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On a New Unit Distribution Incorporating an Adjustable Function and a Shape Parameter

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Abstract

In this article, we introduce a new unit distribution characterized by the inclusion of an adjustable function and a shape parameter, which together provide greater flexibility for modelling proportions, rates, and other fractional data. We investigate several of its fundamental properties, emphasizing its adaptability and broad potential for application. The proposed distribution extends existing classes of unit distributions and offers a promising framework for further theoretical developments and practical applications.

1 Introduction

The term unit distribution typically refers to a (continuous probability) distribution with support the unit interval $(0, 1)$. The unit distributions are widely used in statistical modeling, particularly for representing proportions, rates, and other fractional data. A comprehensive overview of the most notable unit distributions can be found in the survey [1], to which we also refer the reader for a further 185 key references.

In this article, we make a theoretical contribution to the field by introducing a new unit distribution, which incorporates an adjustable function, denoted by ϕ, and a shape parameter, denoted by α. This added flexibility allows the proposed model to capture a wider range of distributional behaviors, thereby enhancing its suitability for analyzing proportions, rates, and other fractional data in various statistical applications.

We investigate the main characteristics of the proposed distribution, including the cumulative distribution function (CDF), the probability density function (PDF), and the quantile function (QF). Plots of selected functions are given. In addition, we derive several fundamental properties, such as distributional and moment results, and propose a general family of distributions that extends the initial construction. A number of illustrative examples are provided to highlight the main features and potential applicability of the model.

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The rest of the article is composed of three sections: Section 2 presents the new unit distribution by its associated CDF. Additional functions and results are given in Section 3. A conclusion is provided in Section 4.

2 A New Unit Distribution

The CDF presented in the theorem below is at the heart of our study.

Theorem 2.1. *The following function is a valid CDF of a unit distribution:*

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2x-1))\right), \quad x \in (0,1)$$

with $\alpha > 0$*, where* ϕ *is a strictly increasing differentiable odd function on* $[-\alpha, \alpha]$ *with* $\phi(0) = 0$*, and* F *is completed by setting* $F(x) = 0$ *for any* $x \leq 0$ *and* $F(x) = 1$ *for any* $x \geq 1$*.*

Proof. Let us examine the limit of F at the points 0 and 1. Using the fact that ϕ is an odd function, we get

$$\begin{aligned}\lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+} \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2x-1))\right) = \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(-\alpha)\right)\\ &= \frac{1}{2}\left(1 - \frac{1}{\phi(\alpha)}\phi(\alpha)\right) = \frac{1}{2}(1-1) = 0 = F(0).\end{aligned}$$

We also have

$$\begin{aligned}\lim_{x\to 1^-} F(x) &= \lim_{x\to 1^-} \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2x-1))\right) = \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha)\right)\\ &= \frac{1}{2}(1+1) = 1 = F(1).\end{aligned}$$

Therefore, F is continuous at 0 and 1, and so in $\mathbb{R}$; 0 and 1 were the only points demanding a special investigation.

Using standard differentiation rules, for any $x \in (0,1)$, we get

$$F'(x) = \frac{\alpha}{\phi(\alpha)}\phi'(\alpha(2x-1)).$$

Since $\alpha > 0$, $\phi(\alpha) > \phi(0) = 0$ and ϕ is a strictly increasing differentiable function so that $\phi'(\alpha(2x-1)) > 0$, we have $F'(x) > 0$. Therefore, F is strictly increasing on $(0,1)$, and it is increasing on $\mathbb{R}$. In particular, this implies that $F(x) \in [0,1]$. As a result, F is a valid CDF of a unit distribution. This completes the proof. □

The distribution defined by F as described in Theorem 2.1 is called the new general unit (NGU) distribution.

Its main novelty lies in the incorporation of the adjustable function ϕ together with the shape parameter α, which provides a high degree of flexibility in modeling a wide variety of distributional behaviors on the unit interval $(0,1)$.

Some special examples based on diverse functions ϕ are developed below.

