of univalence of bounded normalized holomorphic functions together with a covering theorem describing the size of the corresponding image domain. Since its original formulation, Landau's theorem has inspired numerous extensions and refinements for different classes of mappings. Such generalizations not only broaden the applicability of the theorem but also reveal deeper geometric properties of the underlying function classes.

The harmonic analogue of Landau's theorem has been extensively investigated by several authors. Using refined coefficient estimates, Dorff and Nowak [5] established improved versions of Landau's theorem for harmonic mappings under suitable normalization conditions. Subsequently, Grigoryan [8] obtained sharper Landau-type results for bounded harmonic mappings by developing precise coefficient estimates and improving the corresponding univalence radius. These investigations demonstrate that harmonic mappings exhibit geometric behaviour considerably richer than that of holomorphic functions while still admitting quantitative univalence criteria.

The extension of Landau's theorem to biharmonic mappings has also received significant attention. Abdulhadi and Abu Muhanna [1] initiated the study of Landau-type theorems for bounded biharmonic mappings by proving the existence of a Landau constant for this class. Their work established a foundation for subsequent investigations into the geometric theory of biharmonic mappings. Building upon these ideas, Liu [9,10] derived improved coefficient estimates for bounded harmonic and biharmonic mappings, obtained sharper Landau constants and verified the sharpness of several associated results. These developments have considerably strengthened the understanding of Landau-type phenomena in higher-order elliptic mappings.

Despite these advances, comparatively fewer studies have focused on biharmonic mappings generated directly from holomorphic functions. Such mappings inherit structural properties from their holomorphic components while simultaneously exhibiting the richer behaviour characteristic of biharmonic functions. This additional structure suggests that stronger coefficient estimates and improved geometric conclusions may be obtained by combining techniques from classical holomorphic function theory with the biharmonic framework. Consequently, the investigation of Landau-type results for biharmonic mappings generated from holomorphic functions remains a natural and worthwhile problem.

Motivated by the above developments, the present paper investigates a class of biharmonic mappings of the form  $F(\zeta) = |\zeta|^2 G(\zeta)$  generated from holomorphic functions. By exploiting the holomorphic representation together with suitable coefficient estimates, we establish Landau-type theorems that provide

explicit radii of univalence and corresponding covering results. The obtained estimates extend several existing results for biharmonic mappings and further illustrate the close relationship between holomorphic function theory and higher-order elliptic equations. It is expected that these results will contribute to the continuing development of geometric function theory for biharmonic mappings and stimulate further investigations in this direction.

## 2 Preliminaries

In this section, we recall some basic definitions and notation that will be used throughout the paper.

Let  $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$  denote the open unit disk. Unless otherwise specified, we write  $\zeta = re^{i\theta}$ , where  $r = |\zeta| < 1$  and  $\theta = \arg(\zeta)$ .

Let  $D \subset \mathbb{C}$  be a domain and let  $F : D \rightarrow \mathbb{C}$  be a four times continuously differentiable function. The function  $F$  is said to be biharmonic in  $D$  if it satisfies

$$\Delta^2 F = 0 \quad \text{in } D,$$

where the Laplacian operator is given by

$$\Delta = 4 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}}.$$

Equivalently,  $F$  is biharmonic if its Laplacian  $\Delta F$  is harmonic in  $D$ .

Every biharmonic mapping on a simply connected domain admits the representation

$$F(\zeta) = |\zeta|^2 G(\zeta) + H(\zeta),$$

where  $G$  and  $H$  are complex-valued harmonic functions.

Every harmonic mapping  $G$  in  $\mathbb{D}$  admits the canonical decomposition

$$G(\zeta) = h(\zeta) + \overline{g(\zeta)},$$

where  $h$  and  $g$  are holomorphic in  $\mathbb{D}$ . Throughout this paper,  $h'$  and  $g'$  denote the derivatives of  $h$  and  $g$ , respectively.

A biharmonic mapping  $F$  is said to be sense-preserving in  $\mathbb{D} \setminus \{0\}$  if

$$J_F(\zeta) > 0,$$

where the Jacobian determinant of  $F$  is

$$J_F(\zeta) = |F_\zeta(\zeta)|^2 - |F_{\bar{\zeta}}(\zeta)|^2.$$

For convenience, we introduce the quantities

$$\lambda_F(\zeta) = |F_\zeta(\zeta)| - |F_{\bar{\zeta}}(\zeta)|$$

and

$$\Lambda_F(\zeta) = |F_\zeta(\zeta)| + |F_{\bar{\zeta}}(\zeta)|.$$

Consequently,

$$J_F(\zeta) = \lambda_F(\zeta)\Lambda_F(\zeta).$$

We denote by  $\mathcal{A}$  the class of holomorphic functions in  $\mathbb{D}$  satisfying the normalization conditions  $f(0) = 0$ ,  $f'(0) = 1$ . Every function  $f \in \mathcal{A}$  has the Taylor series expansion

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n.$$

The following classical result, known as Landau's theorem, plays a fundamental role in geometric function theory and serves as the motivation for the present work (see [5]).

**Theorem 2.1** (Landau). *Suppose that  $f$  is holomorphic in  $\mathbb{D}$ , satisfies  $f(0) = 0$ ,  $f'(0) = 1$ , and  $|f(\zeta)| < M$  for all  $\zeta \in \mathbb{D}$ . Then  $f$  is univalent in the disk  $\mathbb{D}_{\rho_0}$ , where*

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}.$$

Moreover,  $f(\mathbb{D}_{\rho_0})$  contains the disk  $\mathbb{D}_{R_0}$ , where  $R_0 = M\rho_0^2$

### 3 Main Results

**Theorem 3.1.** *Let  $f \in \mathcal{A}$  and let  $G(\zeta) = f(\zeta) + \overline{f'(\zeta)} - 1$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{n+1}{2}$ , for all  $n \geq 2$ . Then there exists a constant  $\rho_1 \in (0, 1)$  such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_1$ . Specifically,  $\rho_1$  satisfies*

$$\frac{3}{2(1-\rho)^2} + \frac{1}{(1-\rho)} + \frac{1+\rho}{2(1-\rho)^3} + \frac{2(1+\rho)}{\rho(1-\rho)^3} + \frac{1}{\rho(1-\rho)^2} + \frac{2}{(1-\rho)^4} - \frac{3}{\rho} = 4$$

and  $F(\mathbb{D}_{\rho_1}) \subset \mathbb{D}_{R_1}$  with

$$R_1 = 2\rho_1^3 + \rho_1^2 - \frac{\rho_1^2(1+\rho_1)}{2(1-\rho_1)^3} - \frac{\rho_1^2(1+\rho_1)}{2(1-\rho_1)^2} - \frac{\rho_1}{2(1-\rho_1)}.$$

This result is not sharp.

