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Revision and Contribution to Refined Integral Inequalities of the Hilbert Type

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Abstract

The Hilbert integral inequality is a well-known result that forms the basis of analysis. In this article, we critically discuss two existing theorems relating to refinements of this inequality. Subsequently, we present a new result of the same kind. We provide a detailed proof and demonstrate the applicability of the proposed theorem through three illustrative applications.

1 Introduction

The Hilbert integral inequality is a fundamental mathematical result that establishes a sharp upper bound for a specific type of bilinear integral with a homogeneous kernel function. As the continuous analogue of the Hilbert series inequality, it plays a key role in harmonic analysis and the theory of integral operators. Its formal statement is given below. Let $f, g : [0, +\infty) \to [0, +\infty)$ be two functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^{+\infty} f^2(x)dx \right)^{1/2} \left(\int_0^{+\infty} g^2(y)dy \right)^{1/2}, \tag{1}$$

provided that the integrals under consideration converge. Notably, the constant factor π in Equation (1) is sharp and cannot be replaced by a smaller value. Over the years, numerous generalized inequalities have been developed. In particular, the celebrated Hardy-Hilbert integral inequality extends Equation (1) to the L^p setting. Further details and related results can be found in the classical books [3, 9] and the research articles [1, 2, 4–8].

For the purposes of this article, we emphasize the main contributions in [5, 7, 8]. For the special case where $g = f$, a refinement of the Hilbert integral inequality was obtained in [8, Theorem 2.1]. Its formal statement is given below. Let $f : [0, +\infty) \to [0, +\infty)$ be a function. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)f(y)}{x+y} dxdy \leq \pi \left(\left(\int_0^{+\infty} f^2(x)dx \right)^2 - \left(\int_0^{+\infty} w(x)f^2(x)dx \right)^2 \right)^{1/2}, \tag{2}$$

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where

$$w(x) = \frac{1}{1+\sqrt{x}} - \frac{1}{1+x},$$

provided that the integrals under consideration converge. Therefore, in the case where $g = f$, Equation (2) refines Equation (1) by the obvious inequality

$$\left(\left(\int_0^{+\infty} f^2(x)dx \right)^2 - \left(\int_0^{+\infty} w(x) f^2(x)dx \right)^2 \right)^{1/2} \leq \int_0^{+\infty} f^2(x)dx.$$

This result has inspired several further extensions, notably those presented in [5,7]. The main contribution in [7] lies in introducing a symmetric and homogeneous kernel function, and addressing the more general case where $g \neq f$. Subsequently, the work in [5] extends the main result of [7] to the L^p setting, providing a broader functional framework.

Building on this framework, the aim of this article is twofold. First, we revise and critically discuss two specific results from [5, 7], namely [7, Theorem 3.1] and [5, Theorem 2.1]. Our revision focuses on identifying certain theoretical gaps in their proofs and examining how these affect the validity of the corresponding statements. Second, we introduce a new integral inequality inspired by [7, Theorem 3.1] and [5, Theorem 2.1], thereby extending the existing theory. The proof is self-contained and reproducible. The relevance of our findings is further demonstrated through three direct applications.

The remainder of this article is organized as follows: Section 2 examines [7, Theorem 3.1] and [5, Theorem 2.1] through a detailed analytical discussion. The new theorem is presented in Section 3, followed by three illustrative applications in Section 4. Finally, Section 5 concludes the article with some remarks and perspectives for future work.

2 Revision of Two Existing Theorems

2.1 Discussion

For the purpose of revision, [7, Theorem 3.1] is restated below.

