Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 4, July 2026, Pages 577-681
https://doi.org/10.34198/ejms.16426.39.577681

Explicit Identities for Horadam Polynomials: Generalized Fibonacci Formulations and Special Cases

Yüksel Soykan

Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com

Abstract

In this paper, we investigate the generalized Fibonacci (Horadam) polynomials and concentrate on two special subclasses, which we introduce as the (r, s)-Fibonacci and (r, s)-Lucas polynomials. Our primary aim is to present and establish several identities that connect these two families, thereby extending classical relations between Fibonacci and Lucas sequences into a broader polynomial framework. The identities obtained not only highlight the structural interplay between the (r, s)-Fibonacci and (r, s)-Lucas polynomials but also enrich the theory of generalized Horadam polynomials by revealing new algebraic connections. This work is devoted exclusively to the derivation and exposition of such identities, providing a foundation for further exploration of recurrence-based polynomial structures.

1 Introduction: Generalized Fibonacci (Horadam) Polynomials

The study of recurrence sequences has long been a central theme in number theory and combinatorics. Among them, the Horadam sequence, introduced by A. F. Horadam in the 1960s, provides a unifying framework that encompasses many well-known sequences such as Fibonacci, Lucas, Pell, and Jacobsthal numbers. Its general form is defined by a second-order linear recurrence relation with arbitrary initial conditions, thereby offering a versatile structure for both theoretical exploration and practical applications.

In recent years, attention has turned toward polynomial analogues of such sequences, motivated by the desire to extend classical identities and to uncover new algebraic connections. The generalized Horadam polynomials serve as a natural extension, embedding recurrence relations into a polynomial setting enriched by parameters that allow greater flexibility and generalization.

In this paper, we focus on two special subclasses of the generalized Horadam polynomials, which we denote as the (r, s)-Fibonacci and (r, s)-Lucas polynomials. These families extend the classical Fibonacci and Lucas sequences by introducing the parameters r and s, thereby broadening the range of identities that can be established. Our primary aim is to present and prove several identities that connect these two classes, extending familiar relations between Fibonacci and Lucas numbers into a more general polynomial framework. The work is devoted exclusively to the derivation and exposition of such identities, highlighting the structural interplay between the (r, s)-Fibonacci and (r, s)-Lucas polynomials.

In doing so, we enrich the theory of generalized Horadam polynomials by demonstrating how classical identities persist and evolve within this broader setting. Our findings contribute to a deeper understanding of the interplay

Received: April 3, 2026; Revised & Accepted: May 13, 2026; Published online: May 21, 2026
2020 Mathematics Subject Classification: 05A15, 05A19, 11B37, 11B39, 11B83, 11C20, 15B36.
Keywords and phrases: Horadam polynomials, (r, s)-Fibonacci polynomials, (r, s)-Lucas polynomials, generalized Fibonacci polynomials, identities, Fibonacci numbers, Lucas numbers, Fibonacci polynomials, Lucas polynomials.

between these special cases and the general Horadam framework, while providing a solid foundation for further theoretical exploration of recurrence-based polynomial structures.

In this introductory section, we begin by recalling the definitions of the Horadam polynomials together with their two notable subclasses, namely the (r, s)-Fibonacci and (r, s)-Lucas polynomials. These definitions provide the necessary foundation for our subsequent discussion. Following this, we present a collection of identities involving these families, many of which are well established in the literature and serve as classical benchmarks in the study of recurrence-based polynomial structures. By organizing and restating these known identities, we aim to set the stage for the new results developed in the later sections of this paper, thereby highlighting the continuity between established theory and our contributions.

The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{W_n(W_0, W_1; r(x), s(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \qquad W_0(x) = a(x), W_1(x) = b(x), \quad n \geq 2 \tag{1.1}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)}W_{-(n-1)}(x) + \frac{1}{s(x)}W_{-(n-2)}(x)$$

for $n = 1, 2, 3, ...$ when $s(x) \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Note that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For further details on Horadam polynomials, see Soykan [9]. References on special cases of Horadam polynomials include the papers ([3], [4], [5], [11], [12]) and the books ([1], [2], [6], [7], [8], [10])}.

Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{1.2}$$

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}, \tag{1.3}$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r(x), \\ \alpha\beta &= -s(x), \end{aligned}$$

Note also that

$$\begin{aligned} (W_1 - \alpha W_0)(W_1 - \beta W_0) &= W_1^2 + \alpha\beta W_0^2 - \alpha W_0 W_1 - \beta W_0 W_1 \\ &= W_1^2 + \alpha\beta W_0^2 - (\alpha + \beta)\, W_1 W_0 \\ &= W_1^2 - sW_0^2 - rW_1 W_0 \\ &\Rightarrow \\ (W_1 - \alpha W_0)(W_1 - \beta W_0) &= W_1^2 - sW_0^2 - rW_1 W_0. \end{aligned}$$

If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha \neq \beta$ then $r^2(x) + 4s(x) \neq 0$ and if the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha = \beta$ then (1.2) can be written as

$$y^2 - r(x)y - s(x) = (y-\alpha)^2 = y^2 - 2\alpha y + \alpha^2 = 0$$

and, in this case,

$$\begin{aligned} \alpha &= \frac{r(x)}{2}, \\ r(x) &= 2\alpha, \\ s(x) &= -\alpha^2 = -\frac{r^2(x)}{4}, \\ r^2(x) + 4s(x) &= 0. \end{aligned}$$

Now, we define two special cases of the polynomials $W_n(x)$. $(r(x), s(x))$-Fibonacci polynomials $\{G_n(0,1; r(x), s(x))\}_{n\geq 0}$ (or shortly $G_n(x)$) and $(r(x), s(x))$-Lucas polynomials $\{H_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $H_n(x)$) are defined, respectively, by the second-order recurrence relations

$$G_{n+2}(x) = r(x)G_{n+1} + s(x)G_n(x), \qquad G_0(x) = 0, G_1(x) = 1, \tag{1.4}$$

$$H_{n+2}(x) = r(x)H_{n+1} + s(x)H_n(x), \qquad H_0(x) = 2, H_1(x) = r(x). \tag{1.5}$$

The (sequences of polinomials) $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n}(x) &= -\frac{r(x)}{s(x)}G_{-(n-1)}(x) + \frac{1}{s(x)}G_{-(n-2)}(x), \\ H_{-n}(x) &= -\frac{r(x)}{s(x)}H_{-(n-1)}(x) + \frac{1}{s(x)}H_{-(n-2)}(x), \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.4) and (1.5) hold for all integers n.

Note. For the sake of simplicity, throughout the remainder of this paper we adopt the following convention: we write

$$W_n,\ r,\ s,\ W_0,\ W_1,\ \alpha,\ \beta,\ G_n,\ H_n,\ G_0,\ G_1,\ H_0,\ H_1$$

instead of

$$W_n(x),\ r(x),\ s(x),\ W_0(x),\ W_1(x),\ \alpha(x),\ \beta(x),\ G_n(x),\ H_n(x),\ G_0(x),\ G_1(x),\ H_0(x),\ H_1(x),$$

respectively. For example, we write

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a,\ W_1 = b, \quad n \geq 2,$$

for the equation (1.1).

Some notable special cases of the (r, s)-Fibonacci sequence $\{G_n(0,1; r, s)\}_{n\geq 0}$ and the (r, s)-Lucas sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ can be described as follows:

1. $G_n(0,1;1,1) = F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1) = L_n$, Lucas sequence,
3. $G_n(0,1;2,1) = P_n$, Pell sequence,

4. $H_n(2,2;2,1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2) = J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2) = j_n$, Jacobsthal-Lucas sequence.
7. $G_n(0,1;3,-2) = M_n$, Mersenne sequence,
8. $H_n(2,3;3,-2) = H_n$, Mersenne-Lucas sequence,
9. $G_n(0,1;6,-1) = B_n$, balancing sequence,
10. $H_n(2,6;6,-1) = H_n$, modified Lucas-balancing sequence,
11. $G_n(0,1;1,-\frac{1}{4}) = G_n$, modified Oresme sequence,
12. $H_n(2,1;1,-\frac{1}{4}) = H_n$, Oresme-Lucas sequence,
13. $G_n(0,1;x,1) = F_n(x)$, Fibonacci polynomials,
14. $H_n(2,x;x,1) = L_n(x)$, Lucas polynomials,
15. $G_n(0,1;2x,1) = P_n(x)$, Pell polynomials,
16. $H_n(2,2x;2x,1) = Q_n(x)$, Pell-Lucas polynomials,
17. $G_n(0,1;1,2x) = J_n(x)$, Jacobsthal polynomials,
18. $H_n(2,1;1,2x) = j_n(x)$, Jacobsthal-Lucas polynomials.