*Proof.* Let  $\zeta_1, \zeta_2 \in \mathbb{D}$  with  $\zeta_1 \neq \zeta_2$  be such that  $|\zeta_1| < \rho$  and  $|\zeta_2| < \rho$ ,  $0 < \rho < 1$ . Then

$$F(\zeta_1) - F(\zeta_2) = \int_{[\zeta_1, \zeta_2]} F_\zeta(\zeta) d\zeta + F_{\bar{\zeta}} d\bar{\zeta} = \int_{[\zeta_1, \zeta_2]} (\bar{\zeta}G + r^2 \bar{h}') d\zeta + (\zeta G + r^2 g') d\bar{\zeta},$$

where  $[\zeta_1, \zeta_2]$  is the line segment from  $\zeta_1$  to  $\zeta_2$ . Then

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &= \left| h'(0) \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + z^2 d\bar{\zeta}) + \overline{g'(0)} \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + \bar{\zeta}^2 d\bar{\zeta}) \right. \\
 &\quad + \int_{[\zeta_1, \zeta_2]} [(h(\zeta) - h'(0)\zeta) + \overline{(g(\zeta) - g'(0)\zeta)}](\bar{\zeta} d\zeta + \zeta d\bar{\zeta}) \\
 &\quad \left. + \int_{[\zeta_1, \zeta_2]} r^2((h'(\zeta) - h'(0))d\zeta + (\overline{g'(\zeta) - g'(0)})d\bar{\zeta}) \right| \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n |a_n| \rho^{n-2} \right. \\
 &\quad \left. - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2} \right] \tag{A} \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{3}{2} \sum_{n=2}^{\infty} n \rho^{n-1} - \sum_{n=2}^{\infty} \rho^{n-1} - \frac{3}{2} \sum_{n=2}^{\infty} n^2 \rho^{n-2} \right. \\
 &\quad \left. - \sum_{n=2}^{\infty} n \rho^{n-2} - \frac{1}{2} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \frac{1}{2} \sum_{n=2}^{\infty} n^3 \rho^{n-2} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 4 - \frac{3}{2(1-\rho)^2} - \frac{1}{(1-\rho)} - \frac{1+\rho}{2(1-\rho)^3} \right. \\
 &\quad \left. - \frac{2(1+\rho)}{\rho(1-\rho)^2} - \frac{1}{\rho(1-\rho)^2} - \frac{2}{(1-\rho)^4} + \frac{3}{\rho} \right].
 \end{aligned}$$

Choose  $\rho_1$  so that

$$4 - \frac{3}{2(1-\rho_1)^2} + \frac{1}{(1-\rho_1)} + \frac{1+\rho_1}{2(1-\rho_1)^3} + \frac{2(1+\rho_1)}{\rho_1(1-\rho_1)^3} + \frac{1}{\rho_1(1-\rho_1)^2} + \frac{2}{(1-\rho_1)^4} - \frac{3}{\rho_1} = 0.$$

Then  $F$  will be univalent in the disk  $|\zeta| < \rho_1$ . Furthermore, if  $|\zeta| = \rho_1$ , then

$$\begin{aligned}
 |F(\zeta)| &= \rho_1^2 |G(\zeta)| \geq \rho_1^3 - \sum_{n=2}^{\infty} |a_n| \rho_1^n - \sum_{n=2}^{\infty} n |a_n| \rho_1^{n-1} \\
 &\geq 2\rho_1^3 - \frac{\rho_1^2(1+\rho_1)}{2(1-\rho_1)^3} - \frac{\rho_1^2(1+\rho_1)}{2(1-\rho_1)^2} - \frac{\rho_1}{2(1-\rho_1)} + \rho_1^2 = R_1.
 \end{aligned}$$

□

**Theorem 3.2.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f'(\zeta)} - 1$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{(2n+1)(n+1)}{6}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_2$ ,  $0 < \rho_2 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_2$ . Specifically,  $\rho_2$  satisfies

$$\begin{aligned}
 &\frac{13(1+\rho)}{6(1-\rho)^3} + \frac{7}{6(1-\rho)^2} + \frac{1}{3(1-\rho)} + \frac{4\rho}{3(1-\rho)^4} + \frac{1+\rho}{2\rho(1-\rho)^3} + \frac{1}{\rho(1-\rho)^2} \\
 &+ \frac{1}{6\rho(1-\rho)} + \frac{1+11\rho-8\rho-8\rho^2-\rho^3}{3\rho(1-\rho)^5} - \frac{2}{\rho} = 4,
 \end{aligned}$$

and  $F(\mathbb{D}_{\rho_2}) \subset \mathbb{D}_{R_2}$ , where

$$R_2 = \rho_2^2 + 2\rho_2^3 - \frac{\rho_2^3(1+\rho_2)}{3(1-\rho_2)^3} - \frac{\rho_2^3}{(1-\rho_2)^2} - \frac{\rho_2^3}{(1-\rho_2)} - \frac{5(1+\rho_2)\rho_2^2}{6(1-\rho_2)^3} - \frac{4\rho_2^3}{(1-\rho_2)^4} - \frac{(6+\rho_2)\rho_2^2}{6(1-\rho_2)^2}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.1, from A

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n |a_n| \rho^{n-2} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \sum_{n=2}^{\infty} \frac{(2n+1)(n+1)}{6} \rho^{n-1} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n \left( \frac{(2n+1)(n+1)}{6} \right) \rho^{n-2} - \sum_{n=2}^{\infty} n \left( \frac{(2n+1)(n+1)}{6} \right) \rho^{n-1} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n^2 \left( \frac{(2n+1)(n+1)}{6} \right) \rho^{n-2} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{7}{6} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \frac{7}{6} \sum_{n=2}^{\infty} n \rho^{n-1} - \frac{1}{3} \sum_{n=2}^{\infty} \rho^{n-1} \right. \\ &\quad \left. - \frac{1}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-1} - \frac{1}{2} \sum_{n=2}^{\infty} n^2 \rho^{n-2} - \sum_{n=2}^{\infty} n \rho^{n-2} - \frac{1}{6} \sum_{n=2}^{\infty} \rho^{n-1} - \frac{1}{3} \sum_{n=2}^{\infty} n^4 \rho^{n-2} \right], \end{aligned}$$

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 4 - \frac{13(1+\rho)}{6(1-\rho)^3} - \frac{7}{6(1-\rho)^2} - \frac{1}{3(1-\rho)} - \frac{4\rho}{3(1-\rho)^4} \right. \\ &\quad \left. - \frac{1+\rho}{2\rho(1-\rho)^3} - \frac{1}{\rho(1-\rho)^2} - \frac{1}{6\rho(1-\rho)} - \frac{1+11\rho-8\rho-8\rho^2-\rho^3}{3\rho(1-\rho)^5} + \frac{2}{\rho} \right]. \end{aligned}$$