Theorem 2.1. *[7, Theorem 3.1] Let* $a \in \mathbb{R}$, $f, g : [0, +\infty) \to [0, +\infty)$ *be two functions,* $k : [0, +\infty) \to [0, 1]$ *be a function,* $\lambda > 0$, *and* $h : [0, +\infty)^2 \to [0, +\infty)$ *be a function that is*

- *symmetric, i.e., for any* $x, y \geq 0$,

$$h(x, y) = h(y, x),$$

- *homogeneous function of degree* λ, *i.e., for any* $t, x, y \geq 0$, *we have*

$$h(tx, ty) = t^{\lambda} h(x, y).$$

Then we have

$$\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}dxdy\right)^4$$
$$\leq\left(\left(\Omega_a\int_0^{+\infty}x^{1-\lambda}f^2(x)dx\right)^2-\left(\int_0^{+\infty}\left(\Omega_a k(x)-\Xi_a(x)\right)x^{1-\lambda}f^2(x)dx\right)^2\right)$$
$$\times\left(\left(\Omega_a\int_0^{+\infty}y^{1-\lambda}g^2(y)dy\right)^2-\left(\int_0^{+\infty}\left(\Omega_a k(y)-\Xi_a(y)\right)y^{1-\lambda}g^2(y)dy\right)^2\right),$$

where

$$\Omega_a=\int_0^{+\infty}\frac{t^a}{h(1,t)}dt,\quad \Xi_a(x)=\int_0^{+\infty}\frac{t^a k(xt)}{h(1,t)}dt,$$

provided that the integrals under consideration converge.

In [7, page 381], the proof relies on [7, Lemma 2.1], which states that, under the sole assumption that h is symmetric, one has

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}(1-k(x)+k(y))dxdy.$$

However, in the first double integral, replacing $f(y)$ by an arbitrary function $g(y)$ breaks the functional symmetry required for this identity, since in general we do not have $f(x)g(y)=f(y)g(x)$. Consequently, one cannot assert that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}(1-k(x)+k(y))dxdy.$$

Nevertheless, this equality is used at the very beginning of the proof in [7, Theorem 3.1], which introduces a significant gap.

In the present formulation of the theorem, [7, Theorem 3.1] holds only in the special case where $f=g$, which corresponds to [7, Corollary 4.1].

The study in [5] follows the same line of investigation as [7], with the aim of extending the main result to the L^p setting. For the purpose of revision, [5, Theorem 2.1] is restated below.

Theorem 2.2. *[5, Theorem 2.1] Let $a\geq 0$, $p>1$, $q=p/(p-1)$, $f,g:[0,+\infty)\to[0,+\infty)$ be two functions, $k:[0,+\infty)\to[0,1]$ be a function, $\lambda>0$, and $h:[0,+\infty)^2\to[0,+\infty)$ be a symmetric and* *homogeneous function of degree* λ. *Then we have*

$$\begin{aligned}&\left(\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}dxdy\right)^2\\&\leq\left(\left(\Omega_a\int_0^{+\infty}x^{1-\lambda}f^2(x)dx\right)^2-\left(\int_0^{+\infty}\left(\Omega_a k(x)-\Xi_a(x)\right)x^{1-\lambda}f^2(x)dx\right)^2\right)^{1/p}\\&\times\left(\left(\Omega_a\int_0^{+\infty}y^{1-\lambda}g^2(y)dy\right)^2-\left(\int_0^{+\infty}\left(\Omega_a k(y)-\Xi_a(y)\right)y^{1-\lambda}g^2(y)dy\right)^2\right)^{1/q},\end{aligned}$$

where

$$\Omega_a=\int_0^{+\infty}\frac{t^a}{h(1,t)}dt,\quad \Xi_a(x)=\int_0^{+\infty}\frac{t^a k(xt)}{h(1,t)}dt,$$

provided that the integrals under consideration converge.

As for [7, Theorem 3.1], the first step of the proof relies on the equality

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{h(x,y)}(1-k(x)+k(y))dxdy,$$

which, however, is not valid in full generality, as previously discussed.

Furthermore, in the statement of the upper bound, with reference to [5, Proof of Theorem 2.1], the two entries of the function f^2 must be f^p, and the two entries of the function g^2 must be g^q. This makes the influence of the parameters p and q more important.

