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New Huygens Type Trigonometric Inequalities

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Abstract

In this paper, some Huygens type inequalities involving trigonometric functions are refined and sharpened. We thus improve established inequalities and provide new ones.

1 Introduction

The following trigonometric inequalities

$$\frac{2}{3}\left(\frac{\sin x}{x}\right)+\frac{\tan x}{3x}>1,$$

and

$$\frac{1}{2}\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{2x}>1,$$

for $x \in (0, \pi/2)$, are known as the Huygens and Wilker inequalities respectively. They have attracted a lot of researchers, motivating the development of more refined variants. We refer to [12], [4], [3], [13], [14], [16], and [19] for further exploration.

As an extension the following inequality chain is established:

$$\frac{1}{2}\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{2x}>1+\frac{x^3\tan x}{15}>\frac{1}{\cos x}\left(\frac{\sin x}{x}\right)^3$$

$$>1+x^3\left(1-\frac{x^2}{63}\right)\frac{\tan x}{15}>\frac{2}{3}\left(\frac{\sin x}{x}\right)+\frac{\tan x}{3x}.$$

Many extensions of above results have been made as well as many refinements.

It is proved in [2] that

$$\frac{1}{2}\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{2x}>\frac{2}{3}\left(\frac{\sin x}{x}\right)+\frac{\tan x}{3x}>1. \qquad (E)$$

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This inequality chain demonstrates significant importance, revealing the close relationship between Huygens and Wilker inequalities.

The study in [12] pioneered an extension of the basic versions of the Wilker and Huygens inequalities, by connecting them in a series of inequalities dependent on a parameter n. Indeed, as demonstrated in [7], these inequalities are nested in chains of inequalities.

In [15], Sumner and al. improved the Wilker inequality and provided for $x \in (0, \pi/2)$

$$2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45} x^3 \tan x. \tag{1}$$

where the constants $\left(\frac{2}{\pi}\right)^4$ and $\frac{8}{45}$ are the best possible. In [16], Wu and Srivastava established for $x \in (0, \pi/2)$

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2.$$

In [10] and [11], Hua as well as Jiang and al proposed the following inequalities for $|\, x \,| \in (0, \pi/2)$

$$3 + \frac{1}{40} x^3 \sin x < \frac{\sin x}{x} + 4\frac{\tan \frac{x}{2}}{x} < 3 + \frac{80 - 24\pi}{\pi^4} x^3 \sin x. \tag{2}$$

In [2] Theorem 3, Chen and Cheung established the following sharp Huygens-type inequalities for $x \in (0, \pi/2)$

$$3 + \frac{3}{20} x^3 \tan x < 2\frac{\sin x}{x} + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x \tag{3}$$

and

$$3 + \frac{3}{20} x^4 + \frac{3}{56} x^5 \tan x < 2\frac{\sin x}{x} + \frac{\tan x}{x} < 3 + \frac{3}{20} x^4 + \left(\frac{2}{\pi}\right)^4 x^5 \tan x. \tag{4}$$

In [5] Remark 3.4, Chen and Paris showed the inequality for $x \in (0, \pi/2)$

$$3 < 2\frac{x}{\sin x} + \frac{x}{\tan x} < 3 + \frac{1}{60} x^3 \tan x. \tag{5}$$

For more information on this topic, please refer to the newly published articles [3], [4], [6], [7], [8], [12] and closely related references therein.

In this work, inspired by these results, we will establish analogous inequalities involving the trigonometric functions. They are extensions of the inequalities mentioned above.

Let us list below new results about the refinements and sharpness of the Huygens and Wilker type inequalities (some of them are announced in [8]).

