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Survival Class of Truncated Quantile Generated Family of Distributions: Properties and Applications

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Abstract

The $T - X(W)$ family of distributions appeared in [1], and in [2] its survival class was defined for a special model. On the other hand in [6] and [7], the $q_T - X$ family of distributions was defined, and it was observed that the range of the CDF of the $q_T - X$ family of distributions is not always $[0, 1]$, and this leads us to introduce its truncated version in the sense of [8]. In this paper, following [2], we introduce the survival class of the truncated $q_T - X$ family of distributions. Some properties and applications are investigated.

1 Introduction

Let $r(t)$ be the probability density function (PDF) of a random variable $T \in [a, b]$, where $-\infty \le a < b \le \infty$. Let $W(F(x))$ be a function of the cumulative distribution function (CDF) $F(x)$ of a random variable X, such that $W(F(x))$ satisfies the following conditions:

(a) $W(F(x)) \in [a, b]$;

(b) $W(F(x))$ is differentiable and monotonically non-decreasing;

(c) $W(F(x)) \to a$ as $x \to -\infty$, and $W(F(x)) \to b$ as $x \to \infty$.

A method for generating new families of distributions is presented as follows:

Definition 1.1. [1] Let X be a random variable with PDF $f(x)$ and CDF $F(x)$, and let T be a continuous random variable with PDF $r(t)$ defined on $[a, b]$. The CDF of the new family of distributions is given by

$$G(x) = \int_a^{W(F(x))} r(t)\, dt = R(W(F(x))),$$

where $R(t)$ is the CDF of the random variable T.

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Remark 1.2. The PDF of the new family can be obtained by differentiating the CDF above.

Remark 1.3. Different $W(F(x))$ will give a new family of distributions. The definition of $W(F(x))$ depends on the support of the random variable T. The following are examples of $W(\cdot)$.

(*a*) When the support of T is bounded, $W(F(x))$ can be defined as $F(x)$ or $F^{\alpha}(x)$.

(*b*) When the support of T is $[a, \infty)$, $a \geq 0$, $W(F(x))$ can be defined as $-\log(1 - F(x))$, $\frac{F(x)}{1-F(x)}$, $-\log(1 - F^{\alpha}(x))$, and $\frac{F^{\alpha}(x)}{1-F^{\alpha}(x)}$.

(*c*) When the support of T is $(-\infty, \infty)$, $W(F(x))$ can be defined as $\log(-\log(1 - F(x)))$, $\log\left[\frac{F(x)}{1-F(x)}\right]$, $\log[-\log(1 - F^{\alpha}(x))]$, and $\log\left[\frac{F^{\alpha}(x)}{1-F^{\alpha}(x)}\right]$.

On the other hand in [2], the survival class of the $T - X$ family of distributions was introduced with the following CDF

$$H(x) = 1 - \int_{a}^{W^{*}(F(x))} r(t)dt, \tag{1}$$

where $r(t)$ is the PDF of the random variable T, $F(x)$ is the CDF of the random variable X and $W^{*}(\cdot)$ is an appropriate weight depending on the support of the random variable T.

Remark 1.4. By differentiating the CDF in (1), we obtain the PDF of the survival class of the $T - X$ family of distributions.

Different $W^{*}(\cdot)$ will give a new family of distributions. When the support of T is $(0, \infty)$ we may take W^{*} as:

(*a*) $W^{*}(F(x)) = -\log(F(x))$ ([3]- [4]);

(*b*) $W^{*}(F(x)) = \frac{-\log(F(x))}{F(x)}$ [5].

By setting $W^{*}(F(x)) = W(1 - F(x))$, where W is given by Remark 1.3, we obtain some examples of $W^{*}(\cdot)$.

In the sense of [6] and [7], the CDF of $q_T - X$ family of distributions is given by

$$J(x) = Q_T(V(F(x))), \tag{2}$$

where Q_T is the quantile function of the random variable T, V is an appropriate weight depending on the support of T, and $F(x)$ is the CDF of the random variable X. However we observed in practice that the range of the CDF of the $q_T - X$ family of distributions is not always $[0, 1]$, this leads us to introduce its truncated version in the sense of [8] as follows

Definition 1.5. Let T be a random variable with quantile Q_T, V be an appropriate weight depending on the support of T, and $F(x)$ be the CDF of the random variable X, then the CDF of the truncated $q_T - X$ family of distributions is given as

$$Z(x) = \frac{Q_T[V[F(x)]] - Q_T[V(0)]}{Q_T[V(1)] - Q_T[V(0)]}.$$

Remark 1.6. By differentiating the CDF above, we obtain the PDF of the truncated $q_T - X$ family of distributions.

Remark 1.7. Different $V(F(x))$ will give a new family of distributions. The definition of $V(F(x))$ depends on the support of the random variable T. The following are examples of $V(\cdot)$.

(a) When the support of T is bounded, $V(F(x))$ can be defined as $F(x)$ or $F(x)^{\frac{1}{\alpha}}$, where $\alpha > 0$.

(b) When the support of T is $[a, \infty)$, $a \geq 0$, $V(F(x))$ can be defined as $1 - e^{-F(x)}$, $\frac{F(x)}{1+F(x)}$, $\left[1 - e^{-F(x)}\right]^{\frac{1}{\alpha}}$, and $\left[\frac{F(x)}{1+F(x)}\right]^{\frac{1}{\alpha}}$, where $\alpha > 0$.

(c) When the support of T is $(-\infty, \infty)$, $V(F(x))$ can be defined as $1 - e^{-e^{F(x)}}$, $\frac{e^{F(x)}}{1+e^{F(x)}}$, $\left[1 - e^{-e^{F(x)}}\right]^{\frac{1}{\alpha}}$, $\left[\frac{e^{F(x)}}{1+e^{F(x)}}\right]^{\frac{1}{\alpha}}$, where $\alpha > 0$.

2 Survival Class of Truncated Quantile Log-Logistic-X Family of Distributions

2.1 The New Family Defined

Following (1), and using Definition 1.5, we have

Definition 2.1. Let T be a random variable with quantile Q_T, V^* be an appropriate weight depending on the support of T, and $F(x)$ be the CDF of the random variable X, then the CDF of the survival class of the truncated $q_T - X$ family of distributions is given as

$$L(x) = 1 - \left(\frac{Q_T[V^*(F(x))] - Q_T[V^*(0)]}{Q_T[V^*(1)] - Q_T[V^*(0)]}\right).$$

Remark 2.2. The PDF of the survival class of the truncated $q_T - X$ family of distributions can be obtained by differentiating the CDF above.

Remark 2.3. By setting $V^*(F(x)) = V(1 - F(x))$, where $V(\cdot)$ is given by Remark 1.7, we obtain some examples of $V^*(\cdot)$.

In this section we assume T is standard Log-Logistic, thus $Q_T(t) = \frac{t}{1-t}$ and $V^*(F(x)) = V(1 - F(x))$, where $V(F(x)) = 1 - e^{-F(x)}$, then from Definition 2.1, we have the following

Proposition 2.4. *The CDF of the survival class of the Truncated Quantile Log-Logistic-X family of distributions is given by*

$$Y(x) = \frac{e^{F(x)} - 1}{e - 1},$$

where $F(x)$ is the CDF of the random variable X.

By differentiating the CDF above we have the following

Proposition 2.5. *The PDF of the survival class of the Truncated Quantile Log-Logistic-X family of distributions is given by*

$$y(x) = \frac{e^{F(x)}}{e - 1} f(x),$$

where $F(x)$ and $f(x)$ are the CDF and PDF, respectively, of the random variable X.

The survival function (SF) of the survival class of the Truncated Quantile Log-Logistic-X family of distributions is given by $S(x) = 1 - Y(x)$, and the hazard function (HF) of the survival class of the Truncated Quantile Log-Logistic-X family of distributions is given by $H(x) = \frac{y(x)}{1-Y(x)}$, where $Y(x)$ and $y(x)$ are given by the two propositions immediately above.

Remark 2.6. When the random variable X is Normal with mean μ, and standard deviation σ, we write, $G \sim TQLLN(\mu, \sigma)$, when G follows the survival class of the Truncated Quantile Log-Logistic-Normal family of distributions.