As a secondary observation, we may consider $a \in \mathbb{R}$ instead of $a \geq 0$ only.

Taking these corrections into account, it follows that, similar to [7, Theorem 3.1], [5, Theorem 2.1] is valid only in the special case where $f = g$.

Despite these corrections, the proof techniques used in [7, Theorem 3.1] and [5, Theorem 2.1] continue to be an important source of inspiration in the study of integral inequalities, including in this article.

2.2 Inspiration for improvement

The proof of [5, Theorem 2.1] is particularly inspiring, especially with regard to the sharpness of the resulting inequality and how it depends on the parameters p and q. In the original proof, the Hölder inequality is applied twice in the initial steps. However, new observations suggest that applying the Hölder inequality once is sufficient to obtain a sharper upper bound. This observation forms the basis of the new theorem developed in the next section.

3 A New Theorem

The result below supplements our revision of [7, Theorem 3.1] and [5, Theorem 2.1] by offering a new sharp Hilbert-type integral inequality in the L^p setting.

Theorem 3.1. *Let $a \in \mathbb{R}$, $p > 1$, $q = p/(p-1)$, $f : [0, +\infty) \to [0, +\infty)$ be a function, $k : [0, +\infty) \to [0, 1]$ be a function, $\lambda > 0$, and $h : [0, +\infty)^2 \to [0, +\infty)$ be a symmetric and homogeneous function of degree λ. Then we have*

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}dxdy\\
&\leq \left(\Omega_{ap}\int_0^{+\infty} x^{1-\lambda} f^p(x)dx - \int_0^{+\infty} x^{1-\lambda}(\Omega_{ap}k(x) - \Xi_{ap}(x))f^p(x)dx\right)^{1/p}\\
&\times \left(\Omega_{aq}\int_0^{+\infty} y^{1-\lambda} f^q(y)dy + \int_0^{+\infty} y^{1-\lambda}(\Omega_{aq}k(y) - \Xi_{aq}(y))f^q(y)dy\right)^{1/q},
\end{aligned}$$

where, for any $\alpha \in \mathbb{R}$,

$$\Omega_\alpha = \int_0^{+\infty} \frac{t^\alpha}{h(1,t)}dt, \quad \Xi_\alpha(x) = \int_0^{+\infty} \frac{t^\alpha k(xt)}{h(1,t)}dt,$$

provided that the integrals under consideration converge.

Proof of Theorem 3.1. The first step corresponds to [7, Lemma 2.1], that we reformulate in a more natural way below. Let us set

$$F(x,y) = 1 - k(x) + k(y).$$

Then, using the symmetry of h and the interchangeability of the product term $f(x)f(y)$, we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}dxdy - \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}k(x)dxdy\\
&+ \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}k(x)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}dxdy - \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}k(x)dxdy\\
&+ \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}k(y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}(1 - k(x) + k(y))dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)}F(x,y)dxdy. \qquad (3)
\end{aligned}$$

Decomposing suitably the integrand, noting that, for any $x, y \geq 0$, $F(x,y) \geq 0$ because $k(x) \in [0,1]$, and using the Hölder integral inequality, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)f(y)}{h(x,y)} F(x,y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \left(\frac{y}{x}\right)^a \frac{f(x)}{h^{1/p}(x,y)} F^{1/p}(x,y) \times \left(\frac{x}{y}\right)^a \frac{f(y)}{h^{1/q}(x,y)} F^{1/q}(x,y)dxdy \\
&\leq A^{1/p}B^{1/q}, \qquad (4)
\end{aligned}$$

where

$$A = \int_0^{+\infty}\int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{f^p(x)}{h(x,y)} F(x,y)dxdy$$

and

$$B = \int_0^{+\infty}\int_0^{+\infty} \left(\frac{x}{y}\right)^{aq} \frac{f^q(y)}{h(x,y)} F(x,y)dxdy.$$

Let us examine the term A.