More precisely, we prove the following inequalities for $|\, x \,| \in (0, \pi/2)$ analog to (E)

$$x < 3\tan\frac{x}{3} < \frac{40}{x^3}\left(\frac{\sin x}{x} + 4\frac{\tan \frac{x}{2}}{x} - 3\right) < 2\tan\frac{x}{2} < \frac{60}{x^3}\left(\frac{x}{\sin x} + \frac{x}{\tan \frac{x}{2}} - 3\right) < \tan x,$$

$$3\tan\frac{x}{3} < \frac{40}{x^3}\left(\frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} - 3\right) < 2\tan\frac{x}{2} < \frac{720}{17x^3}\left(\frac{x}{\sin x} + \frac{x^2}{4\left(\tan\frac{x}{2}\right)^2} - 2\right) < \tan x.$$

We will use for that computational methods that are often resource-intensive. This will be done using *Maple*, even though these calculations can sometimes be lengthy.

2 Main Results

The following Lemmas will be very useful in the sequel.

Lemma 2.0.0 *For $0 < x < \frac{\pi}{2}$ the following inequalities holds for any $n \geq 0$*

$$\sum_{0}^{2n+1}(-1)^k\frac{x^{2k+1}}{(2k+1)!} < \sin x < \sum_{0}^{2n}(-1)^k\frac{x^{2k+1}}{(2k+1)!},$$

$$\sum_{0}^{2n+1}(-1)^k\frac{x^{2k}}{(2k)!} < \cos x < \sum_{0}^{2n}(-1)^k\frac{x^{2k}}{(2k)!}.$$

Lemma 2.0.1 *For $0 < x < \frac{\pi}{2}$ and for an integer $n > 0$ the following inequalities hold*

$$\sin x < x < .... < (n+1)\tan\frac{x}{n+1} < n\tan\frac{x}{n} < .... < \tan x. \tag{6}$$

Indeed, the function $f(n,x) = n\tan\frac{x}{n}$ is decreasing with n. The derivative with respect to x of $f(n,x) - f(n+1,x)$ is

$$\left(\tan\frac{x}{n}\right)^2 - \left(\tan\frac{x}{n+1}\right)^2 = \sec^2(\frac{x}{n}) - \sec^2(\frac{x}{n+1}) > 0.$$

Therefore one gets inequalities (6).

As we will see in the sequel (6) will be improved

2.1 New trigonometric inequalities

The following which is not listed among the known inequalities refine inequalities (6)

Theorem 2.1.1 *For $0 < x < \frac{\pi}{2}$ the following inequalities hold*

$$3 + \frac{1}{30}x^3\tan\frac{x}{2} < \frac{x}{\sin x} + \frac{x}{\tan\frac{x}{2}} < 3 + \frac{1}{60}x^3\tan x. \tag{7}$$

Proof of Theorem 2.1.1 Let us prove the right side of (7) and consider the difference

$$A(x) = \frac{\left(\frac{x}{\sin x} + \frac{x}{\tan \frac{x}{2}} - 3\right)}{x^3 (\tan x)} =$$

$$\frac{-x\cos 2x - x - 4\,x\cos x + 3\,\sin 2x}{-x^3 + x^3\cos 2x} - \frac{1}{60}.$$

The numerator of $A(x)$ is

$$a(x) = -x\cos 2x - x - 4\,x\cos x + 3\,\sin 2x + \frac{1}{60}\,x^3 - \frac{1}{60}\,x^3\cos 2x =$$

$$\left(-x - \frac{1}{60}\,x^3\right)\cos 2x + \frac{1}{60}\,x^3 - x - 4\,x\cos x + 3\,\sin 2x.$$

It is easy to remark that by Lemma 2.0.0

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!},$$

$$\cos 2x < 1 - 2\,x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8,$$

$$\sin 2x > 2\,x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \frac{4}{2835}x^9 - \frac{8}{155925}x^{11}.$$