Choose  $\rho_2$  so that

$$\begin{aligned} &\left[ 4 - \frac{13(1+\rho_2)}{6(1-\rho_2)^3} - \frac{7}{6(1-\rho_2)^2} - \frac{1}{3(1-\rho_2)} - \frac{4\rho_2}{3(1-\rho_2)^4} - \frac{1+\rho_2}{2\rho_2(1-\rho_2)^3} - \frac{1}{\rho_2(1-\rho_2)^2} \right. \\ &\quad \left. - \frac{1}{6\rho_2(1-\rho_2)} - \frac{1+11\rho_2-8\rho_2-8\rho_2^2-\rho_2^3}{3\rho_2(1-\rho_2)^5} + \frac{2}{\rho_2} \right] = 0, \\ &\frac{13(1+\rho_2)}{6(1-\rho_2)^3} + \frac{7}{6(1-\rho_2)^2} + \frac{1}{3(1-\rho_2)} + \frac{4\rho_2}{3(1-\rho_2)^4} + \frac{1+\rho_2}{2\rho_2(1-\rho_2)^3} + \frac{1}{\rho_2(1-\rho_2)^2} + \frac{1}{6\rho_2(1-\rho_2)} \\ &\quad + \frac{1+11\rho_2-8\rho_2-8\rho_2^2-\rho_2^3}{3\rho_2(1-\rho_2)^5} - \frac{2}{\rho_2} = 4. \end{aligned}$$

Then  $F$  will be univalent in  $|\zeta| < \rho_2$ . Furthermore, if  $|\zeta| = \rho_2$ , then

$$\begin{aligned} |F(\zeta)| &= \rho_2^2 |G(\zeta)| \geq \rho_2^2 \left[ \rho_2 - \sum_{n=2}^{\infty} |a_n| \rho_2^n - \sum_{n=2}^{\infty} n |a_n| \rho_2^{n-1} \right] \\ &\geq \rho_2^2 \left[ \rho_2 - \frac{1}{3} \sum_{n=2}^{\infty} n^2 \rho_2^n - \frac{1}{2} \sum_{n=2}^{\infty} n \rho_2^n - \frac{1}{6} \sum_{n=2}^{\infty} \rho_2^n - \frac{1}{3} \sum_{n=2}^{\infty} n^3 \rho_2^{n-1} \right. \\ &\quad \left. - \frac{1}{2} \sum_{n=2}^{\infty} n^2 \rho_2^{n-1} - \frac{1}{6} n \rho_2^{n-1} \right] \\ &\geq \left[ \rho_2^2 + 2\rho_2^3 - \frac{\rho_2^3(1+\rho_2)}{3(1-\rho_2)^3} - \frac{\rho_2^3}{(1-\rho_2)^2} - \frac{\rho_2^3}{(1-\rho_2)} - \frac{5(1+\rho_2)\rho_2^2}{6(1-\rho_2)^3} \right. \\ &\quad \left. - \frac{4\rho_2^3}{(1-\rho_2)^4} - \frac{(6+\rho_2)\rho_2^2}{6(1-\rho_2)^2} \right] = R_2. \end{aligned}$$

□

**Theorem 3.3.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f'(\zeta)} - 1$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{2n^2+1}{3}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_3$ ,  $0 < \rho_3 < 1$ , such that  $F(\zeta) = r^2G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_3$ . Specifically,  $\rho_3$  satisfies

$$\begin{aligned} \frac{2 + 22\rho_3 - 16\rho_3^2 + \rho_3^3}{3\rho_3(1-\rho_3)^5} + \frac{8(1+\rho_3)}{3(1-\rho_3)^4} + \frac{3(1+\rho_3)}{(1-\rho_3)^2} + \frac{2(1+\rho_3)}{3\rho_3(1-\rho_3)^3} + \frac{1+\rho_3}{3\rho_3(1-\rho_3)^2} \\ + \frac{2}{3(1-\rho_3)} - \frac{2}{\rho_3} = 4, \end{aligned}$$

and  $F(\mathbb{D}_{\rho_3}) \subset \mathbb{D}_{R_3}$ , where

$$R_3 = 2\rho_3^3 - \frac{\rho_3^3}{3(1-\rho_3)} - \frac{2\rho_3^2}{3(1-\rho_3)^2} - \frac{2(1+\rho_3)^2}{3(1-\rho_3)^3} - \frac{8\rho_3^3}{3(1-\rho_3)^4}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.1, from **A**

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n |a_n| \rho^{n-2} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} \left( \frac{2n^2+1}{3} \right) \rho^{n-1} - \sum_{n=2}^{\infty} n \left( \frac{2n^2+1}{3} \right) \rho^{n-2} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n \left( \frac{2n^2+1}{3} \right) \rho^{n-1} - \sum_{n=2}^{\infty} n^2 \left( \frac{2n^2+1}{3} \right) \rho^{n-2} \right], \\ |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{4}{3} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \frac{2}{3} \sum_{n=2}^{\infty} \rho^{n-1} - \frac{2}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-2} \right. \\ &\quad \left. - \frac{1}{3} \sum_{n=2}^{\infty} n \rho^{n-2} - \frac{2}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-1} - \frac{1}{3} \sum_{n=2}^{\infty} n \rho^{n-1} - \frac{2}{3} \sum_{n=2}^{\infty} n^4 \rho^{n-2} - \frac{1}{3} \sum_{n=2}^{\infty} n^2 \rho^{n-2}, \right] \end{aligned}$$

$$|F(\zeta_1) - F(\zeta_2)| \geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 4 - \frac{2 + 22\rho - 16\rho^2 - \rho^3}{3\rho(1-\rho)^5} - \frac{8(1+\rho)}{3(1-\rho)^4} \right. \\ \left. - \frac{3(1+\rho)}{(1-\rho)^2} - \frac{2(1+\rho)}{3\rho(1-\rho)^3} - \frac{1+\rho}{3\rho(1-\rho)^2} - \frac{2}{3(1-\rho)} + \frac{2}{\rho} \right].$$

Choose  $\rho_3$  so that

$$4 - \frac{2 + 22\rho_3 - 16\rho_3^2 - \rho_3^3}{3\rho_3(1-\rho_3)^5} - \frac{8(1+\rho_3)}{3(1-\rho_3)^4} - \frac{3(1+\rho_3)}{(1-\rho_3)^2} - \frac{2(1+\rho_3)}{3\rho_3(1-\rho_3)^3} - \frac{1+\rho_3}{3\rho_3(1-\rho_3)^2} \\ - \frac{2}{3(1-\rho_3)} + \frac{2}{\rho_3} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_3$ . Further, if  $|\zeta| = \rho_3$ , then

$$|F(\zeta)| = \rho_3^2 |G(\zeta)| \geq \rho_3^2 \left[ \rho_3 - \sum_{n=2}^{\infty} |a_n| \rho_3^n - \sum_{n=2}^{\infty} n |a_n| \rho_3^{n-1} \right] \\ \geq \rho_3^2 \left[ \rho_3 - \frac{2}{3} \sum_{n=2}^{\infty} n^2 \rho_3^n - \frac{1}{3} \sum_{n=2}^{\infty} \rho_3^n - \frac{2}{3} \sum_{n=2}^{\infty} n^3 \rho_3^{n-1} - \frac{1}{3} \sum_{n=2}^{\infty} n \rho_3^{n-1} \right] \\ \geq 2\rho_3^3 - \frac{\rho_3^3}{3(1-\rho_3)} - \frac{\rho_3^2}{3(1-\rho_3)^2} - \frac{\rho_3^2}{3(1-\rho_3)^2} - \frac{2(1+\rho_3)^2}{3(1-\rho_3)^3} - \frac{8\rho_3^3}{3(1-\rho_3)^4} = R_3.$$