Using the definition of F, decomposing suitably the integral in two parts, applying the Fubini-Tonelly integral theorem, using the homogeneity of h, and making the change of variables $u = y/x$, we can write

$$\begin{aligned}
A &= \int_0^{+\infty}\int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{f^p(x)}{h(x,y)} F(x,y)dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{f^p(x)}{h(x,y)} (1-k(x))dxdy + \int_0^{+\infty}\int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{f^p(x)}{h(x,y)} k(y)dxdy \\
&= \int_0^{+\infty} f^p(x)(1-k(x)) \int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{1}{h(x,xy/x)} dydx \\
&+ \int_0^{+\infty} f^p(x) \int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{1}{h(x,xy/x)} k\left(x\frac{y}{x}\right) dydx \\
&= \int_0^{+\infty} f^p(x)(1-k(x))x^{1-\lambda} \int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{(1/x)}{h(1,y/x)} dydx \\
&+ \int_0^{+\infty} f^p(x)x^{1-\lambda} \int_0^{+\infty} \left(\frac{y}{x}\right)^{ap} \frac{(1/x)}{h(1,y/x)} k\left(x\frac{y}{x}\right) dydx \\
&= \int_0^{+\infty} f^p(x)(1-k(x))x^{1-\lambda} \int_0^{+\infty} \frac{u^{ap}}{h(1,u)} dudx \\
&+ \int_0^{+\infty} f^p(x)x^{1-\lambda} \int_0^{+\infty} \frac{u^{ap}k(xu)}{h(1,u)} dudx \\
&= \Omega_{ap} \int_0^{+\infty} f^p(x)(1-k(x))x^{1-\lambda}dx + \int_0^{+\infty} f^p(x)x^{1-\lambda}\Xi_{ap}(x)dx \\
&= \Omega_{ap} \int_0^{+\infty} x^{1-\lambda}f^p(x)dx - \int_0^{+\infty} x^{1-\lambda}(\Omega_{ap}k(x) - \Xi_{ap}(x))f^p(x)dx. \qquad (5)
\end{aligned}$$

For the term B, we proceed in a similar manner, but with another initial decomposition, the change of variables $v = x/y$ and the use of the symmetry of h implying that $h(v,1) = h(1,v)$ for any $v \geq 0$. The detailed developments are as follows:

$$
\begin{aligned}
B &= \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{f^q(y)}{h(x,y)}F(x,y)dxdy\\
&= \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{f^q(y)}{h(x,y)}(1+k(y))dxdy - \int_0^{+\infty}\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{f^q(y)}{h(x,y)}k(x)dxdy\\
&= \int_0^{+\infty} f^q(y)(1+k(y))\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{1}{h(yx/y,y)}dxdy\\
&- \int_0^{+\infty} f^q(y)\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{1}{h(yx/y,y)}k\left(y\frac{x}{y}\right)dxdy\\
&= \int_0^{+\infty} f^q(y)(1+k(y))y^{1-\lambda}\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{(1/y)}{h(x/y,1)}dxdy\\
&- \int_0^{+\infty} f^q(y)y^{1-\lambda}\int_0^{+\infty}\left(\frac{x}{y}\right)^{aq}\frac{(1/y)}{h(x/y,1)}k\left(y\frac{x}{y}\right)dxdy\\
&= \int_0^{+\infty} f^q(y)(1+k(y))y^{1-\lambda}\int_0^{+\infty}\frac{v^{aq}}{h(v,1)}dvdy\\
&- \int_0^{+\infty} f^q(y)y^{1-\lambda}\int_0^{+\infty}\frac{v^{aq}k(yv)}{h(v,1)}dvdy\\
&= \Omega_{aq}\int_0^{+\infty} f^q(y)(1+k(y))y^{1-\lambda}dy - \int_0^{+\infty} f^q(y)y^{1-\lambda}\Xi_{aq}(y)dy\\
&= \Omega_{aq}\int_0^{+\infty} y^{1-\lambda}f^q(y)dy + \int_0^{+\infty} y^{1-\lambda}(\Omega_{aq}k(y)-\Xi_{aq}(y))f^q(y)dy. \qquad (6)
\end{aligned}
$$