Example 1 If we take $\phi(x) = x$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\alpha}(\alpha(2x-1))\right) = x, \quad x \in (0,1)$$

(with $\alpha > 0$), and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$. We recognize the CDF associated with the uniform distribution with support $(0,1)$.

Example 2 If we take $\phi(x) = x^3$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\alpha^3}(\alpha(2x-1))^3\right) = x(4x^2 - 6x + 3), \quad x \in (0,1)$$

(with $\alpha > 0$), and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Example 3 If we take $\phi(x) = \tanh(x)$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\tanh(\alpha)}\tanh(\alpha(2x-1))\right), \quad x \in (0,1)$$

with $\alpha > 0$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Example 4 If we take $\phi(x) = \arctan(x)$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\arctan(\alpha)}\arctan(\alpha(2x-1))\right), \quad x \in (0,1)$$

with $\alpha > 0$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Example 5 If we take $\phi(x) = \sinh(x)$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\sinh(\alpha)}\sinh(\alpha(2x-1))\right), \quad x \in (0,1)$$

with $\alpha > 0$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Example 6 If we take $\phi(x) = \sin(x)$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\sin(\alpha)}\sin(\alpha(2x-1))\right), \quad x \in (0,1) \tag{1}$$

with $\alpha \in (0, \pi/2)$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Figure 1 plots this CDF for several values of α.

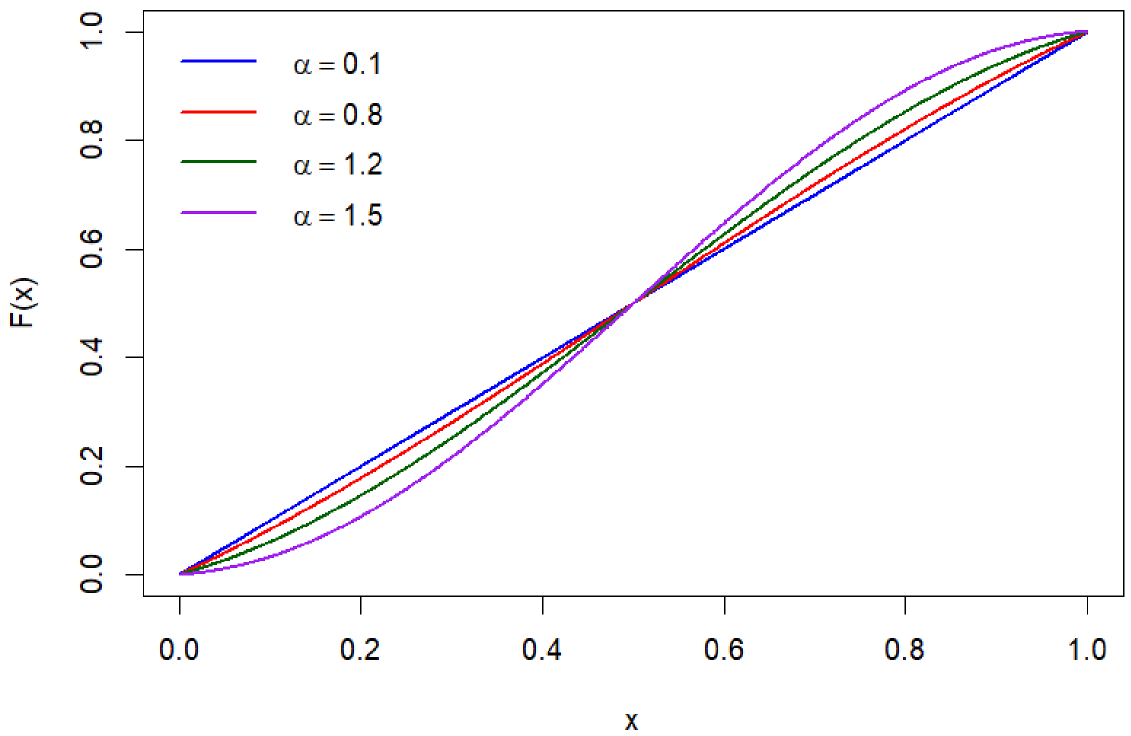

Figure 1: Plot of the CDF in Equation (1) for various values of α.