Using the roots α, β and recurrence relation (1.1), Binet's formula can be given as follows:

Theorem 1.1 ([9], Theorem 1)**.**

(a) *(Distinct Roots Case: $\alpha \neq \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = \frac{p_1\alpha^n}{\alpha-\beta} + \frac{p_2\beta^n}{\beta-\alpha} = \frac{p_1\alpha^n - p_2\beta^n}{\alpha-\beta} \tag{1.6}$$

where

$$p_1 = W_1 - \beta W_0,\ p_2 = W_1 - \alpha W_0.$$

(b) *(Single Root Case: $\alpha = \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = (D_1 + D_2 n)\alpha^n \tag{1.7}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

For all integers n, (r,s)-Fibonacci and (r,s)-Lucas polynomials (using initial conditions in (1.6)) can be expressed using Binet's formulas as

$$G_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases},$$

and

$$H_n = \begin{cases} \alpha^n + \beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases},$$

respectively.

1.1 Identities of Generalized Fibonacci-Type Polynomials

In this subsection, we present several identities involving the generalized Fibonacci (Horadam) polynomials, the (r,s)-Fibonacci polynomials, and the (r,s)-Lucas polynomials. As a starting point, we recall a few basic relations between the sequences $\{G_n\}$ and $\{W_n\}$, which serve as fundamental connections in the theory.

Lemma 1.2 ([9], Lemma 7)**.** *Let $n \in \mathbb{Z}$. The following equalities are true:*

$$\begin{aligned} s^3W_n &= ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+4} + (-r(2s+r^2)W_1 + (3r^2s+r^4+s^2)W_0)G_{n+3}, \quad (1.8) \\ s^2W_n &= (-W_1r + (r^2+s)W_0)G_{n+3} + ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+2}, \\ sW_n &= (W_1 - rW_0)G_{n+2} + (-rW_1 + (s+r^2)W_0)G_{n+1}, \\ W_n &= W_0G_{n+1} + (W_1 - rW_0)G_n, \\ W_n &= W_1G_n + sW_0G_{n-1}, \end{aligned}$$

and

$$\begin{aligned} s^3(-W_1^2 + sW_0^2 + rW_1W_0)G_n &= -((s+r^2)W_1 + srW_0)W_{n+4} + (r(2s+r^2)W_1 + s(s+r^2)W_0)W_{n+3}, \\ s^2(-W_1^2 + sW_0^2 + rW_1W_0)G_n &= (rW_1 + sW_0)W_{n+3} - ((s+r^2)W_1 + srW_0)W_{n+2}, \\ s(-W_1^2 + sW_0^2 + rW_1W_0)G_n &= -W_1W_{n+2} + (rW_1 + sW_0)W_{n+1}, \\ (-W_1^2 + sW_0^2 + rW_1W_0)G_n &= W_0W_{n+1} - W_1W_n, \\ (-W_1^2 + sW_0^2 + rW_1W_0)G_n &= -(W_1 - rW_0)W_n + sW_0W_{n-1}. \end{aligned}$$

Secondly, we recall a few fundamental relations between the sequences $\{H_n\}$ and $\{W_n\}$.

Lemma 1.3 ([9], Lemma 8)**.** *Let $n \in \mathbb{Z}$. The following equalities are true:*

$$\begin{aligned} s^3(4s+r^2)W_n &= (-r(3s+r^2)W_1 + (4r^2s+r^4+2s^2)W_0)H_{n+4} + ((r^4+2s^2+4r^2s)W_1 \\ &\quad -r(5r^2s+r^4+5s^2)W_0)H_{n+3}, \\ s^2(4s+r^2)W_n &= ((2s+r^2)W_1 - r(3s+r^2)W_0)H_{n+3} + (-r(3s+r^2)W_1 + (r^4+2s^2+4r^2s)W_0)H_{n+2}, \\ s(4s+r^2)W_n &= (-rW_1 + (2s+r^2)W_0)H_{n+2} + ((2s+r^2)W_1 - r(3s+r^2)W_0)H_{n+1}, \\ (4s+r^2)W_n &= (2W_1 - rW_0)H_{n+1} + (-rW_1 + (2s+r^2)W_0)H_n, \\ (4s+r^2)W_n &= (rW_1 + 2sW_0)H_n + s(2W_1 - rW_0)H_{n-1}, \end{aligned}$$

and

$$
\begin{aligned}
s^3(-W_1^2+sW_0^2+rW_0W_1)H_n &= (r(3s+r^2)W_1+s(2s+r^2)W_0)W_{n+4}-((r^4+2s^2+4r^2s)W_1 \\
&\quad +rs(3s+r^2)W_0)W_{n+3}, \\
s^2(-W_1^2+sW_0^2+rW_0W_1)H_n &= -((2s+r^2)W_1+rsW_0)W_{n+3}+(r(3s+r^2)W_1+s(2s+r^2)W_0)W_{n+2}, \\
s(-W_1^2+sW_0^2+rW_0W_1)H_n &= (rW_1+2sW_0)W_{n+2}-((2s+r^2)W_1+rsW_0)W_{n+1}, \\
(-W_1^2+sW_0^2+rW_0W_1)H_n &= (-2W_1+rW_0)W_{n+1}+(rW_1+2sW_0)W_n, \\
(-W_1^2+sW_0^2+rW_0W_1)H_n &= (-rW_1+(2s+r^2)W_0)W_n+s(-2W_1+rW_0)W_{n-1}.
\end{aligned}
$$

Thirdly, we recall several well-known relations between the sequences $\{G_n\}$ and $\{H_n\}$.

Lemma 1.4 ([9], Lemma 9)**.** *Let $n \in \mathbb{Z}$. The following equalities are true:*

$$
\begin{aligned}
s^3H_n &= -(3rs+r^3)G_{n+4}+(4r^2s+r^4+2s^2)G_{n+3}, \\
s^2H_n &= (2s+r^2)G_{n+3}-(3rs+r^3)G_{n+2}, \\
sH_n &= -rG_{n+2}+(2s+r^2)G_{n+1}, \\
H_n &= 2G_{n+1}-rG_n, \\
H_n &= rG_n+2sG_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
(r^2s^3+4s^4)G_n &= -(3rs+r^3)H_{n+4}+(4r^2s+r^4+2s^2)H_{n+3}, \\
(r^2s^2+4s^3)G_n &= (2s+r^2)H_{n+3}-(3rs+r^3)H_{n+2}, \\
(r^2s+4s^2)G_n &= -rH_{n+2}+(2s+r^2)H_{n+1}, \\
(r^2+4s)G_n &= 2H_{n+1}-rH_n, \\
(r^2+4s)G_n &= rH_n+2sH_{n-1}.
\end{aligned}
$$

Note that from the last Lemma we have

$$(r^2+4s)G_n = 2H_{n+1}-rH_n \tag{1.9}$$

so if $(r^2+4s)=0$, i.e. $\alpha=\beta$ (in this case $r=2\alpha, s=-\alpha^2$) then

$$2H_{n+1}-rH_n = 0$$

and so $2H_{n+1}-2\alpha H_n=0$ and thus, if $(r^2+4s)=0$, i.e. $\alpha=\beta$, then

$$H_{n+1} = \alpha H_n. \tag{1.10}$$

At this stage, we present several identities that highlight the structural connections among the polynomial families under consideration.