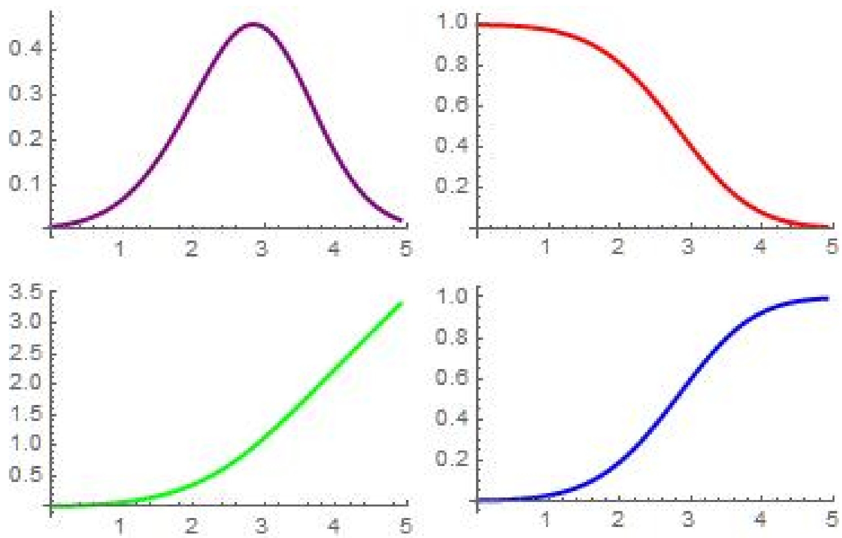

Figure 1: The CDF(blue), PDF(purple), HF(Green), and SF(Red) of $TQLLN(2.50855, 0.895996)$.

2.2 Some Mathematical Properties

2.2.1 Quantile Function

Theorem 2.7. *The quantile function of the survival class of the Truncated Quantile Log-Logistic-X family of distributions is given by*

$$Q(x) = F^{-1}\left[\ln(1 + x(e - 1))\right],$$

where $0 < x < 1$, *and* F^{-1} *is the quantile function of the random variable* X *with CDF* $F(x)$.

Proof. Let $0 < x < 1$. We must solve the following equation for $Q(x)$:

$$x = \frac{e^{F(Q(x))} - 1}{e - 1}.$$

□

Remark 2.8. If the random variable B follows the survival class of the Truncated Quantile Log-Logistic-Exponential distribution, we write $B \sim TQLLE(\lambda)$.

Table 1: Some quantile values of TQLLN and TQLLE

x	$Q(x)$ of TQLLN(2.7,0.8)	$Q(x)$ of TQLLE(0.5)
0.1	1.8997	0.345293
0.2	2.26985	0.700234
0.3	2.52975	1.0748
0.4	2.74642	1.48105
0.5	2.94463	1.93577

2.2.2 Random Number Generation

Random numbers from the survival class of the Truncated Quantile Log-Logistic-X distribution can be obtained via

$$P = F^{-1}[\ln(1 + U(e - 1))],$$

where $U \sim \text{Uniform}(0, 1)$, and F^{-1} is the quantile function of the baseline distribution with CDF F.

2.2.3 rth Non-Central Moments

Theorem 2.9. *The rth non-central moments of the survival class of the Truncated Quantile Log-Logistic-X family of distributions can be expressed as*

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \Omega_{i,n} E(U^n),$$

where $\Omega_{i,n}$ is defined as in the proof of the theorem, and $U \sim$ Uniform$(0,1)$.

Proof. According to [9], we can write

$$Q_X(u) = \sum_{i=0}^{\infty} h_i u^i,$$

where the coefficients are suitably chosen real numbers that depend on the parameters of the $F(x)$ distribution. For a power series raised to a positive integer $r \geq 1$, we have

$$(Q_X(u))^r = \left(\sum_{i=0}^{\infty} h_i u_i \right)^r = \sum_{i=0}^{\infty} \delta_{r,i} u^i,$$

where $\delta_{r,i}$ are obtained from $\delta_{r,i} = (ih_0)^{-1} \sum_{s=1}^{i} [s(r+1) - i] h_s \delta_{r,i-s}$, with $\delta_{r,0} = h_0^r$ for $i = 1, 2, \cdots$ [10]. Thus, we have the following

$$\mu_r' = \sum_{i=0}^{\infty} \delta_{r,i} E\left[\left\{ \ln(1 + U(e-1)) \right\}^i \right],$$

where $E[\cdot]$ is an expectation.

On the other hand, the following power series is well known

$$(\ln(1+u))^r = \sum_{n=r}^{\infty} \frac{r!}{n!} |S_n^r| u^n,$$

where $|S_n^r|$ are the unsigned stirling numbers of the first kind. Thus, we have

$$\left\{ \ln(1 + U(e-1)) \right\}^i = \sum_{n=i}^{\infty} \frac{i!}{n!} |S_n^i| U^n (e-1)^n.$$

Thus

$$\mu_r' = \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \Omega_{i,n} E(U^n),$$

where $\Omega_{i,n} = \delta_{r,i} \frac{i!}{n!} |S_n^i| (e-1)^n$, and $E(\cdot)$ is an expectation. □

For computational purposes, if we let $u = F(x)$, then $du = f(x)dx$, and $x = F^{-1}(u)$, thus the ordinary moments are given by

$$E[X^r] = \int_0^1 (F^{-1}(u))^r du.$$

Table 2: Some ordinary moments of TQLLN and TQLLE

r	$E[X^r]$ of TQLLN(2.7,0.8)	$E[X^r]$ of TQLLE(0.5)
1	2.92245	2.5204
2	9.16002	11.279
3	30.3237	71.6129
4	105.092	589.654
5	379.035	5983.81

2.2.4 Renyi Entropy

Lemma 2.10. *Let $y(x)$ denote the PDF of the survival class of the Truncated Quantile Log-Logistic-X family of distributions, then for $\delta > 0$ and $\delta \neq 1$, $y(x)^{\delta}$ can be expressed as*

$$\sum_{n=0}^{\infty} \Theta_n F(x)^n f(x)^{\delta},$$

where Θ_n is defined as in the proof of the Lemma, $f(x)$ and $F(x)$ are the PDF and CDF, respectively associated with the random variable X.

Proof. By the power series representation of the exponential function we have

$$e^{\delta F(x)} = \sum_{n=0}^{\infty} \frac{\delta^n F(x)^n}{n!}.$$

Thus

$$y(x)^{\delta} = \sum_{n=0}^{\infty} \Theta_n F(x)^n f(x)^{\delta},$$

where $\Theta_n = \frac{\delta^n}{(e-1)^{\delta} n!}$. □

Let X be a random variable with PDF $f(x)$. By definition, the Renyi entropy [11] is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \text{Log}\bigg(\int_{-\infty}^{\infty} f(x)^{\delta} dx \bigg),$$

where $\delta > 0$ and $\delta \neq 1$.. From the above Lemma we have the following

Theorem 2.11. *The Renyi entropy of the survival class of the Truncated Quantile Log-Logistic-X distribution can be expressed as*

$$I_R(\delta) = \frac{1}{1-\delta} Log\bigg(\sum_{n=0}^{\infty} \Theta_n \int_{-\infty}^{\infty} f(x)^{\delta} F(x)^n dx \bigg),$$

where $\delta > 0$, $\delta \neq 1$, and $f(x)$ and $F(x)$ are the PDF and CDF, respectively, associated with the random variable X, and Θ_n is defined as in the proof of the previous Lemma.