The distribution displays a versatile range of concave and convex forms. Additionally, the CDF is symmetrical about $x = 1/2$.

Example 7 If we take $\phi(x) = \tan(x)$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{\tan(\alpha)}\tan(\alpha(2x-1))\right), \quad x \in (0,1) \tag{2}$$

with $\alpha \in (0, \pi/2)$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Figure 2 plots this CDF for several values of α.

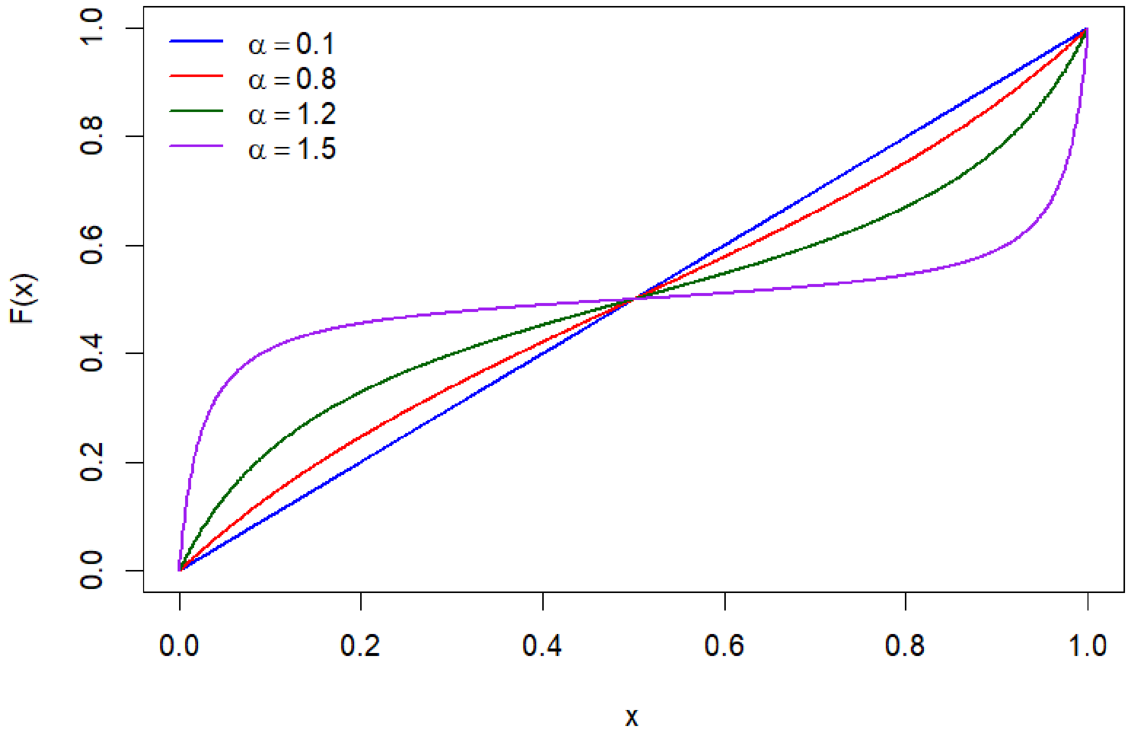

Figure 2: Plot of the CDF in Equation (2) for various values of α.

The distribution displays a versatile range of concave and convex forms. Additionally, the CDF is symmetrical about $x = 1/2$.

Example 8 If we take $\phi(x) = xe^{x^2}$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \frac{1}{e^{\alpha^2}}(2x-1)e^{\alpha^2(2x-1)^2}\right), \quad x \in (0,1)$$

with $\alpha > 0$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

Example 9 If we take $\phi(x) = x/\sqrt{1+x^2}$, then the CDF F of the NGU distribution is given by

$$F(x) = \frac{1}{2}\left(1 + \sqrt{1+\alpha^2}\frac{2x-1}{\sqrt{1+\alpha^2(2x-1)^2}}\right), \quad x \in (0,1)$$

with $\alpha > 0$, and F is completed by setting $F(x) = 0$ for any $x \leq 0$ and $F(x) = 1$ for any $x \geq 1$.