Combining Equations (3), (4), (5) and (6), we obtain

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy\\
&\leq \left(\Omega_{ap}\int_0^{+\infty} x^{1-\lambda}f^p(x)dx - \int_0^{+\infty} x^{1-\lambda}(\Omega_{ap}k(x)-\Xi_{ap}(x))f^p(x)dx\right)^{1/p}\\
&\times \left(\Omega_{aq}\int_0^{+\infty} y^{1-\lambda}f^q(y)dy + \int_0^{+\infty} y^{1-\lambda}(\Omega_{aq}k(y)-\Xi_{aq}(y))f^q(y)dy\right)^{1/q}.
\end{aligned}
$$

This completes the proof of the theorem. □

In comparison with the revised version of [5, Theorem 2.1], the parameters p and q have a nuanced influence on the upper bound, particularly through their roles in Ω_a and Ξ_a. To emphasize this dependence, the inequality in Theorem 3.1 can be expressed in the form

$$
\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy\\
&\leq \left(\Omega_{ap}\int_0^{+\infty} x^{1-\lambda}f^p(x)dx - \Upsilon_a(p)\right)^{1/p}\left(\Omega_{aq}\int_0^{+\infty} y^{1-\lambda}f^q(y)dy + \Upsilon_a(q)\right)^{1/q},
\end{aligned}
$$

where, for any $\beta > 1$,

$$
\Upsilon_a(\beta) = \int_0^{+\infty} x^{1-\lambda}(\Omega_{a\beta}k(x)-\Xi_{a\beta}(x))f^\beta(x)dx.
$$

As a last remark, when $p = 2$, the inequality in Theorem 3.1 reduces to

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{h(x,y)}dxdy$$
$$\leq \left(\left(\Omega_{2a}\int_0^{+\infty}x^{1-\lambda}f^2(x)dx\right)^2-\left(\int_0^{+\infty}x^{1-\lambda}(\Omega_{2a}k(x)-\Xi_{2a}(x))f^2(x)dx\right)^2\right)^{1/2},$$

which also corresponds to [7, Corollary 4.1] with a instead of $2a$, without loss of generality since $a \in \mathbb{R}$.

4 Applications

Three applications of Theorem 3.1 are given below in the form of three corollaries. Each considers a specific function of H involving the degree of homogeneity λ as an explicit parameter.

Corollary 4.1. *Let $a \in \mathbb{R}$, $p > 1$, $q = p/(p-1)$, $f : [0,+\infty) \to [0,+\infty)$ be a function, and $k : [0,+\infty) \to [0,1]$ be a function. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{(x+y)^\lambda}dxdy$$
$$\leq \left(\Omega_{ap}\int_0^{+\infty}x^{1-\lambda}f^p(x)dx-\int_0^{+\infty}x^{1-\lambda}(\Omega_{ap}k(x)-\Xi_{ap}(x))f^p(x)dx\right)^{1/p}$$
$$\times\left(\Omega_{aq}\int_0^{+\infty}y^{1-\lambda}f^q(y)dy+\int_0^{+\infty}y^{1-\lambda}(\Omega_{aq}k(y)-\Xi_{aq}(y))f^q(y)dy\right)^{1/q},$$

where, for any $\alpha \in \mathbb{R}$,

$$\Omega_\alpha=\int_0^{+\infty}\frac{t^\alpha}{(1+t)^\lambda}dt, \quad \Xi_\alpha(x)=\int_0^{+\infty}\frac{t^\alpha k(xt)}{(1+t)^\lambda}dt,$$

provided that the integrals under consideration converge.