For computational purposes if we let $u = F(x)$, then $du = f(x)dx$ and $x = F^{-1}(u)$, thus the Renyi entropy can be expressed as

$$I_R(\delta) = \frac{1}{1-\delta}\log\bigg(\int_0^1 \{f(F^{-1}(u))\}^{\delta-1} du\bigg).$$

Table 3: Some values of the Renyi Entropy for the TQLLN and TQLLE distributions

δ	$I_R(\delta)$ of TQLLN(2.7,0.8)	$I_R(\delta)$ of TQLLE(0.5)
2	1.01857	1.68298
3	0.943877	1.5816
4	0.898633	1.52213
5	0.86768	1.48226
6	0.844909	1.45335

2.3 Parameter Estimation

The methods of maximum likelihood and least squares are used in this paper to estimate model parameters. Here we discuss only the method of maximum likelihood for the survival class of the Truncated Quantile Log-Logistic Weibull distribution (TQLLW for short). We assume in Proposition 2.5, that X follows the two parameter Weibull distribution, that is, $F(x) = 1 - e^{-\left(\frac{x}{b}\right)^a}$, and $f(x) = \frac{ae^{-\left(\frac{x}{b}\right)^a}\left(\frac{x}{b}\right)^{a-1}}{b}$. Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the TQLLW family of distributions. It can be shown that the total log-likelihood function is given by

$$lnL = \sum_{i=1}^{n}\bigg\{-e^{-\left(\frac{x_i}{b}\right)^a} + \log\left(\frac{ae^{-\left(\frac{x_i}{b}\right)^a}\left(\frac{x_i}{b}\right)^{a-1}}{b}\right) + 1 - \log(e-1)\bigg\},$$

where $a, b > 0$. Partial differentiation of the total log-likelihood function with respect to the model parameters gives the following as the score functions

$$\frac{\partial lnL}{\partial a} = \sum_{i=1}^{n}\bigg\{\left(e^{-\left(\frac{x_i}{b}\right)^a}\left(\frac{x_i}{b}\right)^a - \left(\frac{x_i}{b}\right)^a + 1\right)\log\left(\frac{x_i}{b}\right) + \frac{1}{a}\bigg\},$$

$$\frac{\partial lnL}{\partial b} = \sum_{i=1}^{n}\left\{\frac{a\left(-e^{-\left(\frac{x_i}{b}\right)^a}\left(\frac{x_i}{b}\right)^a + \left(\frac{x_i}{b}\right)^a - 1\right)}{b}\right\}.$$

Equating the score functions to zero and numerically solving the resulting system of equations using techniques such as the quasi-Newton-Raphson method yield the maximum likelihood estimates of the model parameters. Let $\triangle = (a, b)$. For the purpose of constructing confidence intervals for the parameters in the TQLLW family of distributions, the observed information matrix, denoted by $J(\triangle)$, can be used due to the complexity of the expected information matrix. The observed information matrix is given by

$$J(\triangle) = -\begin{bmatrix} \frac{\partial^2 lnL}{\partial^2 a} & \frac{\partial^2 lnL}{\partial a \partial b} \\ & \frac{\partial^2 lnL}{\partial^2 b} \end{bmatrix}.$$

When the usual regularity conditions are satisfied and the parameters lie in the interior of the parameter space (i.e., not on the boundary), the distribution of $\sqrt{n}(\widehat{\triangle} - \triangle)$ converges to the multivariate normal distribution $N_2(0, I^{-1}(\triangle))$, where $I(\triangle)$ is replaced by the observed information matrix evaluated at $J(\widehat{\triangle})$. The asymptotic multivariate normal distribution $N_2(0, J^{-1}(\widehat{\triangle}))$ is a very useful tool for constructing approximate $100(1 - \psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

2.4 Monte Carlo Simulation Study

In this section we show that the methods of maximum likelihood and least squares are adequate in estimating the parameters in the $TQLLN(\mu, \sigma)$ distribution. For this a Monte Carlo simulation study is carried out to assess the performance of the estimation method in the $TQLLN(\mu, \sigma)$ model. Samples of sizes 400, 550, 700, and 900 are drawn from the TQLLN distribution, and the samples have been drawn for the following set of parameters

(a) <u>Set I:</u> $(\mu, \sigma) = (2, 5)$

(b) <u>Set II:</u> $(\mu, \sigma) = (5, 5)$

The maximum likelihood and least squares estimators for the parameters μ and σ are obtained. The procedure has been repeated 1000 times and the mean and standard deviation for the estimates are computed, and the results are summarized in Tables 4-7 below for each of sets I and II, respectively, considered above.

2.4.1 Simulation Results Based on the Method of Maximum Likelihood

Table 4: Result of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
400	1.991	0.258
550	1.987	0.2249
700	1.9833	0.1973
900	1.9855	0.1714
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
400	4.9883	0.1765
550	4.992	0.1532
700	4.9942	0.1378
900	4.9957	0.1236

Table 5: Result of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
400	4.9907	0.2578
550	4.987	0.2249
700	4.9833	0.1973
900	4.986	0.1714
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
400	4.9883	0.1765
550	4.992	0.1532
700	4.9942	0.1378
900	4.996	0.1236

From Tables 4-5 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall, the simulation study conducted indicates that the method of maximum likelihood is adequate for estimating the model parameters.

2.4.2 Simulation Results Based on the Method of Least Squares

Table 6: Result of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
400	1.9869	0.2726
550	1.9825	0.2394
700	1.9796	0.2068
900	1.984	0.1802
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
400	4.9939	0.2253
550	4.9964	0.1911
700	4.9957	0.1684
900	4.9957	0.1481

Table 7: Result of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
400	4.9869	0.2726
550	4.9825	0.2394
700	4.9796	0.2068
900	4.984	0.1802
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
400	4.9939	0.2253
550	4.9964	0.1911
700	4.9957	0.1684
900	4.996	0.1481

From Tables 6-7 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall, the simulation study conducted indicates that the method of least squares is adequate for estimating the model parameters.

2.5 Application

Here we demonstrate the usefulness of the TQLLN distribution to the breaking stress of carbon fibers data in Table 2 of [12]. The other distribution we consider is the exponential-N{exponential} distribution in Section 5 of [12]. The CDF of the T-N{exponential} family of distributions is given by

$$F_X(x) = F_T\{-\log(1-\Psi(x))\} \tag{3}$$

and the corresponding PDF is given by

$$f_X(x) = \frac{\psi(x)}{1-\Psi(x)} f_T(-\log(1-\Psi(x))), \tag{4}$$

where f_T and F_T are the PDF and CDF, respectively of the random variable T, and $\psi(x)$ and $\Psi(x)$ are the PDF and CDF, respectively of the standard normal distribution.

Remark 2.12. When the random variable Q follows the exponential-N{exponential} distribution, that is, T follows the exponential distribution in (3) and (4), we write $Q \sim ENE(a)$.

Using the R software, we report below in Table 8, the estimates for the parameters in each of the two distributions alongside their standard errors.

Table 8: Estimates for the parameter of fitted distribution

Distribution	**Parameters**	**Estimates**	**Standard Error**
TQLLN	$\hat{\mu}$	2.5085	0.11204
	$\hat{\sigma}$	0.89596	0.07936
ENE	$\hat{a}$	0.1613	0.01985

The fitted PDF of the TQLLN distribution to the breaking stress of carbon fibers data using the above table is shown below

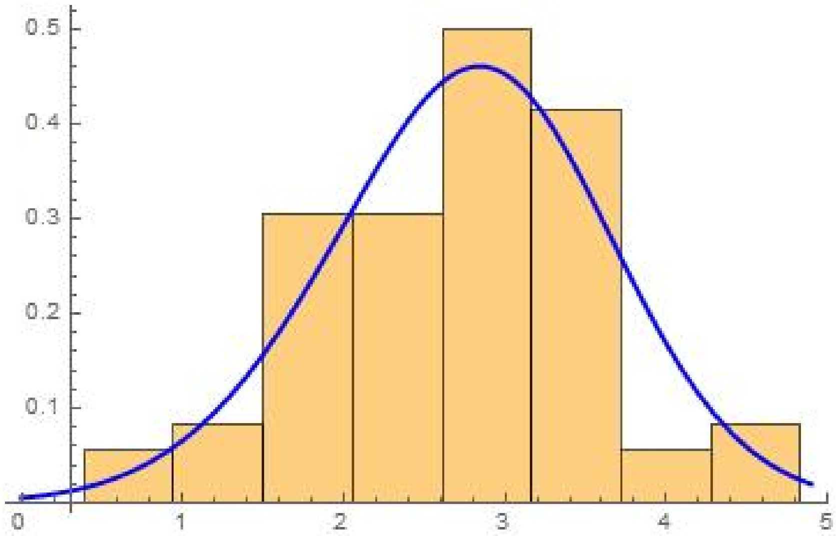

Figure 2: The PDF of TQLLN fitted to the histogram of Table 2 [12].

and the fitted CDF is shown below

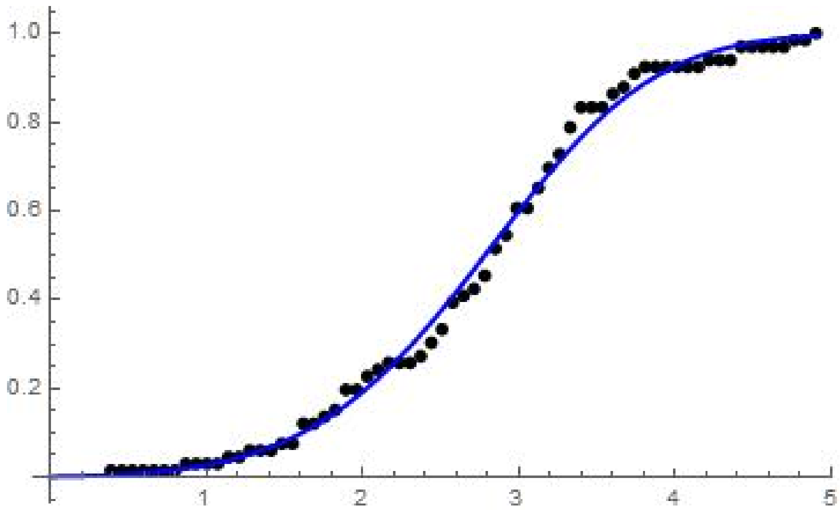

Figure 3: The CDF of TQLLN fitted to the empirical distribution of Table 2 [12].