To the best of our knowledge, these unit distributions are new to the literature, except for the first one, which presents the classical uniform distribution with support $(0, 1)$.

3 Additional Functions and Properties

3.1 PDF of the NGU distribution

The PDF associated with the NGU distribution is determined in the theorem below.

Theorem 3.1. *The PDF associated with the NGU distribution is given by*

$$f(x)=\frac{\alpha}{\phi(\alpha)}\phi'(\alpha(2x-1)), \quad x\in(0,1)$$

and f is completed by setting $f(x)=0$ for any $x\notin(0,1)$.

Proof. Since the NGU distribution is a unit distribution, we have $f(x)=0$ for any $x\notin(0,1)$. For any $x\in(0,1)$, based on the CDF F as described in Theorem 2.1, using standard differentiation rules, we immediately get

$$f(x)=F'(x)=\frac{\alpha}{\phi(\alpha)}\phi'(\alpha(2x-1)).$$

This ends the proof. □

Some special examples are developed below.

Example 1 If we take $\phi(x)=x$, then the PDF f of the NGU distribution is given by

$$f(x)=\frac{\alpha}{\alpha}=1, \quad x\in(0,1)$$

(with $\alpha>0$), and f is completed by setting $f(x)=0$ for any $x\notin(0,1)$.

Example 2 If we take $\phi(x)=x^3$, then the PDF f of the NGU distribution is given by

$$f(x)=\frac{\alpha}{\alpha^3}3(\alpha(2x-1))^2=3(2x-1)^2, \quad x\in(0,1)$$

(with $\alpha>0$), and f is completed by setting $f(x)=0$ for any $x\notin(0,1)$.

Example 3 If we take $\phi(x)=\tanh(x)$, then the PDF f of the NGU distribution is given by

$$f(x)=\frac{\alpha}{\tanh(\alpha)}\left(\frac{1}{(\cosh(\alpha(2x-1)))^2}\right), \quad x\in(0,1)$$

with $\alpha>0$, and f is completed by setting $f(x)=0$ for any $x\notin(0,1)$.

Example 4 If we take $\phi(x)=\arctan(x)$, then the PDF f of the NGU distribution is given by

$$f(x)=\frac{\alpha}{\arctan(\alpha)}\left(\frac{1}{1+\alpha^2(2x-1)^2}\right), \quad x\in(0,1)$$

with $\alpha>0$, and f is completed by setting $f(x)=0$ for any $x\notin(0,1)$.

Example 5 If we take $\phi(x) = \sinh(x)$, then the PDF f of the NGU distribution is given by

$$f(x) = \frac{\alpha}{\sinh(\alpha)} \cosh(\alpha(2x-1)), \quad x \in (0,1)$$

with $\alpha > 0$, and f is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

Example 6 If we take $\phi(x) = \sin(x)$, then the PDF f of the NGU distribution is given by

$$f(x) = \frac{\alpha}{\sin(\alpha)} \cos(\alpha(2x-1)), \quad x \in (0,1) \tag{3}$$

with $\alpha \in (0, \pi/2)$, and f is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

Figure 3 plots this PDF for several values of α.