Corollary 1.5 ([9], Corollary 10)**.** *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i) $H_n = G_{n+1} + sG_{n-1}$.

(ii) $(r^2 + 4s)G_n = H_{n+1} + sH_{n-1}$.

(b)

(i) $(r^2 + 4s)G_n + H_n = (H_{n+1} + G_{n+1}) + s(H_{n-1} + G_{n-1})$.

(ii) $(r^2 + 4s)G_n - H_n = (H_{n+1} - G_{n+1}) + s(H_{n-1} - G_{n-1})$.

We now present several special identities involving the generalized Fibonacci (Horadam) polynomials $\{W_n\}$, the (r,s)-Fibonacci polynomials $\{G_n\}$, and the (r,s)-Lucas polynomials $\{H_n\}$.

Theorem 1.6 ([9], Theorem 11)**.** *Let n, m and k be any integers. Then the following identities are true:*

(a) *Catalan's identity.*

(i) $W_{n+m}W_{n-m} - W_n^2 = -(-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$.

(ii) $G_{n+m}G_{n-m} - G_n^2 = -(-s)^{n-m}G_m^2$.

(iii) $H_{n+m}H_{n-m} - H_n^2 = (-s)^{n-m}(r^2 + 4s)G_m^2$.

(b) *Cassini's identity.*

(i) $W_{n+1}W_{n-1} - W_n^2 = -(-s)^{n-1}(W_1^2 - sW_0^2 - rW_0W_1)$.

(ii) $G_{n+1}G_{n-1} - G_n^2 = -(-s)^{n-1}$.

(iii) $H_{n+1}H_{n-1} - H_n^2 = (-s)^{n-1}(r^2 + 4s)$.

(c) *d'Ocagne's identity.*

(i) $W_{m+1}W_n - W_mW_{n+1} = -(W_1^2 - sW_0^2 - rW_0W_1)(-H_mG_n + G_{m+n})$.

(ii) $G_{m+1}G_n - G_mG_{n+1} = -(-H_mG_n + G_{m+n})$.

(iii) $H_{m+1}H_n - H_mH_{n+1} = (r^2 + 4s)(-H_mG_n + G_{m+n})$.

(d) *Gelin-Cesàro's identity.*

(i) $W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 = (-s)^{n-3}(W_1^2 - sW_0^2 - rW_0W_1)(s(r^2 - s)W_n^2 + r^2(-s)^n W_1^2 - r^2 s(-s)^n W_0^2 - r^3(-s)^n W_0W_1)$.

(ii) $G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = (-s)^{n-3}(r^2(-s)^n - s(s - r^2)G_n^2)$.

(iii) $H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = (-s)^{n-3}(r^2 + 4s)\left(-s(r^2 - s)H_n^2 + r^2(-s)^n(r^2 + 4s)\right)$.

(e) *Melham's identity.*

(i) $W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = -(-s)^{n+1}(W_1^2 - sW_0^2 - rW_0W_1)(r^3W_{n+2} - (-s)^2W_{n+1})$.

(ii) $G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = -(-s)^{n+1})(r^3G_{n+2} - (-s)^2G_{n+1})$.

(iii) $H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = (-s)^{n+1}(r^2 + 4s)(r^3H_{n+2} - (-s)^2H_{n+1})$.

(f) *Vajda's identity (Taguiri's identity (a generalization of Catalan's identity)).*

(i) $W_{n+m}W_{n+k} - W_nW_{n+m+k} = (-s)^n(W_1^2 - sW_0^2 - rW_0W_1)G_kG_m.$

(ii) $G_{n+m}G_{n+k} - G_nG_{n+m+k} = (-s)^nG_kG_m.$

(iii) $H_{n+m}H_{n+k} - H_nH_{n+m+k} = -(-s)^n(r^2 + 4s)G_kG_m.$

The following theorem expresses the generalized Fibonacci (Horadam) polynomials W_n at negative indices in terms of the sequence itself at positive indices.

Theorem 1.7 ([9], Theorem 13). *For $n \in \mathbb{Z}$, the generalized Fibonacci (Horadam) polynomials satisfy*

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= -(-s)^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

As immediate consequences of Theorem 1.7, we obtain the following corollary.

Corollary 1.8. *For $n \in \mathbb{Z}$, we have*

$$H_{-n} = (-s)^{-n}H_n, \qquad G_{-n} = -(-s)^{-n}G_n.$$

Theorem 1.7 can also be reformulated in a more explicit form, as given below.

Theorem 1.9 ([9], Theorem 14). *For $n \in \mathbb{Z}$, the generalized Fibonacci polynomials satisfy*

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Theorem 1.10 ([9], Theorem 25). *(Honsberger's Identity) For all integers m, n we have*

$$W_{n+m} = W_nG_{m+1} + sW_{n-1}G_m. \tag{1.11}$$

By Lemma 1.2, we know that

$$(-W_1^2 + sW_0^2 + rW_1W_0)G_m = W_0W_{m+1} - W_1W_m,$$

so (1.11) can be written in the following form

$$\begin{aligned} (-W_1^2 + sW_0^2 + rW_1W_0)W_{n+m} &= (W_0W_{m+2} - W_1W_{m+1})W_n \\ &\quad +s(W_0W_{m+1} - W_1W_m)W_{n-1} \\ &= ((-W_1 + rW_0)W_{m+1} + sW_0W_m)W_n \\ &\quad +s(W_0W_{m+1} - W_1W_m)W_{n-1}. \end{aligned} \tag{1.12}$$

Corollary 1.11 ([9], Corollary 26). *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_nG_{m+1} + sG_{n-1}G_m, \\ H_{n+m} &= H_nG_{m+1} + sH_{n-1}G_m, \end{aligned}$$

and

$$\begin{aligned} (r^2 + 4s)H_{n+m} &= (2H_{m+2} - rH_{m+1})H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= (2(rH_{m+1} + sH_m) - rH_{m+1})H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= (rH_{m+1} + 2sH_m)H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= 2H_{m+1}H_{n+1} + (r^2 + 2s)H_mH_n - r(H_{m+1}H_n + H_mH_{n+1}), \end{aligned}$$

where

$$\begin{aligned} H_{m+2} &= rH_{m+1} + sH_m, \\ H_{n-1} &= \frac{1}{s}(H_{n+1} - rH_n). \end{aligned}$$

If we replace $m = n$, $m = n + 1$ and $m = 2n$, respectively, in Corollory 1.11 and apply the identities (given in Lemma 1.4)

$$\begin{aligned} G_{n+1} &= rG_n + sG_{n-1}, \\ (r^2 + 4s)G_n &= rH_n + 2sH_{n-1}, \\ H_n &= rG_n + 2sG_{n-1}, \\ H_n &= 2G_{n+1} - rG_n, \end{aligned}$$

we obtain the following result.

Corollary 1.12 ([9], Corollary 27)**.** *For all integers m, n, we have*

$$\begin{aligned} G_{2n} &= G_n\left(G_{n+1} + sG_{n-1}\right) = G_n H_n, \\ H_{2n} &= H_n G_{n+1} + sH_{n-1}G_n, \\ 2H_{2n} &= (r^2 + 4s)G_n^2 + H_n^2, \\ (r^2 + 4s)H_{2n} &= (2H_{n+2} - rH_{n+1})H_n + s(2H_{n+1} - rH_n)H_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_{2n+1} &= G_n G_{n+2} + sG_{n-1}G_{n+1}, \\ H_{2n+1} &= H_n G_{n+2} + sH_{n-1}G_{n+1}, \\ (r^2 + 4s)H_{2n+1} &= (2H_{n+3} - rH_{n+2})H_n + s(2H_{n+2} - rH_{n+1})H_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_{3n} &= G_n G_{2n+1} + sG_{n-1}G_{2n}, \\ H_{3n} &= H_n G_{2n+1} + sH_{n-1}G_{2n}, \\ (r^2 + 4s)H_{3n} &= (2H_{2n+2} - rH_{2n+1})H_n + s(2H_{2n+1} - rH_{2n})H_{n-1}. \end{aligned}$$

We now recall several well-known identities for the generalized Fibonacci (Horadam) polynomials $\{W_n\}$, which are expressed in terms of the roots of their characteristic equation.