The measures of goodness of fit considered include the Bayesian Information Criterion (BIC), negative log-likelihood, Cramér-von Mises statistic (W), Anderson-Darling statistic (A), Kolmogorov-Smirnov statistic (KS), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC); these are reported in Table 9 below. Table 9 and the fitted models above reveal that the TQLLN distribution is the most compatible with the data set, and hence can be considered the best in this instance.

Table 9: Goodness of fit measures

	TQLLN	ENE
W	0.0571	0.0925
A	0.3497	0.4981
KS statistic	0.0707	0.29205
KS p-value	0.8961	0.00002578
AIC	174.5919	231.6584
CAIC	174.7823	231.7209
BIC	178.9712	233.8481
HQIC	176.3223	232.5237
-Log(likelihood)	85.29594	114.8292

3 Survival Class of Truncated Quantile Weibull Exponential-X Family of Distributions

3.1 The New Family Defined

In this section we assume T is Weibull Exponential [13], so that the quantile function in standard form is given by $Q_T(x) = \log(1 - \log(1 - x))$ and $V^*(F(x)) = V(1 - F(x))$, where $V(F(x)) = 1 - e^{-F(x)}$, then from Definition 2.1 we have the following

Proposition 3.1. *The CDF of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions is given by*

$$R(x) = \frac{\log(1 + F(x))}{\log(2)},$$

where $F(x)$ is the CDF of the random variable X.

By differentiating the CDF above, we have the following

Proposition 3.2. *The PDF of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions is given by*

$$r(x) = \frac{f(x)}{\log(2)(1 + F(x))},$$

where $f(x)$ and $F(x)$ are the PDF and CDF, respectively of the random variable X.

The hazard function (HF) of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions is given by $H(x) = \frac{r(x)}{1-R(x)}$, and the survival function (SF) is given by $S(x) = 1 - R(x)$, where $r(x)$ and $R(x)$ are given by the previous two propositions.

Remark 3.3. Assuming X follows the normal distribution with mean μ, and standard deviation σ, we write $T \sim TQWEN(\mu, \sigma)$, when T follows the survival class of the Truncated Quantile Weibull Exponential-Normal family of distributions.

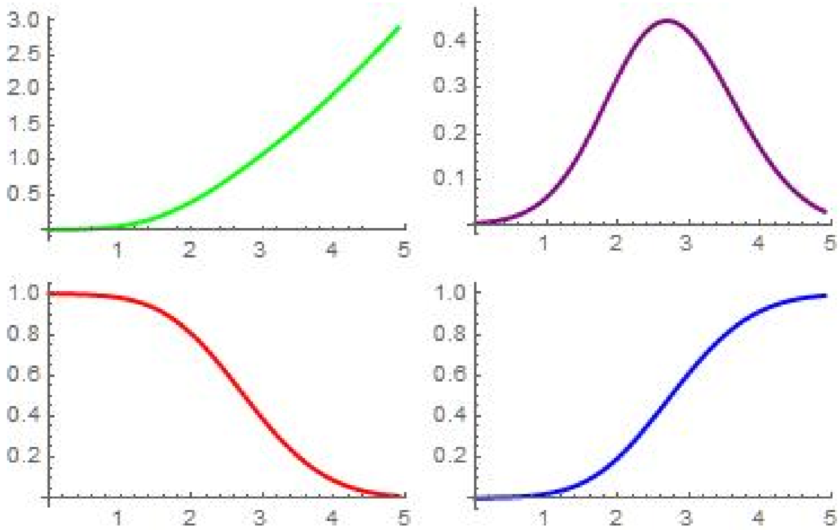

Figure 4: The HF(green), PDF(purple), SF(red), and CDF (blue) of $TQWEN(2.93552, 0.886973)$.

3.2 Some Mathematical Properties

3.2.1 Quantile Function

Theorem 3.4. *The quantile function of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions is given by*

$$Q(x) = F^{-1}(2^x - 1),$$

where $0 < x < 1$ *and* F^{-1} *is the quantile function of the random variable* X *with CDF* $F(x)$.

Proof. Let $0 < x < 1$, we must solve the following equation for $Q(x)$:

$$x = \frac{\log(1 + F(Q(x)))}{\log(2)}.$$

□

Remark 3.5. When the random variable L follows the survival class of the Truncated Quantile Weibull Exponential-Exponential family of distributions we write $L \sim TQWEE(\lambda)$.

Table 10: Some quantile values of the TQWEN and TQWEE distributions

x	$Q(x)$ of TQWEN(2.7,0.8)	$Q(x)$ of TQWEE(0.5)
0.1	1.52983	0.148959
0.2	1.8664	0.321978
0.3	2.11193	0.525704
0.4	2.32474	0.769878
0.5	2.52662	1.0696

3.2.2 Random Number Generation

Random numbers from the survival class of the Truncated Quantile Weibull Exponential -X family of distributions can be obtained via

$$P = F^{-1}(2^U - 1),$$

where $U \sim \text{Uniform}(0, 1)$, and F^{-1} is the quantile function of the random variable X with CDF $F(x)$.

3.2.3 rth Non-Central Moments

Theorem 3.6. *The rth non-central moments of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions can be expressed as*

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{k=0}^{i} \Omega_{i,k} E(2^{kU}),$$

where $\Omega_{i,k}$ is defined as in the proof of the theorem, $U \sim$ Uniform$(0, 1)$, and $E(\cdot)$ denotes an expectation.

Proof. Following the proof of Theorem 2.9, we have

$$\mu'_r = \sum_{i=0}^{\infty} \delta_{r,i} E\left[\left(2^U - 1\right)^i\right].$$

On the other hand by the binomial theorem we have

$$(2^U - 1)^i = \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} 2^{kU}.$$

Thus we obtain

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{k=0}^{i} \Omega_{i,k} E(2^{kU}),$$

where

$$\Omega_{i,k} = \delta_{r,i} \binom{i}{k} (-1)^{i-k}.$$

□

Table 11: Some ordinary moments of TQWEN and TQWEE

r	$E(X^r)$ of TQWEN(2.7,0.8)	$E(X^r)$ of TQWEE(0.5)
1	2.5439	1.67999
2	7.11521	6.20028
3	21.4259	35.8351
4	68.631	281.651
5	231.868	2792.66

3.2.4 Renyi Entropy

Lemma 3.7. *Let $r(x)$ denote the PDF of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions, then for $\delta > 0$, $\delta \neq 1$, $r(x)^\delta$ can be expressed as*

$$\sum_{k=0}^{\infty} \Theta_k F(x)^k f(x)^\delta,$$

where Θ_k is defined as in the proof of the Lemma, $f(x)$ and $F(x)$ are the PDF and CDF, respectively, of the random variable X.

Proof. By the generalized Binomial theorem, we have

$$(1 + F(x))^{-\delta} = \sum_{k=0}^{\infty} \binom{-\delta}{k} F(x)^k.$$

Thus

$$r(x)^\delta = \sum_{k=0}^{\infty} \Theta_k F(x)^k f(x)^\delta,$$

where $\Theta_k = \frac{1}{(\log(2))^\delta} \binom{-\delta}{k}$. □

Theorem 3.8. *The Renyi entropy of the survival class of the Truncated Quantile Weibull Exponential-X family of distributions can be expressed as*

$$I_R(\delta) = \frac{1}{1-\delta} \log \left(\sum_{k=0}^{\infty} \Theta_k \int_{-\infty}^{\infty} F(x)^k f(x)^\delta dx \right),$$

where $\delta > 0$, $\delta \neq 1$, and $f(x)$ and $F(x)$ are the PDF and CDF, respectively, of the random variable X, and Θ_k is defined as in the proof of the previous Lemma.