□

**Theorem 3.4.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f'(\zeta)} - 1$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{M}{n(n-1)}$  for all  $n \geq 2$  and  $M > 0$ . Then there exists a constant  $\rho_4$ ,  $0 < \rho_4 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_4$ . Specifically,  $\rho_4$  satisfies

$$1 + M + 2M \log(1 - \rho) - \frac{M}{1 - \rho} = 0,$$

and  $F(\mathbb{D}_{\rho_4}) \subset \mathbb{D}_{R_4}$ , where

$$R_4 = (1 - M)\rho_4^3 + M\rho_4 \log(1 - \rho_4).$$

This result is not sharp.

*Proof.* By the previous Theorem 3.1, from A

$$|F(\zeta_1) - F(\zeta_2)| \geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n |a_n| \rho^{n-2} \right. \\ \left. - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} - \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2} \right],$$

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho^{n-1} \right. \\
 &\quad - \sum_{n=2}^{\infty} n \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho^{n-2} - \sum_{n=2}^{\infty} n \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho^{n-1} \\
 &\quad \left. - \sum_{n=2}^{\infty} n^2 \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho^{n-2} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2M \sum_{n=2}^{\infty} \frac{\rho^{n-1}}{n-1} + 2M \sum_{n=2}^{\infty} \frac{\rho^{n-1}}{n} \right. \\
 &\quad - M \sum_{n=2}^{\infty} \frac{n\rho^{n-2}}{n-1} + M \sum_{n=2}^{\infty} \rho^{n-2} - M \sum_{n=2}^{\infty} \frac{n\rho^{n-1}}{n-1} + M \sum_{n=2}^{\infty} \rho^{n-1} \\
 &\quad \left. - M \sum_{n=2}^{\infty} \frac{n^2\rho^{n-2}}{n-1} + M \sum_{n=2}^{\infty} n\rho^{n-2} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left( 1 + M + 2M \log(1 - \rho) - \frac{M}{(1 - \rho)} \right),
 \end{aligned}$$

Choose  $\rho_4$  so that  $1 + M + 2M \log(1 - \rho_4) - \frac{M}{(1 - \rho_4)} = 0$ . Then  $F$  will be univalent in  $|\zeta| < \rho_4$ . Further, if  $|\zeta| = \rho_4$ , then

$$\begin{aligned}
 |F(\zeta)| = \rho_4^2 |G(\zeta)| &\geq \rho_4^2 \left[ \rho_4 - \sum_{n=2}^{\infty} |a_n| \rho_4^n - \sum_{n=2}^{\infty} n |a_n| \rho_4^{n-1} \right] \\
 &\geq \rho_4^2 \left[ \rho_4 - \sum_{n=2}^{\infty} \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho_4^n - \sum_{n=2}^{\infty} n \left( \frac{M}{n-1} - \frac{M}{n} \right) \rho_4^{n-1} \right] \\
 &\geq (1 - M) \rho_4^3 + M \rho_4 \log(1 - \rho_4) = R_4.
 \end{aligned}$$

□

**Theorem 3.5.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f(\zeta) - \zeta}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{n+1}{2}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_5$ ,  $0 < \rho_5 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_5$ . Specifically,  $\rho_5$  satisfies

$$\frac{1 + \rho}{(1 - \rho)^3} + \frac{3}{(1 - \rho)^2} + \frac{2}{1 - \rho} = 7,$$

and  $F(\mathbb{D}_{\rho_5}) \subset \mathbb{D}_{R_5}$ , where

$$R_5 = 3\rho_5^3 - \frac{\rho_5^3}{(1 - \rho_5)^2} - \frac{\rho_5^3}{1 - \rho_5}.$$

This result is not sharp.

*Proof.* Let  $\zeta_1, \zeta_2 \in \mathbb{D}$  with  $\zeta_1 \neq \zeta_2$  be such that  $|\zeta_1| < \rho$  and  $|\zeta_2| < \rho$ ,  $0 < \rho < 1$ . Then

$$F(\zeta_1) - F(\zeta_2) = \int_{[\zeta_1, \zeta_2]} F_\zeta(\zeta) d\zeta + F_{\bar{\zeta}} d\bar{\zeta} = \int_{[\zeta_1, \zeta_2]} (\bar{\zeta} G + r^2 \bar{h}') d\zeta + (\zeta G + r^2 g') d\bar{\zeta},$$

where  $[\zeta_1, \zeta_2]$  is the line segment from  $\zeta_1$  to  $\zeta_2$ . Then

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &= \left| h'(0) \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + \zeta^2 d\bar{\zeta}) + \overline{g'(0)} \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + \bar{\zeta}^2 d\bar{\zeta}) \right. \\
 &\quad + \int_{[\zeta_1, \zeta_2]} [(h(\zeta) - h'(0)\zeta) + \overline{(g(\zeta) - g'(0)\zeta)}](\bar{\zeta} d\zeta + \zeta d\bar{\zeta}) \\
 &\quad \left. + \int_{[\zeta_1, \zeta_2]} r^2((h'(\zeta) - h'(0))d\zeta + (\overline{g'(\zeta)} - \overline{g'(0)})d\bar{\zeta}) \right| \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - 2 \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \right] \tag{B} \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 2 \sum_{n=2}^{\infty} n \rho^{n-1} - 2 \sum_{n=2}^{\infty} \rho^{n-1} \right. \\
 &\quad \left. - \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \sum_{n=2}^{\infty} n \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{1 + \rho}{(1 - \rho)^3} - \frac{3}{(1 - \rho)^2} - \frac{2}{(1 - \rho)} \right],
 \end{aligned}$$

Choose  $\rho_5$  so that

$$\begin{aligned}
 7 - \frac{1 + \rho_5}{(1 - \rho_5)^3} - \frac{3}{(1 - \rho_5)^2} - \frac{2}{(1 - \rho_5)} &= 0, \\
 \frac{1 + \rho_5}{(1 - \rho_5)^3} + \frac{3}{(1 - \rho_5)^2} + \frac{2}{(1 - \rho_5)} &= 7.
 \end{aligned}$$

Then  $F$  will be univalent in  $|\zeta| < \rho_5$ . Furthermore, if  $|\zeta| = \rho_5$ , then

$$\begin{aligned}
 |F(\zeta)| = \rho_5^2 |G(\zeta)| &\geq \rho_5^3 - 2\rho_5^2 \sum_{n=2}^{\infty} |a_n| \rho_5^n \\
 &\geq 3\rho_5^3 - \frac{\rho_5^3}{(1 - \rho_5)^2} - \frac{\rho_5^3}{(1 - \rho_5)} = R_5.
 \end{aligned}$$