This corollary is a direct application of Theorem 3.1 to the function $h(x,y) = (x+y)^\lambda$, which is clearly symmetric and homogeneous of degree λ.

Another application of this theorem is given below.

Corollary 4.2. *Let $a \in \mathbb{R}$, $p > 1$, $q = p/(p-1)$, $f : [0,+\infty) \to [0,+\infty)$ be a function, and $k : [0,+\infty) \to [0,1]$ be a function. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)f(y)}{x^\lambda+y^\lambda}dxdy$$
$$\leq \left(\Omega_{ap}\int_0^{+\infty}x^{1-\lambda}f^p(x)dx-\int_0^{+\infty}x^{1-\lambda}(\Omega_{ap}k(x)-\Xi_{ap}(x))f^p(x)dx\right)^{1/p}$$
$$\times\left(\Omega_{aq}\int_0^{+\infty}y^{1-\lambda}f^q(y)dy+\int_0^{+\infty}y^{1-\lambda}(\Omega_{aq}k(y)-\Xi_{aq}(y))f^q(y)dy\right)^{1/q},$$

where, for any $\alpha \in \mathbb{R}$,

$$\Omega_\alpha = \int_0^{+\infty} \frac{t^\alpha}{1+t^\lambda} dt, \quad \Xi_\alpha(x) = \int_0^{+\infty} \frac{t^\alpha k(xt)}{1+t^\lambda} dt,$$

provided that the integrals under consideration converge.

This corollary is a direct application of Theorem 3.1 to the function $h(x, y) = x^\lambda + y^\lambda$, which is clearly symmetric and homogeneous of degree λ.

A final application of this theorem involving a more innovative function H is stated below.

Corollary 4.3. *Let* $a \in \mathbb{R}$, $p > 1$, $q = p/(p-1)$, $f : [0, +\infty) \to [0, +\infty)$ *be a function, and* $k : [0, +\infty) \to [0, 1]$ *be a function. Then we have*

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)f(y)}{(x^{\sqrt{\lambda}} + y^{\sqrt{\lambda}})^{\sqrt{\lambda}}} dxdy \\
&\leq \left(\Omega_{ap} \int_0^{+\infty} x^{1-\lambda} f^p(x)dx - \int_0^{+\infty} x^{1-\lambda} (\Omega_{ap} k(x) - \Xi_{ap}(x)) f^p(x)dx \right)^{1/p} \\
&\times \left(\Omega_{aq} \int_0^{+\infty} y^{1-\lambda} f^q(y)dy + \int_0^{+\infty} y^{1-\lambda} (\Omega_{aq} k(y) - \Xi_{aq}(y)) f^q(y)dy \right)^{1/q},
\end{aligned}$$

where, for any $\alpha \in \mathbb{R}$,

$$\Omega_\alpha = \int_0^{+\infty} \frac{t^\alpha}{(1+t^{\sqrt{\lambda}})^{\sqrt{\lambda}}} dt, \quad \Xi_\alpha(x) = \int_0^{+\infty} \frac{t^\alpha k(xt)}{(1+t^{\sqrt{\lambda}})^{\sqrt{\lambda}}} dt,$$

provided that the integrals under consideration converge.

This corollary is a direct application of Theorem 3.1 to the function $h(x, y) = (x^{\sqrt{\lambda}} + y^{\sqrt{\lambda}})^{\sqrt{\lambda}}$, which is clearly symmetric and homogeneous of degree λ.

5 Conclusion

In this article, we revise two existing results on refinements of the Hilbert integral inequality. This involves identifying theoretical gaps and exploring their implications. Using the L^p setting, we present a new Hilbert-type integral inequality with a self-contained proof, demonstrating its applicability through three examples. These findings may inspire further research, including the extension of the inequality to more general kernel functions, multidimensional frameworks and broader L^p settings.

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