Table 12: Some values of the Renyi entropy of TQWEN and TQWEE distributions

δ	$I_R(\delta)$ of TQWEN(2.7,0.8)	$I_R(\delta)$ of TQWEE(0.5)
2	1.04447	1.14151
3	0.972528	0.965529
4	0.928893	0.859391
5	0.898995	0.787248
6	0.876968	0.734508

3.3 Parameter Estimation

The methods of maximum likelihood and least squares are used in this paper to estimate model parameters. Here we discuss only the method of maximum likelihood for the survival class of the Truncated Quantile Weibull Exponential Weibull distribution (TQWEW for short). We assume in Proposition 3.2, that X follows the two parameter Weibull distribution as defined in Section 2.3. Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the TQWEW family of distributions. It can be shown that the total log-likelihood function is given by

$$\ln(L) = \sum_{i=1}^{n} \left\{ \log \left(\frac{a e^{-\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a}{x_i} \right) - \log \left(\log(4) - \log(2) e^{-\left(\frac{x_i}{b}\right)^a} \right) \right\},$$

where $a, b > 0$. Partial differentiation of the total log-likelihood function with respect to the model parameters gives the following as the score functions

$$\frac{\partial \ln(L)}{\partial a} = \sum_{i=1}^{n} \left\{ \left(\frac{2e^{\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a}{1 - 2e^{\left(\frac{x_i}{b}\right)^a}} + 1 \right) \log\left(\frac{x_i}{b}\right) + \frac{1}{a} \right\},$$

$$\frac{\partial \ln(L)}{\partial b} = \sum_{i=1}^{n} \left\{ \frac{2a\left(\left(\frac{x_i}{b}\right)^a - 1\right) e^{\left(\frac{x_i}{b}\right)^a} + a}{b\left(2e^{\left(\frac{x_i}{b}\right)^a} - 1\right)} \right\}.$$

Equating the score functions to zero and numerically solving the resulting system of equations using techniques such as the quasi-Newton-Raphson method yield the maximum likelihood estimates of the model parameters. Let $\triangle = (a, b)$. For the purpose of constructing confidence intervals for the parameters in the TQWEW family of distributions, the observed information matrix, denoted by $J(\triangle)$, can be used due to the complexity of the expected information matrix. The observed information matrix is given by

$$J(\triangle) = -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial^2 a} & \frac{\partial^2 \ln L}{\partial a \partial b} \\ & \frac{\partial^2 \ln L}{\partial^2 b} \end{bmatrix}.$$

When the usual regularity conditions are satisfied and the parameters lie in the interior of the parameter space (i.e., not on the boundary), the distribution of $\sqrt{n}(\widehat{\triangle} - \triangle)$ converges to the multivariate normal distribution $N_2(0, I^{-1}(\triangle))$, where $I(\triangle)$ is replaced by the observed information matrix evaluated at $J(\widehat{\triangle})$. The asymptotic multivariate normal distribution $N_2(0, J^{-1}(\widehat{\triangle}))$ is a very useful tool for constructing approximate $100(1-\psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

3.4 Monte Carlo Simulation Study

In this section we show that the methods of maximum likelihood and least squares are adequate in estimating the parameters in the $TQWEN(\mu, \sigma)$ distribution. For this a Monte Carlo simulation study is carried out to assess the performance of the estimation method in the $TQWEN(\mu, \sigma)$ model. Samples of sizes 200, 400, 600, 800 are drawn from the TQWEN model, and the samples have been drawn for the following set of parameters

(*a*) Set I: $(\mu, \sigma) = (7, 12)$

(*b*) Set II: $(\mu, \sigma) = (7, 7)$

The maximum likelihood and least square estimators for the parameters μ and σ are obtained. The procedure has been repeated 1000 times and the mean and standard deviation for the estimates are computed, and the results are summarized in Tables 13-16 below for each of sets I and II, respectively, considered above.

3.4.1 Result of Simulation Study Based on the Method of Maximum Likelihood

Table 13: Result of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	6.9478	0.9012
400	6.9601	0.6242
600	6.9573	0.5132
800	6.95557	0.4349
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	11.9448	0.6173
400	11.9695	0.4167
600	11.9782	0.3465
800	11.98326	0.3016

Table 14: Result of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	6.9695	0.5257
400	6.9767	0.3641
600	6.9751	0.2994
800	6.97408	0.2537
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	6.968	0.36008
400	6.9822	0.2431
600	6.9873	0.2021
800	6.99024	0.1759

From Tables 13-14 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall, the simulation study conducted indicates that the method of maximum likelihood is adequate for estimating the model parameters.

3.4.2 Simulation Results Based on the Method of Least Squares

Table 15: Results of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	6.9542	0.9465
400	6.9605	0.6542
600	6.95317	0.5412
800	6.9485	0.462
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	11.957	0.7619
400	11.9808	0.5328
600	11.98101	0.4331
800	11.98123	0.3696

Table 16: Results of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	6.9733	0.5521
400	6.977	0.3816
600	6.973	0.3157
800	6.96998	0.2695
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	6.97482	0.444
400	6.9888	0.3108
600	6.989	0.2526
800	6.989	0.216

From Tables 15-16 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall, the simulation study conducted indicates that the method of least squares is adequate for estimating the model parameters.

3.5 Application

Here we demonstrate the usefulness of the TQWEN distribution to the breaking stress of carbon fibers data in Table 2 of [12]. The other distribution we consider is the exponential-N{exponential} distribution as discussed in Section 2.5 of this paper.

Using the R software, we report below in Table 17, the estimates for the parameters in the TQWEN distribution alongisde their standard errors.

Table 17: Estimates for parameter of fitted distribution

Distribution	**Parameters**	**Estimates**	**Standard error**
TQWEN	$\hat{\mu}$	2.9355	0.11205
	$\hat{\sigma}$	0.886962	0.07723

The fitted PDF of the TQWEN distribution to the breaking stress of carbon fibers data using the above table is shown below

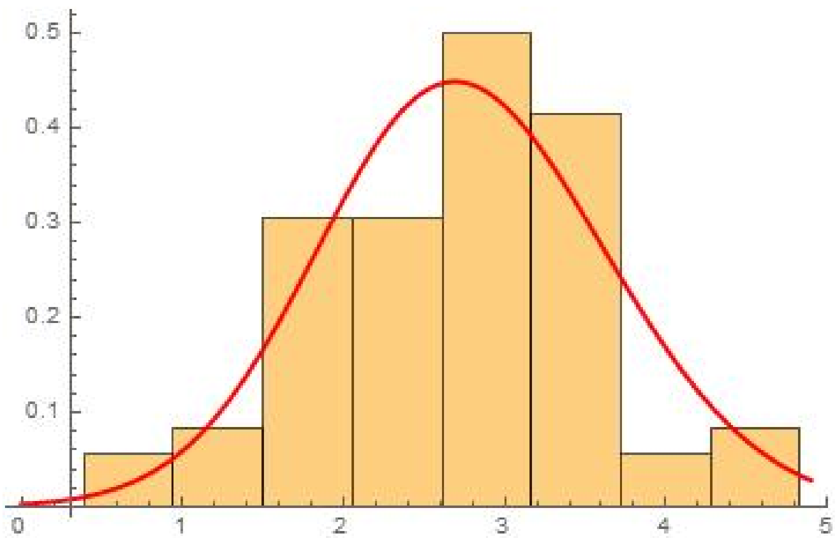

Figure 5: The PDF of TQWEN fitted to the histogram of Table 2 [12].

and the fitted CDF is shown below

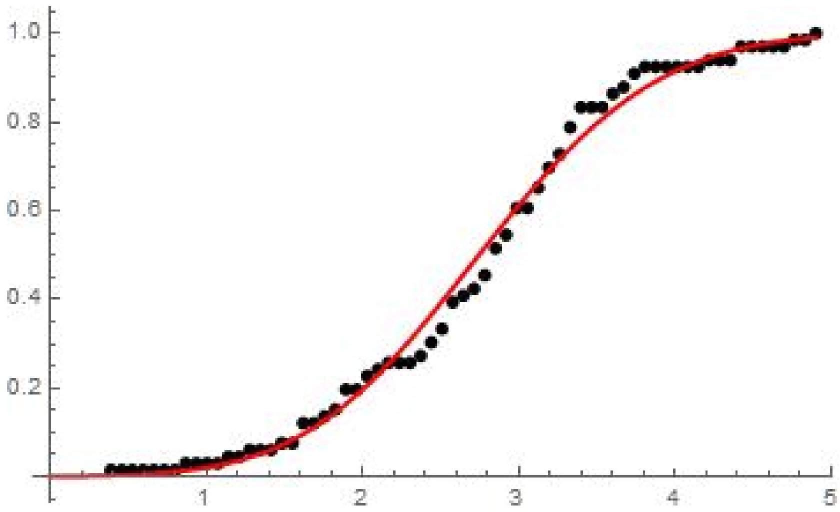

Figure 6: The CDF of TQWEN fitted to the empirical distribution of Table 2 [12].