□

**Theorem 3.6.** *Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f(\zeta) - \zeta}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{(2n + 1)(n + 1)}{6}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_6$ ,  $0 < \rho_6 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_6$ . Specifically,  $\rho_6$  satisfies*

$$\frac{2}{3(1 - \rho)} + \frac{7}{3(1 - \rho)^2} + \frac{1 + \rho}{(1 - \rho)^3} + \frac{8\rho}{3(1 - \rho)^4} = 7,$$

and  $F(\mathbb{D}_{\rho_6}) \subset \mathbb{D}_{R_6}$ , where

$$R_6 = \frac{4}{3}\rho_6^3 + \frac{5}{6}\rho_6^2 - \frac{\rho_6^2(\rho_6^3 - 3)}{3(1 - \rho_6)^2}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.5, from B

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - 2 \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} \left( \frac{n^2}{3} + \frac{n}{2} + \frac{1}{6} \right) \rho^{n-1} \right. \\
 &\quad \left. - 2 \sum_{n=2}^{\infty} n \left( \frac{n^2}{3} + \frac{n}{2} + \frac{1}{6} \right) \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{7}{3} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - 2 \sum_{n=2}^{\infty} n \rho^{n-1} \right. \\
 &\quad \left. - \frac{2}{3} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \frac{2}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-1} - \sum_{n=2}^{\infty} n^2 \rho^{n-1} \right. \\
 &\quad \left. - \frac{1}{3} \sum_{n=2}^{\infty} n \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{2}{3(1-\rho)} - \frac{7}{3(1-\rho)^2} - \frac{1+\rho}{(1-\rho)^3} \right. \\
 &\quad \left. - \frac{8\rho}{3(1-\rho)^4} \right].
 \end{aligned}$$

Choose  $\rho_6$  so that

$$7 - \frac{2}{3(1-\rho_6)} - \frac{7}{3(1-\rho_6)^2} - \frac{1+\rho_6}{(1-\rho_6)^3} - \frac{8\rho_6}{3(1-\rho_6)^4} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_6$ . Further, if  $|\zeta| = \rho_6$ , then

$$\begin{aligned}
 |F(\zeta)| &= \rho_6^2 |G(\zeta)| \geq \rho_6^3 - 2\rho_6^2 \sum_{n=2}^{\infty} |a_n| \rho_6^n \\
 &\geq \rho_6^3 - \frac{2}{3} \rho_6^2 \sum_{n=2}^{\infty} n^2 \rho_6^n - \rho_6^2 \sum_{n=2}^{\infty} n \rho_6^n - \frac{\rho_6^2}{3} \sum_{n=2}^{\infty} \rho_6^n \\
 &\geq \frac{4}{3} \rho_6^3 + \frac{5}{6} \rho_6^2 - \frac{\rho_6^2(\rho_6^3 - 3)}{3(1-\rho)^2} = R_6.
 \end{aligned}$$

□

**Theorem 3.7.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f(\zeta) - \zeta}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{2n^2 + 1}{3}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_7$ ,  $0 < \rho_7 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_7$ . Specifically,  $\rho_7$  satisfies

$$\frac{16\rho}{3(1-\rho)^4} + \frac{4(1+\rho)}{3(1-\rho)^3} + \frac{2}{3(1-\rho)^2} + \frac{4}{3(1-\rho)} = 7,$$

and  $F(\mathbb{D}_{\rho_7}) \subset \mathbb{D}_{R_7}$ , where

$$R_7 = \rho_7^3 + \frac{4}{3} \rho_7^2 - \frac{4\rho_7^2(1+\rho_7)}{3(1-\rho_7)^3} - \frac{2\rho_7^4}{3(1-\rho_7)}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.5, from **B**

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - 2 \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} \frac{(2n^2 + 1)}{3} \rho^{n-1} \right. \\
 &\quad \left. - 2 \sum_{n=2}^{\infty} n \frac{(2n^2 + 1)}{3} \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{8}{3} \sum_{n=2}^{\infty} n^2 \rho^{n-1} - \frac{4}{3} \sum_{n=2}^{\infty} \rho^{n-1} - \frac{4}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-1} \right. \\
 &\quad \left. - \frac{2}{3} \sum_{n=2}^{\infty} n \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{16\rho}{3(1-\rho)^4} - \frac{4(1+\rho)}{3(1-\rho)^3} - \frac{2}{3(1-\rho)^2} \right. \\
 &\quad \left. - \frac{4}{3(1-\rho)} \right].
 \end{aligned}$$

Choose  $\rho_7$  so that

$$7 - \frac{16\rho}{3(1-\rho)^4} - \frac{4(1+\rho)}{3(1-\rho)^3} - \frac{2}{3(1-\rho)^2} - \frac{4}{3(1-\rho)} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_7$ . Further, if  $|\zeta| + \rho_7$ , then

$$\begin{aligned}
 |F(\zeta)| &= \rho_7^2 |G(\zeta)| \geq \rho_7^3 - 2\rho_7^2 \sum_{n=2}^{\infty} |a_n| \rho_7^n \\
 &\geq \rho_7^3 - \frac{4}{3} \rho_7^2 \sum_{n=2}^{\infty} n^2 \rho_7^{n-1} - \frac{2}{3} \rho_7^2 \sum_{n=2}^{\infty} \rho_7^n \\
 &\geq \rho_7^3 + \frac{4}{3} \rho_7^2 - \frac{4\rho_7^2(1+\rho_7)}{3(1-\rho_7)^3} - \frac{2\rho_7^4}{3(1-\rho_7)} = R_7.
 \end{aligned}$$

□

**Theorem 3.8.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{f(\zeta) - \zeta}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{M}{n(n-1)}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_8$ ,  $0 < \rho_8 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_8$ . Specifically,  $\rho_8$  satisfies

$$6M\rho \log(1-\rho) - \frac{4M \log(1-\rho)}{\rho} - (4M-1)\rho = 0,$$

and  $F(\mathbb{D}_{\rho_8}) \subset \mathbb{D}_{R_8}$ , where

$$R_8 = \rho_8^3 + 2M\rho_8^3 \log(1-\rho_8) - 2M \log(1-\rho_8) - 2M\rho_8^2.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.5, from B

$$\begin{aligned}
 |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - 2 \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4 \sum_{n=2}^{\infty} \left( \frac{M}{n(n-1)} \right) \rho^{n-1} \right. \\
 &\quad \left. - 2 \sum_{n=2}^{\infty} n \left( \frac{M}{n(n-1)} \right) \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 4M \sum_{n=2}^{\infty} \frac{\rho^{n-1}}{n-1} + 4M \sum_{n=2}^{\infty} \frac{\rho^{n-1}}{n} \right. \\
 &\quad \left. - 2M \sum_{n=2}^{\infty} \frac{n\rho^{n-1}}{n-1} + 2M \sum_{n=2}^{\infty} \rho^{n-1} \right] \\
 &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 6M\rho \log(1 - \rho) - \frac{4M \log(1 - \rho)}{\rho} \right. \\
 &\quad \left. - (4M - 1)\rho \right].
 \end{aligned}$$