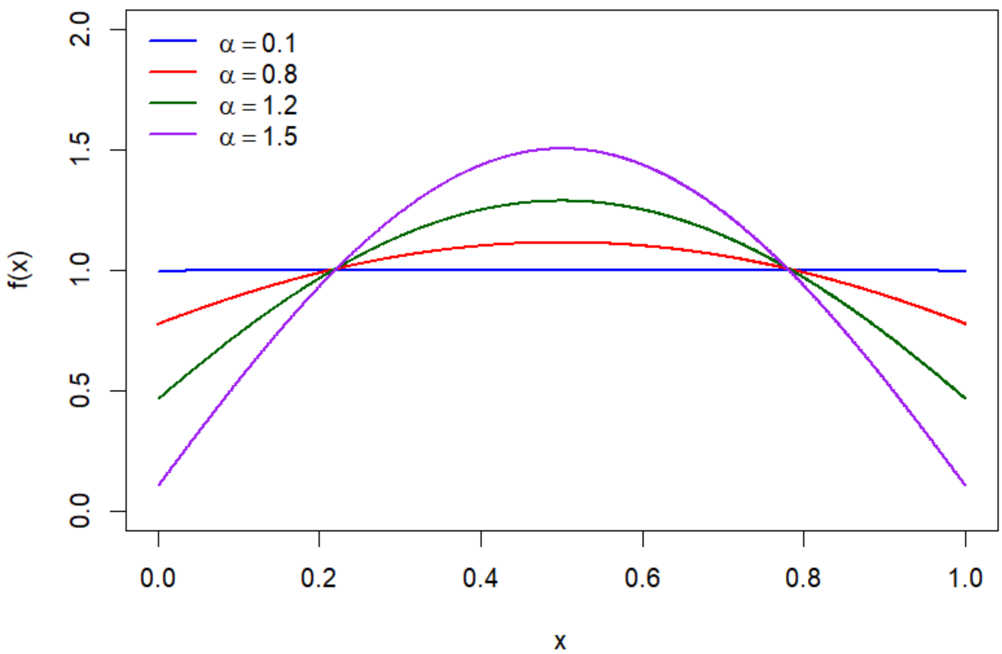

Figure 3: Plot of the PDF in Equation (3) for various values of α.

The distribution displays a versatile range of bell curves. Additionally, the PDF is symmetrical about $x = 1/2$.

Example 7 If we take $\phi(x) = \tan(x)$, then the PDF f of the NGU distribution is given by

$$f(x) = \frac{\alpha}{\tan(\alpha)} \left(\frac{1}{(\cos(\alpha(2x-1)))^2} \right), \quad x \in (0,1) \tag{4}$$

with $\alpha \in (0, \pi/2)$, and f is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

Figure 4 plots this PDF for several values of α.

