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Path Averaged Polynomial Contractions: A New Generalization of Polynomial Contractions, Path-Averaged Contractions, and Banach Contractions

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Abstract

The notion of polynomial contraction appeared in [2], whilst the notion of path-averaged contraction appeared in [3] for metric spaces, in [4, 5] for b-metric spaces and [6] in suprametric spaces. In this paper, we combine both notions to introduce path-averaged polynomial contractions, as a generalization of polynomial contractions, path-averaged contractions, and Banach contractions. We obtain a fixed point theorem for such contractions in the setting of complete metric spaces under continuity and boundedness assumptions on the coefficient functions. We give an example showing path-averaged polynomial contractions are not Banach contractions.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ satisfy*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$ and some $k \in (0, 1)$. Then T has a unique fixed point.

Definition 1.2. [2] Let (X, d) be a metric space, and let $T : X \mapsto X$ be a given mapping. We say that T is a *polynomial contraction* if there exists $\lambda \in [0, 1)$, a natural number $k \geq 1$, and a family of mappings $a_i : X \times X \mapsto [0, \infty)$, $i = 0, \cdots, k$, such that

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)d^i(x, y)$$

for every $x, y \in X$.

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Theorem 1.3. *[2] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a polynomial contraction. Assume that the following conditions hold:*

(i) T is continuous,

(ii) there exists $j \in \{1, \cdots, k\}$ and $A_j > 0$ such that

$$a_j(x, y) \geq A_j$$

for all $x, y \in X$.

Then, T admits a unique fixed point $z^* \in X$. Moreover, for every $z_0 \in X$, the Picard sequence $\{z_n\} \subset X$ defined by $z_{n+1} = Tz_n$ for all $n \geq 0$ converges to z^*.

Definition 1.4. [3] Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is called a *PA-contraction* (*Path Averaged Contraction*) if there exists $\alpha \in (0, 1)$ and $N \in \mathbb{N}$ such that for all $x, y \in X$ and all $n \geq N$

$$\sum_{k=0}^{n-1} d(T^{k+1}x, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} d(T^k x, T^k y).$$

Theorem 1.5. *[3] Let (X, d) be a complete metric space, and let $T : X \mapsto X$ be a continuous PA-contraction. Then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the Picard sequence $x_n = T^n x_0$ converges to x^*.*

Motivation and Novelty. The introduction of averaging along iterates in contraction principles serves two primary purposes: it relaxes strict pointwise contraction requirements by considering cumulative behavior over multiple steps, and it naturally accommodates mappings that may exhibit local expansion but global contractive tendencies. While standard polynomial contractions control distances through weighted polynomial inequalities at each iteration, path-averaged polynomial contractions integrate these polynomial weights along the entire trajectory of the Picard sequence. This conceptual shift allows for a broader class of admissible mappings, particularly useful in iterative algorithms, stability analysis of discrete dynamical systems, and numerical schemes where asymptotic behavior is more relevant than instantaneous contraction. Our framework unifies these approaches and opens pathways for extensions to modular spaces, probabilistic metrics, and nonlinear operator equations.

2 Path-Averaged Polynomial Contractions

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is called a *PA-polynomial contraction* (*Path-Averaged Polynomial Contraction*) if there exists $\alpha \in (0, 1)$ and $N \in \mathbb{N}$ such that for all $x, y \in X$ and all $n \geq N$ we have

$$\sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(T^{k+1}x, T^{k+1}y) d^i(T^{k+1}x, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(T^k x, T^k y) d^i(T^k x, T^k y).$$

Remark 2.2. The PA-polynomial contraction generalizes several known contractions in the literature as follows:

(*i*) If $n = 1$, then the above contraction reduces to

$$\sum_{i=0}^{r} a_i(Tx, Ty)d^i(Tx, Ty) \leq \alpha \sum_{i=0}^{r} a_i(x, y)d^i(x, y),$$

that is, T is a polynomial contraction [2].