By regrouping we then deduce

$$a(x) > \left(-x - \frac{1}{60}\,x^3\right)\left(1 - 2\,x^2 + 2/3\,x^4 - \frac{4}{45}\,x^6 + \frac{2}{315}\,x^8\right) -$$

$$\frac{239}{60}\,x^3 + 5\,x - 4\,x\left(1 - 1/2\,x^2 + 1/24\,x^4 - \frac{1}{720}\,x^6 + \frac{1}{40320}\,x^8\right) +$$

$$4/5\,x^5 - \frac{8}{105}\,x^7 + \frac{4}{945}\,x^9 - \frac{8}{51975}\,x^{11} =$$

$$\frac{1}{140}\,x^7 - \frac{37}{50400}\,x^9 - \frac{1}{3850}\,x^{11} > 0,$$

since the polynomial has no root for $0 < x < \frac{\pi}{2}$. Therefore, $a(x) > 0$ and thus $A(x) < 0$.

Let us prove the left side and consider the quotient

$$B(x) = \frac{\left(\frac{x}{\sin x} + \frac{x}{\tan \frac{x}{2}} - 3\right)}{x^3 \left(\tan \frac{x}{2}\right)} - \frac{1}{30} =$$

$$-\frac{x\cos x + 2\,x - 3\,\sin x}{x^3\,(-1 + \cos x)} - \frac{1}{30}.$$

The numerator of $B(x)$ may be written

$$b(x) = -x\cos x - 2\,x + 3\,\sin x - \frac{x^3}{30}\,(-1 + \cos x) =$$

$$\left(-x-\frac{x^3}{30}\right)\cos x - 2\,x + 3\,\sin x + \frac{x^3}{30}.$$

By Lemma 2.0.0, $b(x)$ is less than

$$b(x) < \left(-x-\frac{x^3}{30}\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\right) + x - \frac{7x^3}{15} + \frac{x^5}{40} - \frac{x^7}{1680} + \frac{x^9}{120960} =$$

$$\frac{11}{201600}\,x^9 - \frac{1}{1680}\,x^7 < 0$$

since $0 < x < \frac{\pi}{2}$. We then deduce $b(x) < 0$ and thus $B(x) > 0$.

Corollary 2.1.2 *For $0 < x < \frac{\pi}{2}$ the following inequalities holds*

$$3 + \frac{1}{60}x^3 \sin x < \frac{x}{\sin x} + \frac{x}{\tan \frac{x}{2}} < 3 + \frac{1}{60}x^3 \tan x.$$

Remark 2.1.0 We may also deduce the following inequalities

$$\sin x < x < 2\tan\frac{x}{2} < \frac{60}{x^3}\left(\frac{x}{\sin x} + \frac{x}{\tan\frac{x}{2}} - 3\right) < \tan x.$$

The following also refine inequalities (6)

Theorem 2.1.3 *For $0 < x < \frac{\pi}{2}$ the following inequalities hold*

$$2 + \frac{17}{360}x^3 \tan\frac{x}{2} < \frac{x}{\sin x} + \frac{1}{4}\left(\frac{x}{\tan\frac{x}{2}}\right)^2 < 2 + \frac{17}{720}x^3 \tan x. \tag{8}$$

Proof of Theorem 2.1.3 For the left inequality of (8) consider the quotient

$$C(x) = \frac{\frac{x}{\sin x} + \frac{1}{4}\left(\frac{x}{\tan\frac{x}{2}}\right)^2 - 2}{x^3 \tan\frac{x}{2}} - \frac{17}{360} =$$

$$-\frac{8\,(\cos x)^2 + x^2\,(\cos x)^2 + 2x^2\,\cos x + 4x\,\sin x - 8 + x^2}{4x^3 \sin x\,(-1+\cos x)} - \frac{17}{360} =$$

$$\frac{8\,\cos 2x - 8 + x^2 \cos 2x + 3\,x^2 + 4x^2\,\cos x + 8x\,\sin x}{8\,x^3 \sin x - 4\,x^3 \sin 2x} - \frac{17}{360}.$$