The measures of goodness of fit considered include the Bayesian Information Criterion (BIC), negative log-likelihood, Cramér-von Mises statistic (W), Anderson–Darling statistic (A), Kolmogorov-Smirnov statistic (KS), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC); these are reported in Table 18 below. Comparing Tables 9 and 18 with the fitted models above, we observe that the TQWEN distribution is more compatible with the data set than the ENE distribution discussed in Section 2.5, and hence can be considered the best in this instance.

Table 18: Goodness of fit measures

	TQWEN
W	0.0894
A	0.4835
KS statistic	0.0798
KS p-value	0.7948
AIC	175.7224
CAIC	175,9129
BIC	180.1018
HQIC	177.4529
-Log(likelihood)	85.86122

4 Survival Class of Truncated Quantile Logistic-X Family of Distributions

4.1 The New Family Defined

In this section we assume that T follows the Logistic distribution, so that the quantile function in standard form is given by $Q_T(x) = \log(\frac{x}{1-x})$, and $V^*(F(x)) = V(1 - F(x))$, where $V(F(x)) = 1 - e^{-e^{F(x)}}$, then from Definition 2.1 we have the following

Proposition 4.1. *The CDF of the survival class of the Truncated Quantile Logistic-X family of distributions is given by*

$$K(x) = \frac{A - \log\left(e^{e^{F(x)}} - 1\right)}{B},$$

where $A = \log(-1 + e)$, $B = \log(-1 + e) - \log(-1 + e^e)$, and $F(x)$ is the CDF of the random variable X.

By differentiating the CDF above, we have the following

Proposition 4.2. *The PDF of the survival class of the Truncated Quantile Logistic-X family of distributions is given by*

$$k(x) = \frac{f(x)e^{F(x)+e^{F(x)}}}{B - Be^{e^{F(x)}}},$$

where $f(x)$ and $F(x)$ are the PDF and CDF, respectively of the random variable X, and B is the constant given in Proposition 4.1.

The survival function (SF) of the survival class of the Truncated Quantile Logistic-X family of distributions is given by $S(x) = 1 - K(x)$, and the hazard function (HF) of the survival class of the Truncated Quantile Logistic-X family of distributions is given by $H(x) = \frac{k(x)}{1-K(x)}$, where $K(x)$ and $k(x)$ are given by the two propositions immediately above.

Remark 4.3. When the random variable X is Normal with mean μ and standard deviation σ, we write $E \sim TQLN(\mu, \sigma)$, when E follows the surivval class of the Truncated Quantile Logistic-Normal family of distributions.

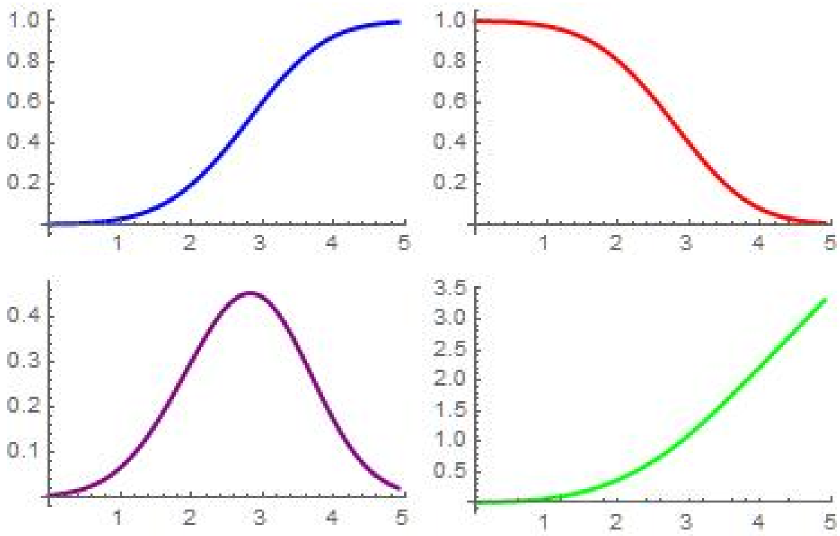

Figure 7: SF(red), PDF(purple), HF(Green), and CDF(Blue) of $TQLN(2.60303, 0.880452)$.

4.2 Some Mathematical Properties

4.2.1 Quantile Function

Theorem 4.4. *The quantile function of the survival class of the Truncated Quantile Logistic-X family of distributions is given by*

$$Q(x) = F^{-1}\left\{ \log(\log(1 + e^{A-Bx})) \right\},$$

where F^{-1} is the quantile function of the random variable X with CDF $F(x)$, and A and B are the constants appearing in Proposition 4.1.

Proof. Let $0 < x < 1$, and A and B be the constants appearing in Proposition 4.1, we must solve the following equation for $Q(x)$

$$x = \frac{A - \log(-1 + e^{e^{F(Q(x))}})}{B}.$$

$\square$

Remark 4.5. If the random variable C follows the survival class of the Truncated Quantile Logistic-Exponential distribution we write $C \sim TQLE(\lambda)$.

Table 19: Some quantile values of TQLN and TQLE distributions

x	$Q(x)$ of TQLN(2.7,0.8)	$Q(x)$ of TQLE(0.5)
0.1	1.79734	0.277582
0.2	2.16504	0.580284
0.3	2.42802	0.914376
0.4	2.65012	1.28926
0.5	2.85493	1.71973

4.2.2 Random Number Generation

Random numbers from the survival class of the Truncated Quantile Logistic-X family of distributions can be obtained via

$$X = F^{-1}\Big\{ \log(\log(1+e^{A-BU})) \Big\},$$

where F^{-1} is the quantile function of the random variable X with CDF $F(x)$, A and B are the constants appearing in Proposition 4.1, and $U \sim \text{Uniform}(0,1)$.

4.2.3 rth Non-Central Moments

Theorem 4.6. *The rth non-central moments of the survival class of the Truncated Quantile Logistic-X family of distributions can be expressed as*

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{n=i}^{\infty}\sum_{k=0}^{n}\sum_{q=k}^{\infty}\sum_{r=0}^{\infty}\sum_{m=0}^{r} \Theta_{i,n,k,q,r,m} E(U^m),$$

where $\Theta_{i,n,k,q,r,m}$ is defined as in the proof of the theorem, $U \sim$ Uniform$(0,1)$, and $E(\cdot)$ denotes an expectation.