Lemma 1.13 ([9], Lemma 33)**.** *Let $n, m \in \mathbb{Z}$. Then*

(a) $\left(W_1 - (r-\alpha)W_0\right)\alpha^n = \alpha W_n + sW_{n-1}$ *and* $\left(W_1 - (r-\alpha)W_0\right)\alpha^{n-1} = W_n - (r-\alpha)W_{n-1}$.

(b) $\left(W_1 - (r-\beta)W_0\right)\beta^n = \beta W_n + sW_{n-1}$ *and* $\left(W_1 - (r-\beta)W_0\right)\beta^{n-1} = W_n - (r-\beta)W_{n-1}$.

(c) *If $y^2 - ry - s = 0$ then*

- **(i)** $\left(W_1 - (r-y)W_0\right)y^n = yW_n + sW_{n-1}$.
- **(ii)** $\left(W_1 - (r-y)W_0\right)y^{n-1} = W_n - (r-y)W_{n-1}$.

(iii) $(W_1-(r-y)W_0)^2\, y^{n+m}=W_mW_ny^2+s(W_nW_{m-1}+W_mW_{n-1})y+s^2W_{m-1}W_{n-1}$.

(d)

(i) $-sW_n^2+s^2W_{n-1}^2+srW_nW_{n-1}=(W_1^2-sW_0^2-rW_1W_0)(-s)^n$.

(ii) $W_{n+1}^2-sW_n^2-rW_{n+1}W_n=(W_1^2-sW_0^2-rW_1W_0)(-s)^n$.

(e)

(i)

1. $\alpha^n=\frac{1}{2}(H_n+\sqrt{r^2+4s}G_n)$.
2. $\alpha^n=\frac{1}{2}(2G_{n+1}+(\sqrt{r^2+4s}-r)G_n)$.
3. $\sqrt{r^2+4s}\alpha^n=\frac{1}{2}(2H_{n+1}+(\sqrt{r^2+4s}-r)H_n)$

(ii)

1. $\beta^n=\frac{1}{2}(H_n-\sqrt{r^2+4s}G_n)$.
2. $\beta^n=\frac{1}{2}(2G_{n+1}-(\sqrt{r^2+4s}+r)G_n)$.
3. $\sqrt{r^2+4s}\beta^n=\frac{1}{2}(-2H_{n+1}+(\sqrt{r^2+4s}+r)H_n)$.

(iii) $H_n^2=(r^2+4s)G_n^2+4(-s)^n$,
i.e.,
$H_n^2-(r^2+4s)G_n^2=4(-s)^n$.

(iv) $H_{2n}=(r^2+4s)G_n^2+2(-s)^n$.

(v) $H_n^2-H_{2n}=2(-s)^n$,
i.e.,
$H_n^2=H_{2n}+2(-s)^n$.

(vi) $(H_m^2-H_{2m})(H_n^2-H_{2n})=4(-s)^{m+n}$.

(f)

(i) $(G_n^2H_m^2+G_m^2H_n^2)-(r^2+4s)(G_m^2G_n^2+G_n^2G_m^2)=4((-s)^mG_n^2+4(-s)^nG_m^2)$.

(ii) $G_n^2H_m^2-G_m^2H_n^2=4((-s)^mG_n^2-(-s)^nG_m^2)$.

(iii) $2H_m^2H_n^2-(r^2+4s)(G_m^2H_n^2+G_n^2H_m^2)=4((-s)^mH_n^2+(-s)^nH_m^2)$.

(iv) $(r^2+4s)(G_n^2H_m^2-G_m^2H_n^2)=4((-s)^mH_n^2-(-s)^nH_m^2)$.

By setting $W_n=G_n$ with $G_0=0$ and $G_1=1$ in the preceding lemma, we obtain the following corollary.

Corollary 1.14 ([9], Corollary 34)**.** *Let $n\in\mathbb{Z}$. Then*

(a) $\alpha^n=\alpha G_n+sG_{n-1}$ *and* $\alpha^{n-1}=G_n-\beta G_{n-1}$.

(b) $\beta^n=\beta G_n+sG_{n-1}$ *and* $\beta^{n-1}=G_n-\alpha G_{n-1}$.

(c) *If* $y^2-ry-s=0$ *then*

(i) $y^n=yG_n+sG_{n-1}$.

(ii) $y^{n-1} = G_n - (r-y)G_{n-1}$.

(iii) $y^{n+m} = G_m G_n y^2 + s(G_n G_{m-1} + G_m G_{n-1})y + s^2 G_{m-1} G_{n-1}$.

(d)

(i) $-sG_n^2 + s^2 G_{n-1}^2 + srG_{n-1}G_n = (-s)^n$.

(ii) $G_{n+1}^2 - sG_n^2 - rG_{n+1}G_n = (-s)^n$.

By setting $W_n = H_n$ with $H_0 = 2$ and $H_1 = r$ in the preceding lemma, we get the following corollary.

Corollary 1.15 ([9], Corollary 35)**.** *Let $n \in \mathbb{Z}$. Then*

(a) $(\alpha - \beta)\alpha^n = \alpha H_n + sH_{n-1}$ *and* $(\alpha-\beta)\alpha^{n-1} = H_n - \beta H_{n-1}$.

(b) $(\beta - \alpha)\beta^n = \beta H_n + sH_{n-1}$ *and* $(\beta-\alpha)\beta^{n-1} = H_n - \alpha H_{n-1}$.

(c) *If* $y^2 - ry - s = 0$ *then*

(i) $2y - ry^n = yH_n + sH_{n-1}$.

(ii) $(2y - r)\, y^{n-1} = H_n - (r-y)H_{n-1}$.

(iii) $(2y-r)^2\, y^{n+m} = H_m H_n y^2 + s(H_n H_{m-1} + H_m H_{n-1})y + s^2 H_{m-1} H_{n-1}$.

(d)

(i) $-sH_n^2 + s^2 H_{n-1}^2 + srH_{n-1}H_n = -(r^2+4s)(-s)^n$.

(ii) $H_{n+1}^2 - sH_n^2 - rH_{n+1}H_n = -(r^2+4s)(-s)^n$.

We know from Lemma 1.3 that

$$(-W_1^2 + sW_0^2 + rW_0W_1)H_n = (-2W_1 + rW_0)W_{n+1} + (rW_1 + 2sW_0)W_n. \tag{1.13}$$

From (1.13) we get

$$(-W_1^2 + sW_0^2 + rW_0W_1)(W_1H_n + sW_0H_{n-1}) = (-W_1^2 + sW_0^2 + rW_0W_1)(rW_n + 2sW_{n-1})$$

i.e.,

$$W_1H_n + sW_0H_{n-1} = rW_n + 2sW_{n-1}. \tag{1.14}$$

We also know from Lemma 1.2 that

$$(-W_1^2 + sW_0^2 + rW_1W_0)G_n = W_0W_{n+1} - W_1W_n. \tag{1.15}$$

We can give Catalan's identity in the following forms.

Theorem 1.16 ([9], Theorem 36)**.** *For all integers m, n, we have the following identities.*

(a)

(i) *(Catalan's identity)* $W_{n+m}W_{n-m} = W_n^2 - (-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$.