Its numerator can be written

$$c(x) = \left(x^2+8\right)\cos 2x + 4x^2\,\cos x - 8 + 3\,x^2 + 8x\,\sin x - \frac{17}{360}\left(8x^3\,\sin x - 4\,x^3 \sin 2x\right) =$$

$$\left(8\,x - \frac{17}{45}\,x^3\right)\sin x + \left(x^2+8\right)\cos 2x + 4x^2\,\cos x - 8 + 3\,x^2 + \frac{17}{90}\,x^3 \sin 2x.$$

Notice that by Lemma 2.0.0 one obtains the following

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!},$$

$$\cos 2x > 1 - 2\,x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8 - \frac{4}{14175}x^{10},$$
$$\sin 2x > 2\,x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{315}x^7 + \frac{4}{2835}x^9 - \frac{8}{155925}x^{11},$$
$$\sin x > x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11}.$$

By combining these calculations we find

$$c(x) > \left(8\,x - \frac{17}{45}\,x^3\right)\left(x - 1/6\,x^3 + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7 + \frac{1}{362880}\,x^9 - \frac{1}{39916800}\,x^{11}\right) +$$
$$\left(x^2+8\right)\left(1 - 2\,x^2 + 2/3\,x^4 - \frac{4}{45}\,x^6 + \frac{2}{315}\,x^8 - \frac{4}{14175}\,x^{10}\right) +$$
$$4\,\left(1 - 1/2\,x^2 + 1/24\,x^4 - \frac{1}{720}\,x^6 + \frac{1}{40320}\,x^8 - \frac{1}{3628800}\,x^{10}\right)x^2 - 8 + 3\,x^2 +$$
$$\frac{17}{90}\,x^3\left(2\,x - 4/3\,x^3 + \frac{4}{15}\,x^5 - \frac{8}{315}\,x^7 + \frac{4}{2835}\,x^9 - \frac{8}{155925}\,x^{11}\right) =$$
$$-\frac{1079}{59875200}\,x^{12} - \frac{527}{54432000}\,x^{14} + \frac{1}{504}\,x^8 - \frac{11}{21600}\,x^{10} > 0,$$

since the polynomial has no root for $0 < x < \frac{\pi}{2}$. Therefore, $c(x) > 0$ and thus $C(x) > 0$.

Turn now to the right side of (8) and consider the quotient

$$D(x) = \frac{\frac{x}{\sin x} + \frac{1}{4}\left(\frac{x}{\tan\frac{x}{2}}\right)^2 - 2}{x^3 \tan x} - \frac{17}{720} =$$
$$\frac{\cos x\left(x^2\left(\cos x\right)^2 + 8\,\left(\cos x\right)^2 + 2x^2\cos x + 4x\sin x - 8 + x^2\right)}{4x^3\left(\sin x\right)^3} - \frac{17}{720}.$$

Its numerator is

$$d(x) = \cos x\left(x^2\left(\cos x\right)^2 + 8\,\left(\cos x\right)^2 + 2x^2\cos x + 4x\sin x - 8 + x^2\right) - \frac{17}{180}\,x^3\left(\sin x\right)^3.$$

Using Lemma 2.0.0 by the same way we find

$$x^2\left(\cos x\right)^2 + 8\,\left(\cos x\right)^2 < 8 - 7\,x^2 + \frac{5}{3}x^4 - \frac{1}{45}x^6 - \frac{2}{105}x^8 + \frac{29}{14175}x^{10},$$
$$2x^2\cos x < 2\,x^2 - x^4 + \frac{1}{12}x^6 - \frac{1}{360}x^8,$$
$$4x\,\sin x < 4\,x^2 - \frac{2}{3}x^4 + \frac{1}{30}x^6 - \frac{1}{1260}x^8,$$
$$\frac{17}{180}\,x^3\left(\sin x\right)^3 > \frac{17}{180}\,x^6 - \frac{17}{360}\,x^8 + \frac{221}{21600}\,x^{10}.$$

By combining these calculations we find after simplifying

$$d(x) < \frac{29}{340200}\,x^{14} - \frac{1783}{907200}\,x^{12} + \frac{1601}{226800}\,x^{10} - \frac{19}{840}\,x^8.$$

The last polynomial is negative for $0 < x < \frac{\pi}{2}$. We then deduce $D(x) < 0$.