Choose  $\rho_8$  so that

$$6M\rho_8 \log(1 - \rho_8) - \frac{4M \log(1 - \rho_8)}{\rho_8} - (4M - 1)\rho_8 = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_8$ . Further, if  $|\zeta| = \rho_8$ , then

$$\begin{aligned}
 |F(\zeta)| = \rho_8^2 |G(\zeta)| &\geq \rho_8^3 - 2\rho_8^2 \sum_{n=2}^{\infty} |a_n| \rho_8^n \\
 &\geq \rho_8^3 - 2M\rho_8^3 \sum_{n=2}^{\infty} \frac{\rho_8^n}{n-1} + 2M\rho_8^3 \sum_{n=2}^{\infty} \frac{\rho_8^n}{n} \\
 &\geq \rho_8^3 + 2M\rho_8^3 \log(1 - \rho_8) - 2M \log(1 - \rho_8) - 2M\rho_8^2 = R_8.
 \end{aligned}$$

□

**Theorem 3.9.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{\left( \int_0^\zeta \frac{f(t)}{t} dt - \zeta \right)}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{n+1}{2}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_9$ ,  $0 < \rho_9 < 1$ , such that  $F(\zeta) = r^2 G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_9$ . Specifically,  $\rho_9$  satisfies

$$\frac{1 + \rho}{2(1 - \rho)^3} + \frac{2}{(1 - \rho)^2} + \frac{5}{2(1 - \rho)} - \frac{\log(1 - \rho)}{\rho} = 7,$$

and  $F(\mathbb{D}_{\rho_9}) \subset \mathbb{D}_{R_9}$ , where

$$R_9 = \frac{7\rho_9^3}{3} - \frac{\rho_9^3}{2(1 - \rho_9)^2} - \frac{\rho_9^3}{1 - \rho_9} + \frac{\rho_9^2 \log(1 - \rho_9)}{2}.$$

This result is not sharp.

*Proof.* Let  $\zeta_1, \zeta_2 \in \mathbb{D}$  with  $\zeta_1 \neq \zeta_2$  be such that  $|\zeta_1| < \rho$  and  $|\zeta_2| < \rho$ ,  $0 < \rho < 1$ . Then

$$F(\zeta_1) - F(\zeta_2) = \int_{[\zeta_1, \zeta_2]} F_\zeta(\zeta) d\zeta + F_{\bar{\zeta}} d\bar{\zeta} = \int_{[\zeta_1, \zeta_2]} (\bar{\zeta}G + r^2\bar{h}') d\zeta + (\zeta G + r^2g') d\bar{\zeta},$$

where  $[\zeta_1, \zeta_2]$  is the line segment from  $\zeta_1$  to  $\zeta_2$ . Then

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &= \left| \int_{[\zeta_1, \zeta_2]} (\bar{\zeta}G + r^2\bar{h}') d\zeta + (\zeta G + r^2g') d\bar{\zeta} \right| \\ &= \left| h'(0) \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + \zeta^2 d\bar{\zeta}) + \overline{g'(0)} \int_{[\zeta_1, \zeta_2]} (2r^2 d\zeta + \zeta^2 d\bar{\zeta}) \right. \\ &\quad \left. + \int_{[\zeta_1, \zeta_2]} [(h(\zeta) - h'(0)\zeta) + \overline{(g(\zeta) - g'(0)\zeta)}](\bar{\zeta} d\zeta + \zeta d\bar{\zeta}) \right. \\ &\quad \left. + \int_{[\zeta_1, \zeta_2]} r^2((h'(\zeta) - h'(0))d\zeta + (\overline{g'(\zeta)} - \overline{g'(0)})d\bar{\zeta}) \right| \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty |a_n| \rho^{n-1} - 2 \sum_{n=2}^\infty \frac{|a_n|}{n} \rho^{n-1} \right. \\ &\quad \left. - \sum_{n=2}^\infty n |a_n| \rho^{n-1} \right] \tag{C} \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty \left( \frac{n+1}{2} \right) \rho^{n-1} - 2 \sum_{n=2}^\infty \frac{(n+1)}{2n} \rho^{n-1} \right. \\ &\quad \left. - \sum_{n=2}^\infty n \left( \frac{n+1}{2} \right) \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{1+\rho}{2(1-\rho)^3} - \frac{2}{(1-\rho)^2} - \frac{5}{2(1-\rho)} \right. \\ &\quad \left. + \frac{\log(1-\rho)}{\rho} \right]. \end{aligned}$$

Choose  $\rho_9$  so that

$$7 - \frac{1+\rho_9}{2(1-\rho_9)^3} - \frac{2}{(1-\rho_9)^2} - \frac{5}{2(1-\rho_9)} + \frac{\log(1-\rho_9)}{\rho_9} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_9$ . Furthermore, if  $|\zeta| = \rho_9$ , then

$$\begin{aligned} |F(\zeta)| = \rho_9^2 |G(\zeta)| &\geq \rho_9^3 - \rho_9^2 \sum_{n=2}^\infty |a_n| \rho_9^n - \sum_{n=2}^\infty \frac{|a_n|}{n} \rho_9^n \\ &\geq \rho_9^3 - \frac{\rho_9^2}{2} \sum_{n=2}^\infty n \rho_9^n - \rho_9^2 \sum_{n=2}^\infty \rho_9^n - \frac{\rho_9^n}{2} \sum_{n=2}^\infty \frac{\rho_9^n}{n} \\ &\geq \frac{7\rho_9^3}{3} - \frac{\rho_9^3}{2(1-\rho_9)^2} - \frac{\rho_9^3}{(1-\rho_9)} + \frac{\rho_9^2 \log(1-\rho_9)}{2} = R_9. \end{aligned}$$

□

**Theorem 3.10.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{\left( \int_0^\zeta \frac{f(t)}{t} dt - \zeta \right)}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{(2n+1)(n+1)}{6}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_{10}$ ,  $0 < \rho_{10} < 1$ , such that

$F(\zeta) = r^2G(\zeta)$  is univalent in the disc  $|\zeta| < \rho_{10}$ . Specifically,  $\rho_{10}$  satisfies

$$\frac{4\rho}{3(1-\rho)^4} + \frac{11(1+\rho)}{6(1-\rho)^3} + \frac{7}{3(1-\rho)^2} + \frac{3}{2(1-\rho)} - \frac{\log(1-\rho)}{3\rho} = 7,$$

and  $F(\mathbb{D}_{\rho_{10}}) \subset \mathbb{D}_{R_{10}}$ , where

$$R_{10} = 3\rho_{10}^3 - \frac{\rho_{10}^3(2\rho_{10} + 7)}{6(1-\rho_{10})^2} - \frac{2\rho_{10}}{1-\rho_{10}} + \frac{\rho_{10}^2 \log(1-\rho_{10})}{6}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.9, from C