(ii) $(-W_1^2+sW_0^2+rW_1W_0)W_{n+m}W_{n-m}=(-W_1^2+sW_0^2+rW_1W_0)W_n^2+(-s)^{n-m}(W_0W_{m+1}-W_1W_m)^2.$

(iii) $(-s)^mW_{n+m}W_{n-m}=(-s)^mW_n^2-(-s)^n(W_1^2-sW_0^2-rW_0W_1)G_m^2.$

(iv) $(-W_1^2+sW_0^2+rW_1W_0)(-s)^mW_{n+m}W_{n-m}=(-W_1^2+sW_0^2+rW_1W_0)(-s)^mW_n^2+(-s)^n(W_0W_{m+1}-W_1W_m)^2.$

(b)

(i) $G_{n+m}G_{n-m}=G_n^2-(-s)^{n-m}G_m^2.$

(ii) $(-s)^mG_{n+m}G_{n-m}=(-s)^mG_n^2-(-s)^nG_m^2.$

(iii) $H_{n+m}H_{n-m}=H_n^2+(r^2+4s)(-s)^{n-m}G_m^2.$

(iv) $(-s)^mH_{n+m}H_{n-m}=(-s)^mH_n^2+(r^2+4s)(-s)^nG_m^2.$

(c)

(i) $G_{n+1}G_{n-1}=G_n^2-(-s)^{n-1}.$

(ii) $sG_{n+1}G_{n-1}=sG_n^2+(-s)^n.$

(iii) $H_{n+1}H_{n-1}=H_n^2+(r^2+4s)(-s)^{n-1}.$

(iv) $sH_{n+1}H_{n-1}=sH_n^2-(r^2+4s)(-s)^n.$

(d)

(i) $G_{n+m}G_{n-m}+H_{n+m}H_{n-m}=G_n^2+H_n^2+((r^2+4s)-1)(-s)^{n-m}G_m^2.$

(ii) $G_{n+m}G_{n-m}-H_{n+m}H_{n-m}=G_n^2-H_n^2-((r^2+4s)+1)(-s)^{n-m}G_m^2.$

(iii) $G_{n+m}G_{n-m}H_{n+m}H_{n-m}=(G_n^2-(-s)^{n-m}G_m^2)(H_n^2+(r^2+4s)(-s)^{n-m}G_m^2).$

(iv) $G_{n-m-1}G_{n-m}G_{n+m}G_{n+m+1}=(G_n^2-(-s)^{n-m}G_m^2)(G_n^2-(-s)^{n-m-1}G_{m+1}^2).$

(v) $H_{n-m-1}H_{n-m}H_{n+m}H_{n+m+1}=(H_n^2+(r^2+4s)(-s)^{n-m}G_m^2)(H_n^2+(r^2+4s)(-s)^{n-m-1}G_{m+1}^2).$

(e)

(i) $G_{n+1}G_{n-1}+H_{n+1}H_{n-1}=G_n^2+H_n^2+((r^2+4s)-1)(-s)^{n-1}.$

(ii) $G_{n+1}G_{n-1}-H_{n+1}H_{n-1}=G_n^2-H_n^2-((r^2+4s)+1)(-s)^{n-1}.$

(iii) $G_{n+1}G_{n-1}H_{n+1}H_{n-1}=(G_n^2-(-s)^{n-1})(H_n^2+(r^2+4s)(-s)^{n-1}).$

(iv) $G_{n-2}G_{n-1}G_{n+1}G_{n+2}=(G_n^2-(-s)^{n-1})(G_n^2-r^2(-s)^{n-2}).$

(v) $H_{n-2}H_{n-1}H_{n+1}H_{n+2}=(H_n^2+(r^2+4s)(-s)^{n-1})(H_n^2+r^2(r^2+4s)(-s)^{n-2}).$

(f)

(i) $G_{n+m}G_{n-m}+(-s)^mH_{n+m}H_{n-m}=G_n^2+(-s)^mH_n^2+((r^2+4s)(-s)^n-(-s)^{n-m})G_m^2.$

(ii) $G_{n+m}G_{n-m}-(-s)^mH_{n+m}H_{n-m}=G_n^2-(-s)^mH_n^2-((r^2+4s)(-s)^n+(-s)^{n-m})G_m^2.$

(iii) $H_{n+m}H_{n-m}+(-s)^mG_{n+m}G_{n-m}=H_n^2+(-s)^mG_n^2+((r^2+4s)(-s)^{n-m}-(-s)^n)G_m^2.$

(iv) $H_{n+m}H_{n-m}-(-s)^mG_{n+m}G_{n-m}=H_n^2-(-s)^mG_n^2+((r^2+4s)(-s)^{n-m}+(-s)^n)G_m^2.$

(g)

(i) $G_{n+1}G_{n-1} - sH_{n+1}H_{n-1} = G_n^2 - sH_n^2 + ((r^2+4s)(-s)^n - (-s)^{n-1})$.

(ii) $G_{n+1}G_{n-1} + sH_{n+1}H_{n-1} = G_n^2 + sH_n^2 - ((r^2+4s)(-s)^n + (-s)^{n-1})$.

(iii) $H_{n+1}H_{n-1} - sG_{n+1}G_{n-1} = H_n^2 - sG_n^2 + ((r^2+4s)(-s)^{n-1} - (-s)^n)$.

(iv) $H_{n+1}H_{n-1} + sG_{n+1}G_{n-1} = H_n^2 + sG_n^2 + ((r^2+4s)(-s)^{n-1} + (-s)^n)$.

Having recalled the fundamental definitions and classical identities, we now turn to our original findings. In the following sections, we present new identities and structural relations for the (r,s)-Fibonacci and (r,s)-Lucas polynomials, extending the established framework of Horadam-type sequences. These results highlight the deeper interplay between the two families and demonstrate how classical properties evolve within the generalized polynomial setting.

2 Identities: Group I

In what follows, we establish a collection of original identities for the generalized Fibonacci (Horadam) polynomials, which form the basis of our main contributions.

Theorem 2.1. *Let $n, m, p, q \in \mathbb{Z}$. Then*

(a)

(i)

$$\begin{aligned}(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} &= (W_0W_{m+1} - W_1W_m)W_{n+1} + (-W_1W_{m+1} + (rW_1 + sW_0)W_m)W_n \\ &= (W_0W_{m+1} - W_1W_m)W_{n+1} - (W_1W_{m+1} - W_2W_m)W_n \\ &= -\begin{vmatrix} W_0 & W_1 & W_m \\ W_1 & W_2 & W_{m+1} \\ W_n & W_{n+1} & 0 \end{vmatrix}.\end{aligned}$$

(ii)

$$\begin{aligned}(-W_1^2 + sW_0^2 + rW_0W_1)(-s)^m W_{n-m} &= -\left(W_0W_{m+1} - W_1W_m\right)W_{n+1} \\ &\quad + (-(W_1 - rW_0)W_{m+1} + sW_0W_m)W_n \\ &= -((W_0W_{m+1} - W_1W_m)W_{n+1} - s\left(-W_{-1}W_{m+1} + W_0W_m\right)W_n) \\ &= -\begin{vmatrix} W_{-1} & W_0 & W_{n+1} \\ W_0 & W_1 & -sW_n \\ W_m & W_{m+1} & 0 \end{vmatrix}.\end{aligned}$$

(iii) $(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} = ((-W_1 + rW_0)W_{m+1} + sW_0W_m)W_n + s(W_0W_{m+1} - W_1W_m)W_{n-1}$.

(iv) $(-W_1^2 + sW_0^2 + rW_0W_1)(-s)^m W_{n-m} = (-W_1W_{m+1} + W_m(rW_1 + sW_0))W_n - s(W_0W_{m+1} - W_1W_m)W_{n-1}$.

(b)

(i) $(-W_1^2 + sW_0^2 + rW_0W_1)^2(-s)^m W_{n+m}W_{n-m} = -(W_0W_{m+1} - W_1W_m)^2 W_{n+1}^2 + ((-W_1 + rW_0)W_{m+1} + sW_0W_m)(-W_1W_{m+1} + (rW_1 + sW_0)W_m)W_n^2 + r(W_0W_{m+1} - W_1W_m)^2 W_{n+1}W_n$.

(ii) $(-W_1^2+sW_0^2+rW_0W_1)^2(-s)^mW_{n+m}W_{n-m}=(-W_1W_{m+1}+(rW_1+sW_0)W_m)((-W_1+rW_0)W_{m+1}+sW_0W_m)W_n^2-s^2(W_0W_{m+1}-W_1W_m)^2W_{n-1}^2-rs(W_0W_{m+1}-W_1W_m)^2W_nW_{n-1}$.