By direct computation we may easily derive the following which improves slightly inequalities (6)

Corollary 2.1.4 *For* $0 < x < \frac{\pi}{2}$ *the following inequalities holds*

$$\sin x < x < 2\tan\frac{x}{2} < \frac{720}{17x^3}\left(\frac{x}{\sin x} + \frac{x^2}{4\left(\tan\frac{x}{2}\right)^2} - 2\right) < \tan x.$$

2.2 On the Inequality (2)

Hua proved the following trigonometric inequalities, Theorem 1 p. 496 of [10]

$$3 + \frac{1}{40}x^3\sin x < \frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} < 3 + \frac{80 - 24\pi}{\pi^4}x^3\sin x. \qquad (2)$$

Next result provide trigonometric inequalities which improve slightly this result as well as inequalities (6).

Theorem 2.2.1 *For* $0 < x < \frac{\pi}{2}$ *the following inequalities hold*

$$\frac{3}{40}x^3\tan\frac{x}{3} \;<\; \frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} - 3 \;<\; \frac{1}{20}x^3\tan\frac{x}{2}. \qquad (9)$$

Proof of Theorem 2.2.1 For the right inequality of (8) consider the function

$$E(x) = \left(\frac{\sin x}{x} + 4\,\frac{\tan\frac{x}{2}}{x} - 3\right)\frac{1}{x^3\tan\frac{x}{2}} =$$

$$\frac{\sin x\cos x - 3x\,\cos x + 5\,\sin x - 3\,x}{x^4\sin x} = \frac{\sin 2x - 6x\,\cos x + 10\,\sin x - 6\,x}{2x^4\sin x}.$$

Its derivative is

$$\frac{2\,\cos 2x + 6\,x\sin x + 4\,\cos x - 6}{2x^4\sin x} - 2\,\frac{\sin 2x - 6x\,\cos x + 10\,\sin x - 6\,x}{x^5\sin x} -$$

$$\frac{(\sin 2x - 6x\,\cos x + 10\,\sin x - 6\,x)\cos x}{2x^4\left(\sin x\right)^2} =$$

$$-\frac{4\,\left(\cos x\right)^2 + x\sin x\cos x + 16\,\cos x + 8\,x\sin x + 3\,x^2 - 20}{\left(-1 + \cos x\right)x^5}.$$

Let us consider the numerator

$$e(x) = 4\,\left(\cos x\right)^2 + x\sin x\cos x + 16\,\cos x + 8\,x\sin x + 3\,x^2 - 20 =$$

$$2\,\cos 2x - 18 + \frac{1}{2}x\sin 2x + 16\,\cos x + 8\,x\sin x + 3\,x^2.$$

Its derivatives are

$$e'(x) = -\frac{7}{2}\sin 2x + x\cos 2x - 8\,\sin x + 8x\cos x + 6x,$$

$$e''(x) = -6\,\cos 2x - 2\,x\sin 2x - 8\,x\sin x + 6.$$

By Lemma 2.0.0, it is easy to remark that for $0 < x < \frac{\pi}{2}$

$$\cos 2x > 1 - 2\,x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8 - \frac{4}{14175}x^{10},$$

$$2x\sin 2x > 4\,x^2 - \frac{8}{3}x^4 + \frac{8}{15}x^6 - \frac{16}{315}x^8,$$

$$8x\sin x > 8\,y^2 - \frac{4}{3}y^4 + \frac{1}{15}y^6 - \frac{1}{630}y^8.$$

It follows

$$e''(x) < -6(1 - 2\,x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{315}x^8 - \frac{4}{14175}x^{10})-$$