Proof. Following the proof of Theorem 2.9, we have

$$\mu_r' = \sum_{i=0}^{\infty} \delta_{r,i} E\left[\Big\{ \log(\log(1+e^{A-BU})) \Big\}^i\right].$$

On the other hand the following power series is well known

$$(\ln(x))^r = \sum_{n=r}^{\infty} s(n,r)\frac{r!}{n!}(x-1)^n,$$

where $s(n, r)$ is the signed Stirling numbers of the first kind. Thus, we have

$$\left\{ \log(\log(1+e^{A-BU})) \right\}^i = \sum_{n=i}^{\infty} s(n,i) \frac{i!}{n!} \left(\log(1+e^{A-BU}) - 1 \right)^n.$$

By the Binomial theorem, we have

$$\left(\log(1+e^{A-BU}) - 1 \right)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\log(1+e^{A-BU}))^k.$$

On the other hand we have

$$(\log(1+e^{A-BU}))^k = \sum_{q=k}^{\infty} s(q,k) \frac{k!}{q!} e^{q(A-BU)}.$$

By the power series representation for the exponential function, we have

$$e^{q(A-BU)} = \sum_{r=0}^{\infty} \frac{q^r (A-BU)^r}{r!}$$

and by the Binomial theorem, we have

$$(A-BU)^r = \sum_{m=0}^{r} \binom{r}{m} A^{r-m} (-1)^m B^m U^m.$$

Thus

$$\mu'_r = \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \sum_{k=0}^{n} \sum_{q=k}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{r} \Theta_{i,n,k,q,r,m} E(U^m)$$

where

$$\Theta_{i,n,k,q,r,m} = \delta_{r,i} s(n,i) \frac{i!}{n!} \binom{n}{k} (-1)^{n-k} s(q,k) \frac{k!}{q!} \frac{q^r}{r!} \binom{r}{m} A^{r-m} (-1)^m B^m,$$

where A and B are the constants appearing in Proposition 4.1. □

Table 20: Some ordinary moments of TQLN and TQLE distributions

r	$E(X^r)$ of TQLN(2.7,0.8)	$E(X^r)$ of TQLE(0.5)
1	2.83883	2.32953
2	8.70253	10.1042
3	28.3131	63.2727
4	96.7879	517.617
5	345.307	5236.37

4.2.4 Renyi Entropy

Lemma 4.7. *Let $k(x)$ denote the PDF of the survival class of the Truncated Quantile Logistic-X family of distributions. Then for $\delta > 0$, $\delta \neq 1$, $k(x)^\delta$ can be expressed as*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{q,z,l,j=0}^{\infty}\Omega_{n,k,q,z,l,j}F(x)^{n-k+q+j}f(x)^\delta,$$

where $F(x)$ and $f(x)$ are the CDF and PDF, respectively of the random variable X, and $\Omega_{n,k,q,z,l,j}$ is defined as in the proof of the Lemma.

Proof. By the power series representation of the exponential function, we have

$$e^{\delta(F(x)+e^{F(x)})} = \sum_{n=0}^{\infty}\frac{\delta^n(F(x)+e^{F(x)})^n}{n!}.$$

By the binomial theorem, we have

$$(F(x)+e^{F(x)})^n = \sum_{k=0}^{n}\binom{n}{k}F(x)^{n-k}e^{kF(x)}.$$

By the power series representation of the exponential function, we have

$$e^{kF(x)} = \sum_{q=0}^{\infty}\frac{k^qF(x)^q}{q!}.$$

On the other hand by the generalized binomial theorem, we have

$$(B-Be^{e^{F(x)}})^{-\delta} = \sum_{z=0}^{\infty}\binom{-\delta}{z}B^{-\delta-z}(-1)^zB^ze^{ze^{F(x)}}.$$

By the power series representation of the exponential function, we have the following

$$e^{ze^{F(x)}} = \sum_{l=0}^{\infty}\frac{z^le^{lF(x)}}{l!}$$

and

$$e^{lF(x)} = \sum_{j=0}^{\infty}\frac{l^jF(x)^j}{j!}.$$

Thus, collectively, we have

$$k(x)^\delta = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{q,z,l,j=0}^{\infty}\Omega_{n,k,q,z,l,j}F(x)^{n-k+q+j}f(x)^\delta,$$

where

$$\Omega_{n,k,q,z,l,j} = \frac{\delta^n}{n!}\binom{n}{k}\frac{k^q}{q!}\binom{-\delta}{z}B^{-\delta}(-1)^z\frac{z^l}{l!}\frac{l^j}{j!}$$

and B is the constant appearing in Proposition 4.1. ∎

Theorem 4.8. *The Renyi entropy of the survival class of the Truncated Quantile Logistic-X family of distributions can be expressed as*

$$I_R(\delta) = \frac{1}{1-\delta} \log \left(\sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{q,z,l,j=0}^{\infty} \Omega_{n,k,q,z,l,j} \int_{-\infty}^{\infty} F(x)^{n-k+q+j} f(x)^{\delta} dx \right),$$

where $\delta > 0$, $\delta \neq 1$, and $f(x)$ and $F(x)$ are the PDF and CDF, respectively of the random variable X, and $\Omega_{n,k,q,z,l,j}$ is defined as in the proof of the previous Lemma.

Table 21: Some values of the Renyi entropy for the TQLN and TQLE distributions

δ	$I_R(\delta)$ of TQLN(2.7,0.8)	$I_R(\delta)$ of TQLE(0.5)
2	1.04449	1.58179
3	0.972562	1.46113
4	0.928932	1.38785
5	0.899036	1.3374
6	0.877009	1.30003

4.3 Parameter Estimation

The methods of maximum likelihood and least squares are used in this paper to estimate model parameters. Here we discuss only the method of maximum likellihood for the survival class of the Truncated Quantile Logistic-Weibull distribution (TQLW for short). We assume in Proposition 4.2, that X follows the two parameter Weibull distribution as defined in Section 2.3. Suppose $x_1, x_2, \cdots, x_n$ is a random sample of size n from the TQLW family of distributions. It can be shown that the total log-liikelihood function is given by

$$\ln(L) = \sum_{i=1}^{n} \Bigg\{ - e^{-\left(\frac{x_i}{b}\right)^a} + e^{1-e^{-\left(\frac{x_i}{b}\right)^a}} + \log\left(\frac{a e^{-\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a}{x_i} \right) - \log\left(\left(\log\left(e^e - 1\right) - \log(e-1)\right) \left(e^{e^{1-e^{-\left(\frac{x_i}{b}\right)^a}}} - 1 \right) \right) + 1 \Bigg\},$$

where $a, b > 0$. Partial differentiation of the total log-likelihood function with respect to the model parameters gives the following as the score functions

$$\frac{\partial \ln(L)}{\partial a} = \sum_{i=1}^{n} \left\{ \left(1 - \frac{\left(e^{-\left(\frac{x_i}{b}\right)^a} \left(-e^{e^{1-e^{-\left(\frac{x_i}{b}\right)^a}}} \left(e^{\left(\frac{x_i}{b}\right)^a} - 1 \right) - e^{1-e^{-\left(\frac{x_i}{b}\right)^a}} - 1 \right) + 1 \right) \left(\frac{x_i}{b}\right)^a}{1 - e^{e^{1-e^{-\left(\frac{x_i}{b}\right)^a}}}} \right) \log\left(\frac{x_i}{b}\right) + \frac{1}{a} \right\}$$

$$\begin{aligned} \frac{\partial \ln(L)}{\partial b} = \sum_{i=1}^{n} \Bigg\{ & \frac{a\left(\left(\frac{x_i}{b}\right)^a - 1\right)}{b} - \frac{a e^{-\left(\frac{x_i}{b}\right)^a} \left(\frac{x_i}{b}\right)^a}{b} - \frac{a e^{-\left(\frac{x_i}{b}\right)^a - e^{-\left(\frac{x_i}{b}\right)^a} + 1} \left(\frac{x_i}{b}\right)^a}{b} \\ & - \frac{a\left(\frac{x_i}{b}\right)^a \exp\left(-\left(\frac{x_i}{b}\right)^a - e^{-\left(\frac{x_i}{b}\right)^a} + e^{1-e^{-\left(\frac{x_i}{b}\right)^a}} + 1 \right)}{b - b e^{e^{1-e^{-\left(\frac{x_i}{b}\right)^a}}}} \Bigg\}. \end{aligned}$$

Equating the score functions to zero and numerically solving the resulting system of equations using techniques such as the quasi-Newton-Raphson method yield the maximum likelihood estimates of the model parameters. Let $\triangle = (a, b)$. For the purpose of constructing confidence intervals for the parameters in the TQLW family of distributions, the observed information matrix, denoted by $J(\triangle)$, can be used due to the complexity of the expected information matrix. The observed information matrix is given by

$$J(\triangle) = -\begin{bmatrix} \frac{\partial^2 \ln(L)}{\partial^2 a} & \frac{\partial^2 \ln(L)}{\partial a \partial b} \\ & \frac{\partial^2 \ln(L)}{\partial^2 b} \end{bmatrix}.$$

When the usual regularity conditions are satisfied and the parameters lie in the interior of the parameter space (i.e., not on the boundary), the distribution of $\sqrt{n}(\widehat{\triangle} - \triangle)$ converges to the multivariate normal distribution $N_2(0, I^{-1}(\triangle))$, where $I(\triangle)$ is replaced by the observed information matrix evaluated at $J(\widehat{\triangle})$. The asymptotic multivariate normal distribution $N_2(0, J^{-1}(\widehat{\triangle}))$ is a very useful tool for constructing approximate $100(1-\psi)\%$ two-sided confidence intervals for the model parameters, where ψ is the significance level.