(iii) $(-W_1^2+sW_0^2+rW_0W_1)^{p+q}(-s)^{qm}W_{n+m}^pW_{n-m}^q=((W_0W_{m+1}-W_1W_m)W_{n+1}+(-W_1W_{m+1}+(rW_1+sW_0)W_m)W_n)^p(-(W_0W_{m+1}-W_1W_m)W_{n+1}+(-(W_1-rW_0)W_{m+1}+sW_0W_m)W_n)^q$.

(iv) $(-W_1^2+sW_0^2+rW_0W_1)^{p+q}(-s)^{qm}W_{n+m}^pW_{n-m}^q=(((-W_1+rW_0)W_{m+1}+sW_0W_m)W_n+s(W_0W_{m+1}-W_1W_m)W_{n-1})^p((-W_1W_{m+1}+W_m(rW_1+sW_0))W_n-s(W_0W_{m+1}-W_1W_m)W_{n-1})^q$.

(v) $-(W_0W_{m+1}-W_1W_m)^2W_{n+1}^2+(-W_1W_{m+1}+(rW_1+sW_0)W_m)((rW_0-W_1)W_{m+1}+sW_0W_m)W_n^2-(-W_1^2+sW_0^2+rW_1W_0)^2(-s)^mW_n^2+r(W_0W_{m+1}-W_1W_m)^2W_{n+1}W_n=(-W_1^2+sW_0^2+rW_1W_0)(W_0W_{m+1}-W_1W_m)^2(-s)^n$.

(vi) $(-W_1W_{m+1}+(rW_1+sW_0)W_m)((-W_1+rW_0)W_{m+1}+sW_0W_m)W_n^2-(-W_1^2+sW_0^2+rW_1W_0)^2(-s)^mW_n^2-s^2(W_0W_{m+1}-W_1W_m)^2W_{n-1}^2-rs(W_0W_{m+1}-W_1W_m)^2W_nW_{n-1}=(-W_1^2+sW_0^2+rW_1W_0)(W_0W_{m+1}-W_1W_m)^2(-s)^n$.

(c)

(i) $(-W_1^2+sW_0^2+rW_0W_1)(W_{n+m}+(-s)^mW_{n-m})=((-2W_1+rW_0)W_{m+1}+(rW_1+2sW_0)W_m)W_n$.

(ii)

$$\begin{aligned}(-W_1^2+sW_0^2+rW_0W_1)(W_{n+m}-(-s)^mW_{n-m}) &= (W_0W_{m+1}-W_1W_m)(2W_{n+1}-rW_n)\\ &= (W_0W_{m+1}-W_1W_m)(rW_n+2sW_{n-1}).\end{aligned}$$

(iii) $(-W_1^2+sW_0^2+rW_0W_1)^p(W_{n+m}^p+(-s)^{pm}W_{n-m}^p)=((W_0W_{m+1}-W_1W_m)W_{n+1}+(-W_1W_{m+1}+(rW_1+sW_0)W_m)W_n)^p+(-(W_0W_{m+1}-W_1W_m)W_{n+1}+(-(W_1-rW_0)W_{m+1}+sW_0W_m)W_n)^p$.

(iv) $(-W_1^2+sW_0^2+rW_0W_1)^p(W_{n+m}^p-(-s)^{pm}W_{n-m}^p)=((W_0W_{m+1}-W_1W_m)W_{n+1}+(-W_1W_{m+1}+(rW_1+sW_0)W_m)W_n)^p-(-(W_0W_{m+1}-W_1W_m)W_{n+1}+(-(W_1-rW_0)W_{m+1}+sW_0W_m)W_n)^p$.

(d)

(i) $(-W_1^2+sW_0^2+rW_0W_1)W_{n+m}=(-W_1W_m-sW_0W_{m-1}+W_0(W_1H_m+sW_0H_{m-1}))W_{n+1}-s(W_0W_m+(W_1-rW_0)W_{m-1}-W_0^2H_m+W_0(-W_1+rW_0)H_{m-1})W_n$.

(ii) $(-W_1^2+sW_0^2+rW_0W_1)(-s)^mW_{n-m}=(W_1W_m+sW_0W_{m-1}-W_0(W_1H_m+sW_0H_{m-1}))W_{n+1}+(-W_1(rW_m+sW_{m-1})-sW_0W_m+W_0((rW_1+sW_0)H_m+sW_1H_{m-1}))W_n$.

(iii) $(-W_1^2+sW_0^2+rW_0W_1)^2(-s)^mW_{n+m}W_{n-m}=-(W_1W_m+sW_0W_{m-1}-W_0(W_1H_m+sW_0H_{m-1}))^2W_{n+1}^2+s(W_0W_m+(W_1-rW_0)W_{m-1}-W_0^2H_m+W_0(-W_1+rW_0)H_{m-1})((rW_1+sW_0)W_m+sW_1W_{m-1}-W_0(rW_1+sW_0)H_m-sW_0W_1H_{m-1})W_n^2+r(W_1W_m+sW_0W_{m-1}-W_0(W_1H_m+sW_0H_{m-1}))^2W_{n+1}W_n$

(iv) $(-W_1^2+sW_0^2+rW_0W_1)(W_{n+m}+(-s)^mW_{n-m})=(-(rW_1+2sW_0)W_m+s(-2W_1+rW_0)W_{m-1}+W_0(rW_1+2sW_0)H_m+sW_0(2W_1-rW_0)H_{m-1})W_n$.

(v) $(-W_1^2+sW_0^2+rW_0W_1)(W_{n+m}-(-s)^mW_{n-m})=(-W_1W_m-sW_0W_{m-1}+W_0(W_1H_m+sW_0H_{m-1}))(2W_{n+1}-rW_n)$.

Proof.

(a) Note that

$$\begin{vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{vmatrix} = W_0W_2 - W_1^2 = W_0(rW_1 + sW_0) - W_1^2 = -W_1^2 + sW_0^2 + rW_0W_1,$$

$$\begin{vmatrix} W_{-1} & W_0 \\ W_0 & W_1 \end{vmatrix} = W_1W_{-1} - W_0^2 = W_1\frac{1}{s}(W_1 - rW_0) - W_0^2 = -\frac{1}{s}(-W_1^2 + sW_0^2 + rW_0W_1).$$

(i)-(ii) These can be proved either by induction or by using the Binet formula for W_n.

(iii) Apply the recurrence $W_{n+1} = rW_n + sW_{n-1}$ to (i).

(iv) Apply the recurrence $W_{n+1} = rW_n + sW_{n-1}$ to (ii).

(b)

(i) Compare (a)(i) and (a)(ii) side by side..

(ii) Use the recurrence $W_{n+1} = rW_n + sW_{n-1}$ in (i), or compare (a)(iii) and (a)(iv).

(iii) Use (a) (i) and (a) (ii).

(iv) Use (a) (iii) and (a) (iv).

(v)-(vi) By Theorem 1.16 (b), we obtain

$$(-W_1^2 + sW_0^2 + rW_1W_0)^2W_{n+m}W_{n-m} = (-W_1^2 + sW_0^2 + rW_1W_0)^2W_n^2 + (W_0W_{m+1} - W_1W_m)^3(-s)^{n-m}.$$

Comparing the right-hand sides with (i) and (ii), respectively, yields the required results.

(c) **(i)** Adding (a)(i) and (a)(ii) side by side gives the required identity; alternatively, add (a)(iii) and (a)(iv).

(ii) Subtract (a)(i) and (a)(ii), then apply $W_{n+1} = rW_n + sW_{n-1}$; alternatively, subtract (a)(iii) and (a)(iv).

(iii) Use (a) (i) and (a) (ii).

(iv) Use (a) (i) and (a) (ii).