$$(4\,x^2 - \frac{8}{3}x^4 + \frac{8}{15}x^6 - \frac{16}{315}x^8) - (8\,y^2 - \frac{4}{3}y^4 + \frac{1}{15}y^6 - \frac{1}{630}y^8) =$$

$$-\frac{1}{15}\,y^6 + \frac{1}{70}\,y^8 + \frac{8}{4725}\,y^{10}.$$

The last polynomial has no root for $0 < x < \frac{\pi}{2}$. Then $e''(x) < 0$ implying $e'(x) < 0$ and $B(x)$ is decreasing. $E(0) = 0$ implies $E(x) < 0$ then $E(x)$ is decreasing. Thus, $E(x) < lim_{x\to 0}E(x) = \frac{1}{20}$.

Prove now the left inequality of (8). Let us consider the function

$$\frac{\sin x}{x} + 4\,\frac{\tan\frac{x}{2}}{x} - 3 - \frac{3}{40}\,x^3\tan\frac{x}{3} = \frac{A(x)}{B(x)},$$

$$A(x) = 20\,\sin 2x + 20\,\sin\frac{4x}{3} + 180\,\sin\frac{2x}{3} + 20\,\sin\frac{5x}{3} + 200\,\sin x +$$

$$180\,\sin\frac{x}{3} - 120\,x\cos x - 120\,x\cos\frac{x}{3} - 3\,x^4\sin x + 3\,\sin\frac{x}{3}x^4 -$$

$$120\,x\cos\frac{2x}{3} - 120\,x - 3\,x^4\sin\frac{2x}{3},$$

$$B(x) = 40\,x\cos\frac{x}{3} + 40\,x\cos(x) + 40\,x\cos\frac{2x}{3} + 40\,x.$$

After changing $x = 3y$ the numerator $A(x)$ can be written

$$A(y) = 20\,\sin 6y + 20\,\sin 4y + 180\,\sin 2y + 20\,\sin 5y + 200\,\sin 3y +$$

$$180\,\sin y - 360\,y\cos 3y - 360\,y\cos y - 243\,y^4\sin 3y + 243y^4\sin y -$$

$$360\,y\cos 2y - 360\,y - 243\,y^4\sin 2y =$$

$$40\,\sin 2y\,(\cos 4y + \cos 2y + 5 + \cos 3y + 10\,\cos y) -$$

$$360\,y\,(\cos 3y + \cos y + \cos 2y + 1) - 243\,y^4\,(\sin 3y - \sin y + \sin 2y)\,.$$

Moreover, by Lemma 2.0.0 the following inequalities hold

$$40\,\sin 2y\,(\cos 4y + \cos 2y + 5 + \cos 3y + 10\,\cos y) <$$

$$1440\,y - 2520\,y^3 + 2442\,y^5 - \frac{11041}{7}y^7 + \frac{5509}{8}y^9,$$

$$360\,y\,(\cos 3y + \cos y + \cos 2y + 1) > 1440\,y - 2520\,y^3 + 1470\,y^5 - 397\,y^7 + \frac{487}{8}y^9 - \frac{4291}{720}y^{11},$$

$$243\,y^4\,(\sin 3y - \sin y + \sin 2y) > (972\,y^5 - 1377\,y^7 + \frac{11097}{20}y^9 - \frac{31239}{280}y^{11}.$$

Regrouping these inequalities we deduce that

$$A(y) < \frac{1377}{7}\,y^7 + \frac{729}{10}\,y^9 - \frac{11600261}{55440}\,y^{11} < 0,$$

since the last polynomial has no root for $0 < x < \pi/2$ or for $0 < y < \pi/6$.