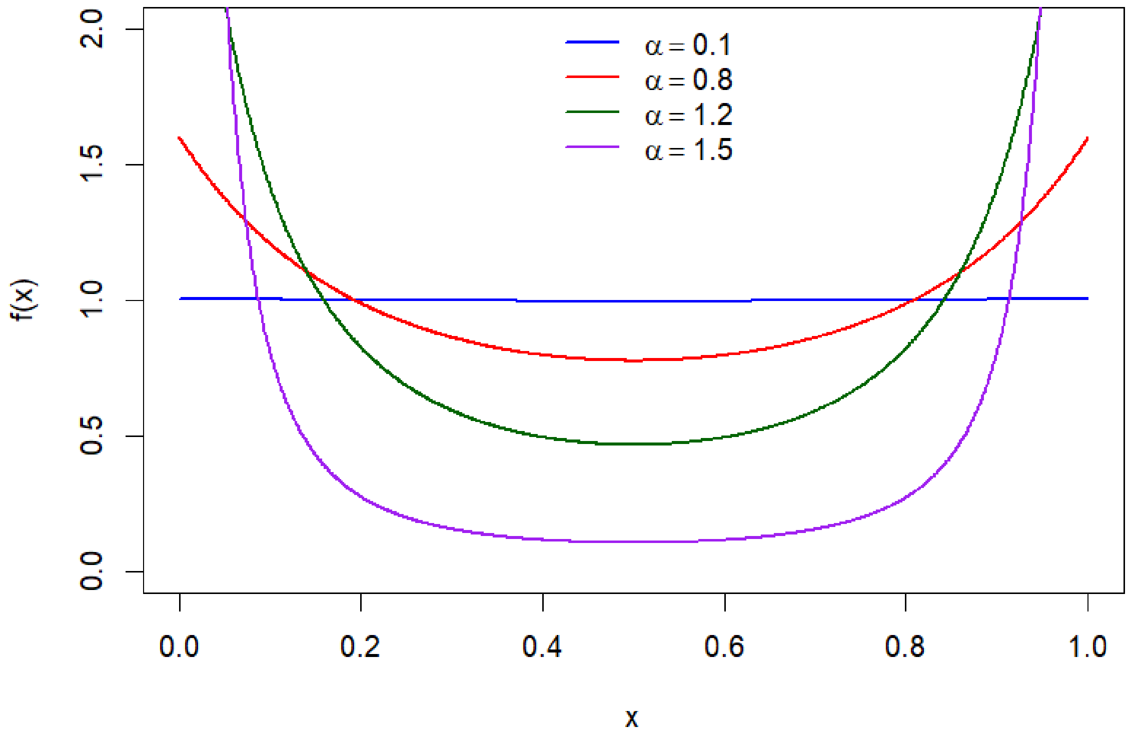

Figure 4: Plot of the PDF in Equation (4) for various values of α.

The distribution displays a versatile range of bathtub curves. Additionally, the PDF is symmetrical about $x = 1/2$.

Note that the plots in Figures 3 and 4 are really different, showing the flexibility of the approach.

Example 8 If we take $\phi(x) = xe^{x^2}$, then the PDF f of the NGU distribution is given by

$$f(x) = \frac{1}{e^{\alpha^2}}(2\alpha^2(2x-1)^2 + 1)e^{\alpha^2(2x-1)^2}, \quad x \in (0,1)$$

with $\alpha > 0$, and f is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

Example 9 If we take $\phi(x) = x/\sqrt{1+x^2}$, then the PDF f of the NGU distribution is given by

$$f(x) = \sqrt{1+\alpha^2}\frac{1}{(1+\alpha^2(2x-1)^2)^{3/2}}, \quad x \in (0,1)$$

with $\alpha > 0$, and f is completed by setting $f(x) = 0$ for any $x \notin (0,1)$.

We may underline the relative simplicity of these PDFs.

3.2 QF of the NGU distribution

The QF associated with the NGU distribution is given by a closed-form expression. This is determined by the theorem below.

Theorem 3.2. *The QF associated with the NGU distribution is given by*

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\phi^{-1}\left((2u-1)\phi(\alpha)\right)\right), \quad u \in (0,1).$$

Proof. The quantile function is given by inverting the CDF F as described in Theorem 2.1, which is equivalent in solving $F(x) = u$, that is

$$\begin{aligned}
&\frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2x-1))\right) = u \;\Leftrightarrow\; \frac{1}{\phi(\alpha)}\phi(\alpha(2x-1)) = 2u-1 \\
&\Leftrightarrow\; \phi(\alpha(2x-1)) = (2u-1)\phi(\alpha) \;\Leftrightarrow\; \alpha(2x-1) = \phi^{-1}\left((2u-1)\phi(\alpha)\right) \\
&\Leftrightarrow\; 2x-1 = \frac{1}{\alpha}\phi^{-1}\left((2u-1)\phi(\alpha)\right) \\
&\Leftrightarrow\; x = \frac{1}{2}\left(1 + \frac{1}{\alpha}\phi^{-1}\left((2u-1)\phi(\alpha)\right)\right).
\end{aligned}$$

Therefore, we have

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\phi^{-1}\left((2u-1)\phi(\alpha)\right)\right), \quad u \in (0,1).$$

This concludes the proof. □

In particular, using $\phi^{-1}(0) = 0$, the median of the NGU distribution is given by

$$\begin{aligned}
\text{median} = Q\left(\frac{1}{2}\right) &= \frac{1}{2}\left(1 + \frac{1}{\alpha}\phi^{-1}\left(\left(2 \times \frac{1}{2} - 1\right)\phi(\alpha)\right)\right) \\
&= \frac{1}{2}\left(1 + \frac{1}{\alpha}\phi^{-1}(0)\right) = \frac{1}{2}.
\end{aligned}$$

Therefore, it does not depend on ϕ and α.