(*ii*) If $n = 1$, $r = 1$, $a_0 \equiv 0$, and $a_1 \equiv 1$, then the above contraction reduces to

$$d(Tx, Ty) \leq \alpha d(x, y),$$

that is, T is a Banach contraction [1]

(*iii*) If $r = 1$, $a_0 \equiv 0$, and $a_1 \equiv 1$, then the above contraction reduces to

$$\sum_{k=0}^{n-1} d(T^{k+1}x, T^{k+1}y) \leq \alpha \sum_{k=0}^{n-1} d(T^k x, T^k y),$$

that is, T is a path-averaged contraction [3].

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \mapsto X$ be a path-averaged polynomial contraction. Assume the following conditions hold:*

(i) T is continuous.

(ii) There exists an index $j = 1$ and a constant $A_1 > 0$ such that

$$a_1(x, y) \geq A_1 \tag{1}$$

for all $x, y \in X$.

Then, T admits a unique fixed point $z^* \in X$. Moreover, for every $z_0 \in X$, the Picard sequence $\{z_n\} \subset X$ defined by $z_n = T^n z_0$ for all $n \geq 0$, converges to z^*.

Proof. **Existence:** Let $z_0 \in X$ be arbitrary. Define the Picard sequence $z_n = T^n z_0$ for $n \geq 0$. Define P_n as the polynomial evaluation along the sequence:

$$P_n = \sum_{i=0}^{r} a_i(z_n, z_{n+1})d^i(z_n, z_{n+1}). \tag{2}$$

Applying the PA-polynomial contraction condition to $x = z_0$ and $y = z_1 = Tz_0$, we have for all $n \geq N$:

$$\sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(T^{k+1}z_0, T^{k+2}z_0)d^i(T^{k+1}z_0, T^{k+2}z_0)$$
$$\leq \alpha \sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(T^k z_0, T^{k+1}z_0)d^i(T^k z_0, T^{k+1}z_0). \tag{3}$$

Substituting $z_k = T^k z_0$, this becomes $\sum_{k=0}^{n-1} P_{k+1} \leq \alpha \sum_{k=0}^{n-1} P_k$. Let $S_n = \sum_{k=0}^{n-1} P_k$. The inequality rewrites as $S_{n+1} - P_0 \leq \alpha S_n$, hence $S_{n+1} \leq \alpha S_n + P_0$. By induction,

$$S_n \leq \alpha^{n-1} S_1 + P_0 \sum_{m=0}^{n-2} \alpha^m \leq \alpha^{n-1} S_1 + \frac{P_0}{1-\alpha}.$$

Since $\alpha \in (0,1)$ and $P_k \geq 0$, $\{S_n\}$ is bounded and non-decreasing, so $\sum_{k=0}^{\infty} P_k$ converges. Thus, $\lim_{k\to\infty} P_k = 0$. From condition (ii) with $j = 1$, $P_k \geq a_1(z_k, z_{k+1}) d(z_k, z_{k+1}) \geq A_1 d(z_k, z_{k+1})$. Hence $\sum_{k=0}^{\infty} d(z_k, z_{k+1}) < \infty$. By the triangle inequality, for any $m \geq 1$,

$$d(z_k, z_{k+m}) \leq \sum_{i=k}^{k+m-1} d(z_i, z_{i+1}) \to 0 \quad \text{as } k \to \infty, \tag{4}$$

so $\{z_n\}$ is Cauchy. Completeness yields $z_n \to z^* \in X$. Continuity of T gives $Tz^* = z^*$.

Uniqueness: Suppose z^{**} is another fixed point with $z^* \neq z^{**}$. Then $T^k z^* = z^*$ and $T^k z^{**} = z^{**}$. Applying the contraction condition:

$$\sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(z^*, z^{**}) d^i(z^*, z^{**}) \leq \alpha \sum_{k=0}^{n-1} \sum_{i=0}^{r} a_i(z^*, z^{**}) d^i(z^*, z^{**}). \tag{5}$$

Let $Q = \sum_{i=0}^{r} a_i(z^*, z^{**}) d^i(z^*, z^{**}) > 0$ (by condition (ii)). Then $nQ \leq \alpha n Q \implies 1 \leq \alpha$, a contradiction. Hence $z^* = z^{**}$. □

Remark 2.4. The continuity of T is assumed to ensure that the limit of the Picard sequence is preserved under the mapping, i.e., $T(\lim z_n) = \lim T z_n$. While this assumption is standard in polynomial and path-averaged contraction theory, it can sometimes be relaxed by assuming orbital continuity, closed graph properties, or by working in spaces where weak convergence suffices. We retain continuity here to maintain direct comparability with the foundational results in [2, 3].