4.4 Monte Carlo Simulation Study

In this section we show that the methods of maximum likelihood and least squares are adequate in estimating the parameters in the $TQLN(\mu, \sigma)$ distribution. For this a Monte Carlo simulation study is carried out to assess the performance of the estimation method in the $TQLN(\mu, \sigma)$ model. Samples of sizes 200, 400, 600, and 800 are drawn from the TQLN model, and the samples have been drawn for the following set of parameters

(a) Set I: $(\mu, \sigma) = (0.5, 2.7)$

(b) Set II: $(\mu, \sigma) = (2.7, 2.7)$

The maximum likelihood and least squares estimators for the parameters μ and σ are obtained. The procedure has been repeated 1000 times and the mean and standard deviation for the estimates are computed, and the results are summarized in Tables 22-25 below for each of sets I and II, respectively considered above.

4.4.1 Simulation Results Based on the Method of Maximum Likelihood

Table 22: Result of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	0.4938	0.2023
400	0.4942	0.1391
600	0.4925	0.1147
800	0.4915	0.0979
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	2.688	0.1389
400	2.69357	0.0937
600	2.6956	0.0779
800	2.6968	0.068

Table 23: Result of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	2.6938	0.2023
400	2.6941	0.1391
600	2.6925	0.1147
800	2.6915	0.0979
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	2.688	0.1389
400	2.69357	0.0937
600	2.6956	0.0779
800	2.6968	0.068

From Tables 23–24 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall, the simulation study conducted indicates that the method of maximum likelihood is adequate for estimating the model parameters.

4.4.2 Simulation Results Based on the Method of Least Squares

Table 24: Result of simulation study for Set I

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	0.493	0.2133
400	0.4924	0.1465
600	0.4908	0.121
800	0.4898	0.1032
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	2.691	0.1718
400	2.6964	0.11995
600	2.6966	0.0975
800	2.6968	0.0832

Table 25: Result of simulation study for Set II

	Parameter μ	
Sample Size	Average Estimate	Standard Deviation
200	2.6929	0.2133
400	2.6924	0.1465
600	2.6908	0.121
800	2.6898	0.1032
	Parameter σ	
Sample Size	Average Estimate	Standard Deviation
200	2.691	0.1718
400	2.6964	0.11995
600	2.6966	0.0975
800	2.697	0.0832

From Tables 24-25 above, we find that the simulated estimates are close to the true values of the parameters; hence, the estimation method is adequate. We also observe that the estimated standard deviation consistently decreases with increasing sample size, as can be seen by plotting the standard deviation against the sample size.

Overall the simulation study conducted, indicated that using the method of least squares in estimating model parameters is adequate.

4.5 Application

Here we demonstrate the usefulness of the TQLN distribution to the breaking stress of carbon fibers data in Table 2 of [12]. The other distribution we consider is the exponential-N{exponential} distribution as discussed in Section 2.5 of this paper.

Using the R software, we report below in Table 26, the estimates for the parameters in the TQLN distribution alongside their standard errors.

Table 26: Estimates for the parameter of fitted distribution

Distribution	**Parameters**	**Estimates**	**Standard error**
TQLN	$\hat{\mu}$	2.6030	0.1099
	$\hat{\sigma}$	0.8804	0.0766

The fitted PDF of the TQLN distribution to the breaking stress of carbon fibers data using the above table is shown below

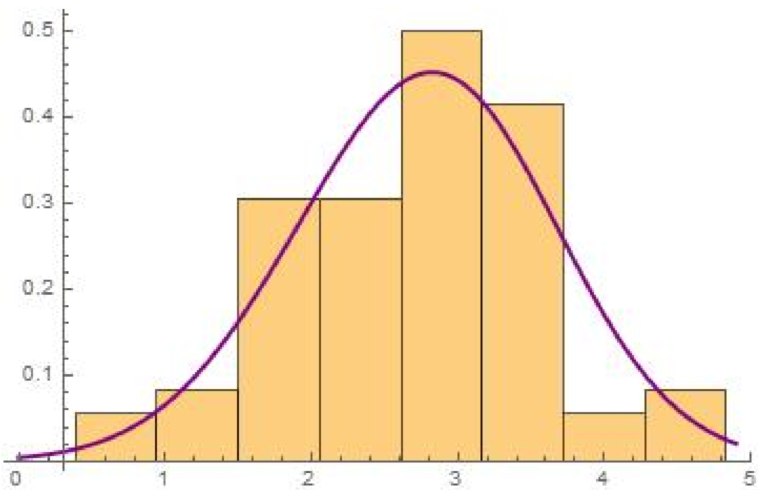

Figure 8: The PDF of TQLN fitted to the histogram of Table 2 [12].

and the fitted CDF is shown below

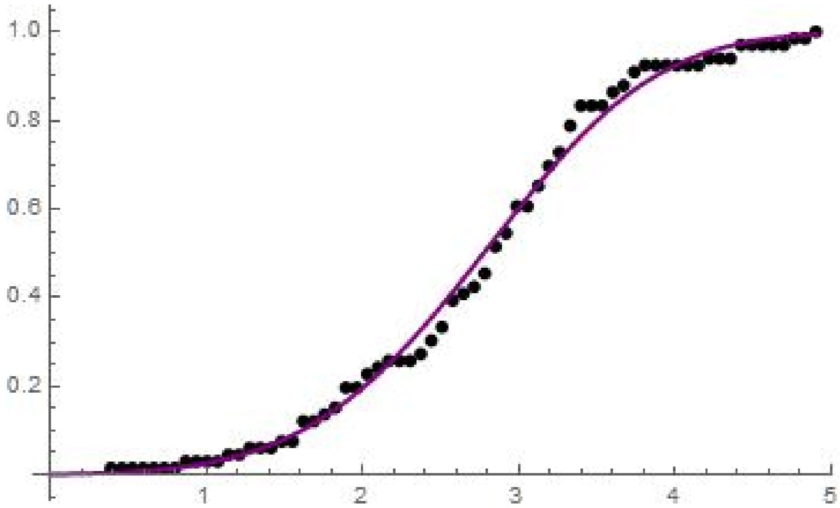

Figure 9: The CDF of TQLN fitted to the empirical distribution of Table 2 [12].

The measures of goodness of fit we consider are discusseed in Sections 2.5 and 3.5 of this paper, and they are reported in Table 27 below for the TQLN distribution. Comparing Table 9 and Table 27 with the fits above, we see that the TQLN distribution is most compatible with the data set as compared to the ENE distribution of Section 2.5, and hence can be considered the best in this instance.

Table 27: Goodness of fit measures

	TQLN
W	0.0638
A	0.379
KS statistic	0.0718
KS p-value	0.8852
AIC	174.8376
CAIC	175.0281
BIC	179.217
HQIC	176,5681
-Log(likelihood)	85.41882

5 Further Recommendation

In this section we assume that T follows the exponentiated Gumbel distribution [14], so that the quantile function in standard form is given by $Q_T(x) = -\log(-\log(x))$, and $V^*(F(x)) = V(1 - F(x))$, where $V(F(x)) = 1 - e^{-e^{F(x)}}$, then from Definition 2.1, we have the following

Proposition 5.1. *The CDF of the survival class of the Truncated Quantile Exponentiated Gumbel-X family of distributions is given by*

$$Z(x) = -\frac{\log\left(\frac{\log\left(1-e^{-e^{F(x)}}\right)}{\log(e-1)-1}\right)}{\log\left(\frac{\log(e-1)-1}{\log(e^e-1)-e}\right)},$$

where $F(x)$ is the CDF of the random variable X.

By differentiating the CDF above, we have the following

Proposition 5.2. *The PDF of the survival class of the Truncated Quantile Exponentiated Gumbel-X family of distributions is given by*

$$z(x) = -\frac{f(x)e^{F(x)}}{\left(e^{e^{F(x)}}-1\right)\log\left(\frac{\log(e-1)-1}{\log(e^e-1)-e}\right)\log\left(1-e^{-e^{F(x)}}\right)},$$

where $f(x)$ and $F(x)$ are the PDF and CDF, respectively, of the random variable X.

One of the future interesting problems is to obtain some properties and applications of the new family of distributions.

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