Some special examples of the QF of the NGU distribution are developed below.

Example 1 If we take $\phi(x) = x$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\left((2u-1)\alpha\right)\right) = u, \quad u \in (0,1)$$

(with $\alpha > 0$). We recognize the QF of the uniform distribution with support $(0,1)$.

Example 2 If we take $\phi(x) = x^3$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\left((2u-1)\alpha^3\right)^{1/3}\right) = \frac{1}{2}\left(1 + (2u-1)^{1/3}\right), \quad u \in (0,1),$$

(with $\alpha > 0$).

Example 3 If we take $\phi(x) = \tanh(x)$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\tanh^{-1}\left((2u-1)\tanh(\alpha)\right)\right), \quad u \in (0,1)$$

with $\alpha > 0$.

Example 4 If we take $\phi(x) = \arctan(x)$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\tan\left((2u-1)\arctan(\alpha)\right)\right), \quad u \in (0,1)$$

with $\alpha > 0$.

Example 5 If we take $\phi(x) = \sinh(x)$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\sinh^{-1}\left((2u-1)\sinh(\alpha)\right)\right), \quad u \in (0,1)$$

with $\alpha > 0$.

Example 6 If we take $\phi(x) = \sin(x)$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\arcsin\left((2u-1)\sin(\alpha)\right)\right), \quad u \in (0,1)$$

with $\alpha \in (0, \pi/2)$.

Example 7 If we take $\phi(x) = \tan(x)$, then the QF Q of the NGU distribution is given by

$$Q(u) = \frac{1}{2}\left(1 + \frac{1}{\alpha}\arctan\left((2u-1)\tan(\alpha)\right)\right), \quad u \in (0,1)$$

with $\alpha \in (0, \pi/2)$.

3.3 A distributional theorem

The theorem below presents a distributional result relating a random variable that follows the NGU distribution to its symmetrical random variable counterpart.

Theorem 3.3. *Let X be a random variable that follows the NGU distribution. Then $Y = 1 - X$ also follows the NGU distribution.*

Proof. Let F_Y be the CDF of Y and F be the CDF of the NGU distribution as defined in Theorem 2.1. Since the NGU distribution is a unit distribution, the distribution of Y is also a unit distribution, which implies that $F_Y(x) = 0$ for any $x \leq 0$ and $F_Y(x) = 1$ for any $x \geq 1$, so that $F_Y(x) = F(x)$ for any $x \notin (0,1)$. For any $x \in (0,1)$, using the property that ϕ is an odd function, we have

$$\begin{aligned} F_Y(x) &= P(Y \leq x) = P(1 - X \leq x) = P(X \geq 1 - x) = 1 - P(X < 1 - x) \\ &= 1 - F(1 - x) = 1 - \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2(1 - x) - 1))\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{\phi(\alpha)}\phi(\alpha(1 - 2x))\right) = \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2x - 1))\right) \\ &= F(x). \end{aligned}$$

Therefore, Y and X follow the same NGU distribution. This concludes the proof. □

This theorem shows that the NGU distribution is structurally robust to the transformation $1 - X$.

3.4 A moment theorem

The mean associated with the NGU distribution is determined in the theorem below.

Theorem 3.4. *Let X be a random variable that follows the NGU distribution. Then the mean of X is given by*

$$E(X) = \frac{1}{2}.$$

Proof. It follows from Theorem 3.3 that the random variables X and $1 - X$ follow the same NGU distribution. Therefore, we have

$$E(X) = E(1 - X) \;\Leftrightarrow\; E(X) = 1 - E(X) \;\Leftrightarrow\; 2E(X) = 1 \;\Leftrightarrow\; E(X) = \frac{1}{2}.$$

This completes the proof. □

It is interesting to see that $E(X)$ does not depend on ϕ and α. In this sense, the NGU distribution is stable with respect to its mean.