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^{\infty} |a_n| \rho^{n-1} - 2 \sum_{n=2}^{\infty} \frac{|a_n|}{n} \rho^{n-1} \right. \\ &\quad \left. - \sum_{n=2}^{\infty} n |a_n| \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^{\infty} \left( \frac{(2n+1)(n+1)}{6} \right) \rho^{n-1} \right. \\ &\quad \left. - 2 \sum_{n=2}^{\infty} \left( \frac{(2n+1)(n+1)}{6n} \right) \rho^{n-1} - \sum_{n=2}^{\infty} n \left( \frac{(2n+1)(n+1)}{6} \right) \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{1}{3} \sum_{n=2}^{\infty} n^3 \rho^{n-1} - \frac{3}{2} \sum_{n=2}^{\infty} n^2 \rho^{n-1} \right. \\ &\quad \left. - \frac{7}{3} \sum_{n=2}^{\infty} n \rho^{n-1} - \frac{3}{2} \sum_{n=2}^{\infty} \rho^6 n - 1 - \frac{1}{3} \frac{\rho^{n-1}}{n} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{4\rho}{3(1-\rho)^4} - \frac{11(1+\rho)}{6(1-\rho)^3} - \frac{7}{3(1-\rho)^2} \right. \\ &\quad \left. - \frac{3}{2(1-\rho)} + \frac{\log(1-\rho)}{3\rho} \right]. \end{aligned}$$

Choose  $\rho_{10}$  so that

$$7 - \frac{4\rho_{10}}{3(1-\rho_{10})^4} - \frac{11(1+\rho_{10})}{6(1-\rho_{10})^3} - \frac{7}{3(1-\rho_{10})^2} - \frac{3}{2(1-\rho_{10})} + \frac{\log(1-\rho_{10})}{3\rho_{10}} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_{10}$ . Further, if  $|\zeta| < \rho_{10}$ , then

$$\begin{aligned} |F(z)| = \rho_{10}^2 |G(z)| &\geq \rho_{10}^3 - \rho_{10}^2 \sum_{n=2}^{\infty} |a_n| \rho_{10}^n - \sum_{n=2}^{\infty} \frac{|a_n|}{n} \rho_{10}^n \\ &\geq \rho_{10}^3 - \frac{\rho_{10}^2}{3} \sum_{n=2}^{\infty} n^2 \rho_{10}^n - \rho_{10}^2 \sum_{n=2}^{\infty} n \rho_{10}^n - \frac{2\rho_{10}^n}{3} \sum_{n=2}^{\infty} \rho_{10}^n - \frac{1}{6} \sum_{n=2}^{\infty} \frac{\rho_{10}^n}{n} \\ &\geq 3\rho_{10}^3 - \frac{\rho_{10}^3(2\rho_{10} + 7)}{6(1-\rho_{10})^2} - \frac{2\rho_{10}}{(1-\rho_{10})} + \frac{\rho_{10}^2 \log(1-\rho_{10})}{6} = R_{10}. \end{aligned}$$

□

**Theorem 3.11.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{\left(\int_0^\zeta \frac{f(t)}{t} dt - \zeta\right)}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{2n^2 + 1}{3}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_{11}$ ,  $0 < \rho_{11} < 1$ , such that  $F(\zeta) = r^2G(\zeta)$  is univalent in the disc  $|\zeta| < \rho_{11}$ . Specifically,  $\rho_{11}$  satisfies

$$\frac{8\rho}{3(1-\rho)^4} + \frac{8(1+\rho)}{3(1-\rho)^3} + \frac{5}{3(1-\rho)^2} + \frac{1}{1-\rho} - \frac{\log(1-\rho)^2}{3\rho} = 7,$$

and  $F(\mathbb{D}_{\rho_{11}}) \subset \mathbb{D}_{R_{11}}$ , where

$$R_{11} = 3\rho_{11}^3 - \frac{2\rho_{11}^3(1+\rho_{11})}{3(1-\rho_{11})^2} - \frac{\rho_{11}^3}{1-\rho_{11}} + \frac{\rho_{11}^2 \log(1-\rho_{11})}{3}.$$

This result is not sharp.

*Proof.* By the previous Theorem 3.9, from **C**

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty |a_n| \rho^{n-1} - 2 \sum_{n=2}^\infty \frac{|a_n|}{n} \rho^{n-1} - \sum_{n=2}^\infty n |a_n| \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty \left( \frac{2n^2 + 1}{3} \right) \rho^{n-1} \right. \\ &\quad \left. - 2 \sum_{n=2}^\infty \left( \frac{2n^2 + 1}{3n} \right) \rho^{n-1} - \sum_{n=2}^\infty n \left( \frac{2n^2 + 1}{3} \right) \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - \frac{2}{3} \sum_{n=2}^\infty n^3 \rho^{n-1} - 2 \sum_{n=2}^\infty n^2 \rho^{n-1} \right. \\ &\quad \left. - \frac{5}{3} \sum_{n=2}^\infty n \rho^{n-1} - \sum_{n=2}^\infty \rho^{n-1} - \frac{2}{3} \sum_{n=2}^\infty \frac{\rho^{n-1}}{n} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 7 - \frac{8\rho}{3(1-\rho)^4} - \frac{8(1+\rho)}{3(1-\rho)^3} - \frac{5}{3(1-\rho)^2} \right. \\ &\quad \left. - \frac{1}{(1-\rho)} + \frac{\log(1-\rho)^2}{3\rho} \right]. \end{aligned}$$

Choose  $\rho_{11}$  so that

$$7 - \frac{8\rho_{11}}{3(1-\rho_{11})^4} - \frac{8(1+\rho_{11})}{3(1-\rho_{11})^3} - \frac{5}{3(1-\rho_{11})^2} - \frac{1}{(1-\rho_{11})} + \frac{\log(1-\rho_{11})^2}{3\rho_{11}} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_{11}$ . Further, if  $|\zeta| = \rho_{11}$ , then

$$\begin{aligned} |F(\zeta)| &= \rho_{11}^2 |G(\zeta)| \geq \rho_{11}^3 - \rho_{11}^2 \sum_{n=2}^\infty |a_n| \rho_{11}^n - \sum_{n=2}^\infty \frac{|a_n|}{n} \rho_{11}^n \\ &\geq \rho_{11}^3 - \frac{2\rho_{11}^2}{3} \sum_{n=2}^\infty n^2 \rho_{11}^n - \frac{\rho_{11}^2}{3} \sum_{n=2}^\infty \rho_{11}^n - \frac{2\rho_{11}^n}{3} \sum_{n=2}^\infty n \rho_{11}^n - \frac{1}{3} \sum_{n=2}^\infty \frac{\rho_{11}^n}{n} \\ &\geq 3\rho_{11}^3 - \frac{2\rho_{11}^3(1+\rho_{11})}{3(1-\rho_{11})^2} - \frac{\rho_{11}^3}{(1-\rho_{11})} + \frac{\rho_{11}^2 \log(1-\rho_{11})}{3} = R_{11}. \end{aligned}$$