3 Generalization of Polynomial Contractions

Proposition 3.1. *Every polynomial contraction is a path-averaged polynomial contraction.*

Proof. If T is a polynomial contraction, then

$$\sum_{i=0}^{r} a_i(Tx, Ty) d^i(Tx, Ty) \leq \alpha \sum_{i=0}^{r} a_i(x, y) d^i(x, y)$$

for all $x, y \in X$. Replacing x, y with $T^k x, T^k y$ and summing from $k = 0$ to $n-1$ immediately yields the PA-polynomial contraction inequality with $N = 1$. Thus, T is a PA-polynomial contraction. □

Example 3.2. (Path-averaged polynomial contraction not Banach). Let $X = \{0, 1, 2\}$ with the discrete metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Define $T(0) = 1,\ T(1) = 2,\ T(2) = 2$. T is not a Banach contraction since $d(T0, T1) = d(1, 2) = 1 = d(0, 1)$ and $d(T0, T2) = d(1, 2) = 1 = d(0, 2)$, so no $\alpha < 1$ satisfies $d(Tx, Ty) \leq \alpha d(x, y)$ for all x, y.

However, T is a PA-polynomial contraction with $\alpha = \frac{1}{2}$, $N = 2$, $r = 2$, $a_0 \equiv 0$, and $a_1 \equiv a_2 \equiv 1$. For any $x, y \in X$ and $n \geq 2$:

$$\sum_{k=0}^{n-1} \left[d(T^{k+1}x, T^{k+1}y) + d^2(T^{k+1}x, T^{k+1}y)\right] \leq \frac{1}{2} \sum_{k=0}^{n-1} \left[d(T^k x, T^k y) + d^2(T^k x, T^k y)\right].$$

Verification for pair $(0, 1)$ at $n = 2$:

$$\begin{aligned} \text{LHS} &= d(T0, T1) + d^2(T0, T1) + d(T^2 0, T^2 1) + d^2(T^2 0, T^2 1) \\ &= d(1, 2) + d^2(1, 2) + d(2, 2) + d^2(2, 2) = 1 + 1 + 0 + 0 = 2, \\ \text{RHS} &= \frac{1}{2}\left[d(0, 1) + d^2(0, 1) + d(T0, T1) + d^2(T0, T1)\right] \\ &= \frac{1}{2}[1 + 1 + 1 + 1] = 2. \end{aligned}$$

Thus, $2 \leq 2$.

Verification for pair $(0, 2)$ at $n = 2$:

$$\begin{aligned} \text{LHS} &= d(T0, T2) + d^2(T0, T2) + d(T^2 0, T^2 2) + d^2(T^2 0, T^2 2) \\ &= d(1, 2) + d^2(1, 2) + d(2, 2) + d^2(2, 2) = 1 + 1 + 0 + 0 = 2, \\ \text{RHS} &= \frac{1}{2}\left[d(0, 2) + d^2(0, 2) + d(T0, T2) + d^2(T0, T2)\right] \\ &= \frac{1}{2}[1 + 1 + 1 + 1] = 2. \end{aligned}$$

Thus, $2 \leq 2$.

Verification for pair $(1, 2)$ at $n = 2$:

$$\begin{aligned} \text{LHS} &= d(T1, T2) + d^2(T1, T2) + d(T^2 1, T^2 2) + d^2(T^2 1, T^2 2) \\ &= d(2, 2) + d^2(2, 2) + d(2, 2) + d^2(2, 2) = 0, \\ \text{RHS} &= \frac{1}{2}\left[d(1, 2) + d^2(1, 2) + d(T1, T2) + d^2(T1, T2)\right] \\ &= \frac{1}{2}[1 + 1 + 0 + 0] = 1. \end{aligned}$$

Thus, $0 < 1$. For $n > 2$ and any pair, $T^k x = T^k y = 2$ for $k \geq 2$, so all subsequent distances vanish. Hence T is a PA-polynomial contraction but not a Banach contraction.

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