3.5 The NGU family of distributions

In line with the approach taken in [2], since the NGU distribution is a unit distribution, we can define the NGU family of distributions. This is formalized in the theorem below.

Theorem 3.5. *Let G be the CDF of a continuous distribution. Based on G and the CDF F of the NGU distribution, the following function is a valid CDF:*

$$\begin{aligned} F_{\dagger}(x) &= F(G(x)) \\ &= \frac{1}{2}\left(1 + \frac{1}{\phi(\alpha)}\phi(\alpha(2G(x) - 1))\right), \quad x \in \mathbb{R} \end{aligned}$$

with $\alpha > 0$, where ϕ is a strictly increasing differentiable odd function on $[-\alpha, \alpha]$ with $\phi(0) = 0$.

Proof. Since $G(x) \in [0,1]$ for any $x \in \mathbb{R}$ and F is the CDF of a unit distribution, the composition $F_\dagger(x) = F(G(x))$ is valid from the mathematical point of view. Moreover, the composition of two continuous functions is a continuous function, the composition of two increasing functions is an increasing function and clearly $F_\dagger(x) \in [0,1]$ for any $x \in \mathbb{R}$ because $F(x) \in [0,1]$. Therefore, $F_\dagger$ is a valid CDF. This completes the proof. □

Some special examples are developed below.

Example 1 If we take $\phi(x) = x$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = G(x), \quad x \in \mathbb{R}.$$

We recognize the baseline CDF.

Example 2 If we take $\phi(x) = x^3$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = G(x)\left(4(G(x))^2 - 6G(x) + 3\right), \quad x \in \mathbb{R}.$$

Example 3 If we take $\phi(x) = \tanh(x)$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{\tanh(\alpha)}\tanh(\alpha(2G(x)-1))\right), \quad x \in \mathbb{R}$$

with $\alpha > 0$.

Example 4 If we take $\phi(x) = \arctan(x)$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{\arctan(\alpha)}\arctan(\alpha(2G(x)-1))\right), \quad x \in \mathbb{R}$$

with $\alpha > 0$.

Example 5 If we take $\phi(x) = \sinh(x)$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{\sinh(\alpha)}\sinh(\alpha(2G(x)-1))\right), \quad x \in \mathbb{R}$$

with $\alpha > 0$.

Example 6 If we take $\phi(x) = \sin(x)$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{\sin(\alpha)}\sin(\alpha(2G(x)-1))\right), \quad x \in \mathbb{R}$$

with $\alpha \in (0, \pi/2)$.

Example 7 If we take $\phi(x) = \tan(x)$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{\tan(\alpha)}\tan(\alpha(2G(x)-1))\right), \quad x \in \mathbb{R}$$

with $\alpha \in (0, \pi/2)$.

Example 8 If we take $\phi(x) = xe^{x^2}$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \frac{1}{e^{\alpha^2}}(G(x) - 1)e^{\alpha^2(G(x)-1)^2}\right), \quad x \in \mathbb{R}$$

with $\alpha > 0$.

Example 9 If we take $\phi(x) = x/\sqrt{1 + x^2}$, then the CDF $F_\dagger$ of the NGU family of distributions is given by

$$F_\dagger(x) = \frac{1}{2}\left(1 + \sqrt{1 + \alpha^2}\frac{2G(x) - 1}{\sqrt{1 + \alpha^2(2G(x) - 1)^2}}\right), \quad x \in \mathbb{R}$$

with $\alpha > 0$.

Each of these families of distributions could form the basis of an independent study.

4 Conclusion

In this article, we introduced a new type of unit distribution called the NGU distribution. It is defined through an adjustable function, ϕ, and a shape parameter, α, which provide greater flexibility for modeling proportions, rates and other fractional data. This new construction expands the range of existing unit distributions and provides a flexible framework for theoretical and practical research. Future work could involve studying the statistical properties and inference procedures, exploring applications to real-world data and extending the model to multivariate settings or alternative functional forms.

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