□

**Theorem 3.12.** Let  $f \in \mathcal{A}$  and  $G(\zeta) = f(\zeta) + \overline{\left(\int_0^\zeta \frac{f(t)}{t} dt - \zeta\right)}$  be harmonic in  $\mathbb{D}$  such that  $G(0) = 0$ , with  $|a_n| \leq \frac{M}{n(n-1)}$  for all  $n \geq 2$ . Then there exists a constant  $\rho_{12}$ ,  $0 < \rho_{12} < 1$ , such that  $F(\zeta) = r^2G(\zeta)$  is univalent in the disk  $|\zeta| < \rho_{12}$ . Specifically,  $\rho_{12}$  satisfies

$$\frac{6M \log(1 - \rho)}{\rho} - 6M \log(1 - \rho) + \frac{M}{1 - \rho} + 5M - \frac{\text{Li}_2(\rho)}{\rho} = 2,$$

and  $F(\mathbb{D}_{\rho_{12}}) \subset \mathbb{D}_{R_{12}}$ , where

$$R_{12} = (1 - 3M)\rho_{12}^3 + 2M(\rho_{12} - 1) \log(1 - \rho_{12}) + M\rho_{12}^2 \text{Li}_2(\rho_{12}).$$

This result is not sharp.

*Proof.* By the previous Theorem 3.9, from C

$$\begin{aligned} |F(\zeta_1) - F(\zeta_2)| &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty |a_n| \rho^{n-1} - 2 \sum_{n=2}^\infty \frac{|a_n|}{n} \rho^{n-1} - \sum_{n=2}^\infty n |a_n| \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 3 \sum_{n=2}^\infty \left( \frac{M}{n(n-1)} \right) \rho^{n-1} \right. \\ &\quad \left. - 2 \sum_{n=2}^\infty \frac{1}{n} \times \left( \frac{M}{n(n-1)} \right) \rho^{n-1} - \sum_{n=2}^\infty n \left( \frac{M}{n(n-1)} \right) \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 1 - 6M \sum_{n=2}^\infty \frac{\rho^{n-1}}{n-1} + 6M \sum_{n=2}^\infty \frac{\rho^{n-1}}{n} \right. \\ &\quad \left. + 2M \sum_{n=2}^\infty \frac{\rho^{n-1}}{n^2} - M \sum_{n=2}^\infty \rho^{n-1} \right] \\ &\geq |\zeta_2 - \zeta_1| \left( \int_0^1 r^2 dt \right) \left[ 2 - \frac{6M \log(1 - \rho)}{\rho} + 6M \log(1 - \rho) \right. \\ &\quad \left. - \frac{M}{1 - \rho} - 5M + \frac{\text{Li}_2(\rho)}{\rho} \right]. \end{aligned}$$

Choose  $\rho_{12}$  so that

$$2 - \frac{6M \log(1 - \rho)}{\rho} + 6M \log(1 - \rho) - \frac{M}{1 - \rho} - 5M + \frac{\text{Li}_2(\rho)}{\rho} = 0.$$

Then  $F$  will be univalent in  $|\zeta| < \rho_{12}$ . Further, if  $|\zeta| = \rho_{12}$ , then

$$\begin{aligned} |F(\zeta)| &= \rho_{12}^2 |G(\zeta)| \geq \rho_{12}^3 - \rho_{12}^2 \sum_{n=2}^\infty |a_n| \rho_{12}^n - \sum_{n=2}^\infty \frac{|a_n|}{n} \rho_{12}^n \\ &\geq \rho_{12}^3 - 2M\rho_{12}^2 \sum_{n=2}^\infty \frac{\rho_{12}^n}{n-1} + 2M\rho_{12}^2 \sum_{n=2}^\infty \frac{\rho_{12}^n}{n} + \rho_{12}^2 \sum_{n=2}^\infty \frac{\rho_{12}^n}{n^2} \\ &\geq (1 - 3M)\rho_{12}^3 + 2M(\rho_{12} - 1) \log(1 - \rho_{12}) + M\rho_{12}^2 \text{Li}_2(\rho_{12}) = R_{12}. \end{aligned}$$

□

## 4 Conclusion

In this paper, we established several versions of Landau's theorem for biharmonic mappings constructed from holomorphic functions under different coefficient conditions. Explicit estimates for the radius of univalence and the corresponding covering results were obtained, thereby extending the classical Landau theorem to this class of biharmonic mappings. The methods developed in this work provide a framework for studying the geometric properties of biharmonic mappings. Future research may focus on extending these results to broader classes of polyharmonic mappings and investigating the sharpness of the obtained bounds.

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## References

- [1] Abdulhadi, Z., & Abu Muhanna, Y. (2008). Landau's theorem for biharmonic mappings. *Journal of Mathematical Analysis and Applications*, 338(1), 705–709. <https://doi.org/10.1016/j.jmaa.2007.05.065>
- [2] Chen, S., Ponnusamy, S., & Wang, X. (2009). Landau's theorem for certain biharmonic mappings. *Applied Mathematics and Computation*, 208(2), 427–433. <https://doi.org/10.1016/j.amc.2008.12.016>
- [3] Chen, S., Ponnusamy, S., & Wang, X. (2010). Coefficient estimates and Landau–Bloch's constant for planar harmonic mappings. *Bulletin of the Malaysian Mathematical Sciences Society*, 33(2), 277–284.
- [4] Clunie, J., & Sheil-Small, T. (1984). Harmonic univalent functions. *Annales Academiae Scientiarum Fennicae Series A I Mathematica*, 9, 3–25. <https://doi.org/10.5186/aasfm.1984.0905>
- [5] Dorff, M., & Nowak, M. (2004). Landau theorem for planar harmonic mappings. *Computational Methods and Function Theory*, 4(1), 151–158. <https://doi.org/10.1007/BF03321062>
- [6] Duren, P. L. (1983). *Univalent functions*. Springer.
- [7] Duren, P. L. (2004). *Harmonic mappings in the plane*. Cambridge University Press. <https://doi.org/10.1017/CB09780511546600>
- [8] Grigoryan, A. (2006). Landau and Bloch theorems for harmonic mappings. *Complex Variables and Elliptic Equations*, 51(1), 81–87. <https://doi.org/10.1080/02781070500369214>
- [9] Liu, M.-S. (2008). Landau's theorems for biharmonic mappings. *Complex Variables and Elliptic Equations*, 53(9), 843–855. <https://doi.org/10.1080/17476930802166337>

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- [10] Liu, M.-S. (2009). Landau's theorem for planar harmonic mappings. *Computers & Mathematics with Applications*, 57(7), 1142–1146. <https://doi.org/10.1016/j.camwa.2009.01.009>
- [11] Liu, M.-S., & Liu, Z.-W. (2011). Landau's theorem for certain biharmonic mappings. *Acta Mathematica Sinica (Chinese Series)*, 54(1), 69–80.

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