(d) **(i)** Replace m with $-m$ in (a)(ii) and use the identity

$$W_{-n} = -(-s)^{-n}(W_n - H_nW_0),$$

given in Theorem 1.7.

(ii) Replace m with $-m$ in (a)(i) and use the identity from (i)

(iii) Combine (i) and (ii).

(iv) Using (i) together with (1.13), we obtain the required identity.

(v) Using (ii) together with (1.13), we obtain the required identity. □

Taking $W_n = G_n$ with $G_0 = 0, G_1 = 1$ in Theorem 2.1, we get the following Corollary. This Corollary illustrates the structural link between the (r, s)-Fibonacci polynomials $\{G_n\}$ and the (r, s)-Lucas polynomials $\{H_n\}$. In particular, it shows how the identities governing G_n naturally extend to H_n, thereby reinforcing the deep interplay between these two families within the Horadam framework.

Corollary 2.2. *Let $n, m, p, q \in \mathbb{Z}$. Then*

(a)

(i)

$$G_{n+m}=G_mG_{n+1}+sG_{m-1}G_n=\begin{vmatrix} 0 & 1 & G_m \\ 1 & r & G_{m+1} \\ G_n & G_{n+1} & 0 \end{vmatrix},$$

i.e.,

$$G_{n+m}=G_mG_{n+1}+(G_{m+1}-rG_m)G_n=\begin{vmatrix} 0 & 1 & G_m \\ 1 & r & G_{m+1} \\ G_n & G_{n+1} & 0 \end{vmatrix}.$$

(ii)

$$(-s)^mG_{n-m}=-G_mG_{n+1}+G_{m+1}G_n=\begin{vmatrix} \frac{1}{s} & 0 & G_{n+1} \\ 0 & 1 & -sG_n \\ G_m & G_{m+1} & 0 \end{vmatrix}.$$

(iii) $G_{n+m}=G_{m+1}G_n+sG_mG_{n-1}$.

(iv) $(-s)^mG_{n-m}=sG_{m-1}G_n-sG_mG_{n-1}$.

(b)

(i) $(-s)^mG_{n+m}G_{n-m}=-G_m^2G_{n+1}^2+sG_{m-1}G_{m+1}G_n^2+rG_m^2G_{n+1}G_n$.

(ii) $(-s)^mG_{n+m}G_{n-m}=sG_{m-1}G_{m+1}G_n^2-s^2G_m^2G_{n-1}^2-rsG_m^2G_nG_{n-1}$.

(iii) $(-1)^{p+q}(-s)^{qm}G_{n+m}^pG_{n-m}^q=(-G_mG_{n+1}-sG_{m-1}G_n)^p(G_mG_{n+1}-G_{m+1}G_n)^q$.

(iv) $(-1)^{p+q}(-s)^{qm}G_{n+m}^pG_{n-m}^q=(-G_{m+1}G_n-sG_mG_{n-1})^p(-sG_{m-1}G_n+sG_mG_{n-1})^q$.

(v) $-G_m^2G_{n+1}^2+(sG_{m-1}G_{m+1}-(-s)^m)G_n^2+rG_m^2G_nG_{n+1}=-(-s)^nG_m^2$.

(vi) $(sG_{m-1}G_{m+1}-(-s)^m)G_n^2-s^2G_m^2G_{n-1}^2-rsG_m^2G_nG_{n-1}=-(-s)^nG_m^2$.

(c)

(i) $G_{n+m}+(-s)^mG_{n-m}=H_mG_n$.

(ii) $G_{n+m}-(-s)^mG_{n-m}=G_mH_n$.

(iii) $(-1)^p(G_{n+m}^p+(-s)^{pm}G_{n-m}^p)=(-G_mG_{n+1}-sG_{m-1}G_n)^p+(G_mG_{n+1}-G_{m+1}G_n)^p$.

(iv) $(-1)^p(G_{n+m}^p-(-s)^{pm}G_{n-m}^p)=(-G_mG_{n+1}-sG_{m-1}G_n)^p-(G_mG_{n+1}-G_{m+1}G_n)^p$.

(v)

1. $G_{n+m}^2+s^{2m}G_{n-m}^2=(2G_{m+1}^2+r^2G_m^2-2rG_{m+1}G_m)G_n^2+2sG_m^2G_{n-1}G_{n+1}$.
2. $G_{n+m}^2+s^{2m}G_{n-m}^2=(2G_{m+1}^2+(r^2+2s)G_m^2-2rG_{m+1}G_m)G_n^2+2\left(-s\right)^nG_m^2$.
3. $G_{n+m}^2+s^{2m}G_{n-m}^2=(H_m^2-2(-s)^m)G_n^2+2(-s)^nG_m^2$.
4. $G_{n+m}^2+s^{2m}G_{n-m}^2=2(-s)^mG_n^2+(H_n^2-2(-s)^n)G_m^2$.

(vi) $G_{n+m}^2-s^{2m}G_{n-m}^2=H_mG_mH_nG_n=G_{2m}G_{2n}$.

(vii)

1. $G_{n+m}^3+(-s)^{3m}G_{n-m}^3=(G_{m+1}^2+r^2G_m^2-rG_{m+1}G_m)H_mG_n^3+3sH_mG_m^2G_{n-1}G_nG_{n+1}$.
2. $G_{n+m}^3+(-s)^{3m}G_{n-m}^3=(G_{m+1}^2+(r^2+3s)G_m^2-rG_{m+1}G_m)H_mG_n^3+3(-s)^nG_m^2H_mG_n$.

3. $G_{n+m}^3+(-s)^{3m}G_{n-m}^3=H_m^3G_n^3-3((-s)^mG_n^2-(-s)^nG_m^2)H_mG_n$.

(viii)

1. $G_{n+m}^3-(-s)^{3m}G_{n-m}^3=(3G_{m+1}^2+r^2G_m^2-3rG_{m+1}G_m)G_mH_nG_n^2+sG_m^3H_nG_{n-1}G_{n+1}$.
2. $G_{n+m}^3-(-s)^{3m}G_{n-m}^3=(3G_{m+1}^2+(r^2+s)G_m^2-3rG_{m+1}G_m)G_mH_nG_n^2+(-s)^nH_nG_m^3$.
3. $G_{n+m}^3-(-s)^{3m}G_{n-m}^3=H_n^3G_m^3+3((-s)^mG_n^2-(-s)^nG_m^2)G_mH_n$.

(ix)

1. $G_{n+m}^4+s^{4m}G_{n-m}^4=H_m^4G_n^4-2((-s)^mG_n^2-(-s)^nG_m^2)(2H_m^2G_n^2-((-s)^mG_n^2-(-s)^nG_m^2))$.
2. $G_{n+m}^4+s^{4m}G_{n-m}^4=H_m^4G_n^4-2(-s)^mG_{n-m}G_{n+m}(G_m^2G_{n+1}^2-(G_{m+1}^2-2H_m^2-rG_mG_{m+1})G_n^2-rG_m^2G_{n+1}G_n)$.

(x)

1. $G_{n+m}^4-s^{4m}G_{n-m}^4=(2G_{m+1}^2+r^2G_m^2-2rG_{m+1}G_m)H_mG_mH_nG_n^3+2sH_mG_m^3H_nG_{n-1}G_nG_{n+1}$.
2. $G_{n+m}^4-s^{4m}G_{n-m}^4=(2G_{m+1}^2+(r^2+2s)G_m^2-2rG_{m+1}G_m)H_mG_mH_nG_n^3+2(-s)^nH_mG_m^3H_nG_n$.

(d)

(i) $2G_{n+m}=H_mG_n+G_mH_n$.

(ii) $2(-s)^mG_{n-m}=H_mG_n-G_mH_n$.