Remark 2.2.2 (i) - Comparing the two right inequalities of (2) and (8) we find for $x < x_1 = 1.512377337 < \frac{\pi}{2}$

$$\frac{\sin x}{x} + 4\,\frac{\tan\frac{x}{2}}{x} - 3 < \frac{1}{20}\tan\frac{x}{2} < \frac{(80 - 24\,\pi)\sin x}{\pi^4},$$

and for $x_1 < x < \frac{\pi}{2}$ we have

$$\frac{\sin x}{x} + 4\,\frac{\tan\frac{x}{2}}{x} - 3 < \frac{(80 - 24\,\pi)\sin x}{\pi^4} < \frac{1}{20}\tan\frac{x}{2}.$$

(ii) - Comparing now the two left inequalities of (2) and (8), the one given by Theorem 2.2.1 is better than the one of Theorem 1, [10]. Indeed, we may verify for $0 < x < \frac{\pi}{2}$

$$\frac{1}{40}x^3\sin x < \frac{3}{40}x^3\tan\frac{x}{3} < \frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} - 3.$$

Indeed, a quick computation yields

$$\sin x - 3\tan\frac{x}{3} = \frac{\left(-3 + 4\,\left(\cos\frac{x}{3}\right)^3 - \cos\frac{x}{3}\right)\sin\frac{x}{3}}{\cos\frac{x}{3}} =$$

$$\frac{\left(\cos\frac{x}{3} - 1\right)\left(4\,\left(\cos\frac{x}{3}\right)^2 + 4\,\cos\frac{x}{3} + 3\right)\sin\frac{x}{3}}{\cos\frac{x}{3}} < 0$$

for $0 < x < \frac{\pi}{2}$.

As a corollary we may derive the following

Corollary 2.2.3 *For $0 < x < \frac{\pi}{2}$ the following inequalities hold*

$$x < 3\tan\frac{x}{3} < \frac{40}{x^3}\left(\frac{\sin x}{x} + 4\,\frac{\tan\frac{x}{2}}{x} - 3\right) < 2\tan\frac{x}{2} < \tan x.$$

3 Conclusion

Finally, by combining all these results, we can highlight inequalities of the type (E) mentioned earlier in the introduction. This allows us to compare inequalities of the Huygens type. We may derive from above ((6) and Theorems 2.1.1, 2.1.3, and 2.2.1) following results

Theorem 3.1.1 *For $0 < x < \frac{\pi}{2}$ the following inequality chain holds*

$$3\tan\frac{x}{3} < \frac{40}{x^3}\left(\frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} - 3\right) < 2\tan\frac{x}{2} < \frac{60}{x^3}\left(\frac{x}{\sin x} + \frac{x}{\tan\frac{x}{2}} - 3\right) < \tan x.$$

Theorem 3.1.2 *For $0 < x < \frac{\pi}{2}$ the following inequality chain holds*

$$3\tan\frac{x}{3} < \frac{40}{x^3}\left(\frac{\sin x}{x} + 4\frac{\tan\frac{x}{2}}{x} - 3\right) < 2\tan\frac{x}{2} < \frac{720}{17x^3}\left(\frac{x}{\sin x} + \frac{x^2}{4\left(\tan\frac{x}{2}\right)^2} - 2\right) < \tan x.$$

As corollaries

Corollary 3.1.3 *For $0 < x < \frac{\pi}{2}$ the following inequality holds*

$$\frac{2\sin x}{x} + \frac{8\tan\frac{x}{2}}{x} < \frac{3x}{\sin x} + \frac{3x}{\tan\frac{x}{2}} - 3.$$

Corollary 3.1.4 *For $0 < x < \frac{\pi}{2}$ the following inequality holds*

$$\frac{17\sin x}{x} + \frac{68\tan\frac{x}{2}}{x} < 15 + \frac{18x}{\sin x} + \frac{9x^2}{2\left(\tan\frac{x}{2}\right)^2}.$$

However, one question remains: which theorems, 2.1.3 or 2.2.1, is stronger?

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