(iii)

1. $2G_{n+m}^2=(2G_{m+1}^2+r^2G_m^2-2rG_{m+1}G_m)G_n^2+2sG_m^2G_{n-1}G_{n+1}+H_mG_mH_nG_n$.
2. $2G_{n+m}^2=(2G_{m+1}^2+(r^2+2s)G_m^2-2rG_{m+1}G_m)G_n^2+2(-s)^nG_m^2+H_mG_mH_nG_n$.
3. $2G_{n+m}^2=(H_m^2-2(-s)^m)G_n^2+2(-s)^nG_m^2+H_mG_mH_nG_n$.
4. $2G_{n+m}^2=2(-s)^mG_n^2+(H_n^2-2(-s)^n)G_m^2+H_mG_mH_nG_n$.

(iv)

1. $2s^{2m}G_{n-m}^2=(2G_{m+1}^2+r^2G_m^2-2rG_{m+1}G_m)G_n^2+2sG_m^2G_{n-1}G_{n+1}-H_mG_mH_nG_n$.
2. $2s^{2m}G_{n-m}^2=(2G_{m+1}^2+(r^2+2s)G_m^2-2rG_{m+1}G_m)G_n^2+2(-s)^nG_m^2-H_mG_mH_nG_n$.
3. $2s^{2m}G_{n-m}^2=(H_m^2-2(-s)^m)G_n^2+2(-s)^nG_m^2-H_mG_mH_nG_n$.
4. $2s^{2m}G_{n-m}^2=2(-s)^mG_n^2+(H_n^2-2(-s)^n)G_m^2-H_mG_mH_nG_n$.

(v) $2G_{n+m}^3=G_m^3H_n^3+G_n^3H_m^3+3(G_mH_n-G_nH_m)((-s)^mG_n^2-(-s)^nG_m^2)$.

(vi) $2(-s)^{3m}G_{n-m}^3=G_n^3H_m^3-G_m^3H_n^3-3(G_mH_n+G_nH_m)((-s)^mG_n^2-(-s)^nG_m^2)$.

In Corollary 2.2, we employed the following identities:

$$\begin{aligned}
H_n &= 2G_{n+1}-rG_n,\\
sG_{n-1} &= G_{n+1}-rG_n,\\
G_{n+1}G_{n-1}-G_n^2 &= -(-s)^{n-1},\\
(-s)^mG_{n+m}G_{n-m} &= (-s)^mG_n^2-(-s)^nG_m^2.
\end{aligned}$$

By setting $W_n=H_n$ with $H_0=2$ and $H_1=r$ in Theorem 2.1, we obtain the following corollary. This corollary highlights the intrinsic relationship between the (r,s)-Fibonacci polynomials $\{G_n\}$ and the (r,s)-Lucas polynomials $\{H_n\}$. In particular, it demonstrates that the structural identities satisfied by $\{H_n\}$ naturally extend to $\{G_n\}$, thereby underscoring the close correspondence of these two families within the Horadam framework.

Corollary 2.3. *Let $n, m, p, q \in \mathbb{Z}$. Then*

(a)

(i)

$$\begin{aligned}(r^2+4s)H_{n+m} &= (2H_{m+1}-rH_m)H_{n+1}-(rH_{m+1}-(r^2+2s)H_m)H_n \\ &= -\begin{vmatrix} H_0 & H_1 & H_m \\ H_1 & H_2 & H_{m+1} \\ H_n & H_{n+1} & 0 \end{vmatrix}.\end{aligned}$$

i.e.,

$H_{n+m} = G_m H_{n+1} + sG_{m-1}H_n.$

(ii)

$$\begin{aligned}(r^2+4s)(-s)^m H_{n-m} &= -(2H_{m+1}-rH_m)H_{n+1}+(rH_{m+1}+2sH_m)H_n \\ &= -\begin{vmatrix} -\frac{r}{s} & H_0 & H_{n+1} \\ H_0 & H_1 & -sH_n \\ H_m & H_{m+1} & 0 \end{vmatrix},\end{aligned}$$

i.e.,

$(-s)^m H_{n-m} = -G_m H_{n+1} + G_{m+1}H_n.$

(iii) $H_{n+m} = G_{m+1}H_n + sG_m H_{n-1}.$

(iv) $(-s)^m H_{n-m} = sG_{m-1}H_n - sG_m H_{n-1}.$

(b)

(i) $(-s)^m H_{n+m}H_{n-m} = -G_m^2 H_{n+1}^2 + sG_{m+1}G_{m-1}H_n^2 + rG_m^2 H_{n+1}H_n.$

(ii) $(-s)^m H_{n+m}H_{n-m} = sG_{m+1}G_{m-1}H_n^2 - s^2G_m^2H_{n-1}^2 - rsG_m^2H_nH_{n-1}.$

(iii) $(-s)^{qm} H_{n+m}^p H_{n-m}^q = (G_mH_{n+1}+sG_{m-1}H_n)^p(-G_mH_{n+1}+G_{m+1}H_n)^q.$

(iv) $(-s)^{qm} H_{n+m}^p H_{n-m}^q = (G_{m+1}H_n+sG_mH_{n-1})^p(sG_{m-1}H_n-sG_mH_{n-1})^q.$

(v) $-G_m^2H_{n+1}^2 + (sG_{m+1}G_{m-1}-(-s)^m)H_n^2 + rG_m^2H_{n+1}H_n = (r^2+4s)G_m^2(-s)^n$

(vi) $(sG_{m+1}G_{m-1}-(-s)^m)H_n^2 - s^2G_m^2H_{n-1}^2 - rsG_m^2H_nH_{n-1} = (r^2+4s)G_m^2(-s)^n.$

(c)

(i) $H_{n+m} + (-s)^m H_{n-m} = H_m H_n.$

(ii) $H_{n+m} - (-s)^m H_{n-m} = (r^2+4s)G_mG_n.$

(iii) $H_{n+m}^p + (-s)^{pm}H_{n-m}^p = (G_mH_{n+1}+sG_{m-1}H_n)^p + (-G_mH_{n+1}+G_{m+1}H_n)^p.$

(iv) $H_{n+m}^p - (-s)^{pm}H_{n-m}^p = (G_mH_{n+1}+sG_{m-1}H_n)^p - (-G_mH_{n+1}+G_{m+1}H_n)^p.$

(v)

1. $H_{n+m}^2 + s^{2m}H_{n-m}^2 = (2G_{m+1}^2 + r^2G_m^2 - 2rG_mG_{m+1})H_n^2 + 2sG_m^2H_{n-1}H_{n+1}.$
2. $H_{n+m}^2 + s^{2m}H_{n-m}^2 = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_mG_{m+1})H_n^2 - 2\left(-s\right)^n(r^2+4s)G_m^2.$
3. $H_{n+m}^2 + s^{2m}H_{n-m}^2 = (H_m^2 - 2(-s)^m)H_n^2 - 2(r^2+4s)(-s)^nG_m^2.$

4. $H_{n+m}^2 + s^{2m} H_{n-m}^2 = 2(-s)^m H_n^2 + (r^2+4s)((r^2+4s)G_n^2 + 2(-s)^n)G_m^2$.

(vi) $H_{n+m}^2 - s^{2m} H_{n-m}^2 = (r^2+4s)G_m H_m G_n H_n$.

(vii)

1. $H_{n+m}^3 + (-s)^{3m} H_{n-m}^3 = (G_{m+1}^2 + r^2 G_m^2 - rG_{m+1}G_m)H_m H_n^3 + 3sG_m^2 H_m H_{n-1} H_n H_{n+1}$.
2. $H_{n+m}^3 + (-s)^{3m} H_{n-m}^3 = (G_{m+1}^2 + (r^2+3s)G_m^2 - rG_{m+1}G_m)H_m H_n^3 - 3(-s)^n (r^2+4s)G_m^2 H_m H_n$.
3. $H_{n+m}^3 + (-s)^{3m} H_{n-m}^3 = (H_m^2 - 3(-s)^m)H_m H_n^3 - 3(r^2+4s)(-s)^n G_m^2 H_m H_n$.

(viii)

1. $H_{n+m}^3 - (-s)^{3m} H_{n-m}^3 = (r^2+4s)(3G_{m+1}^2 + r^2 G_m^2 - 3rG_{m+1}G_m)G_m G_n H_n^2 + s(r^2+4s)G_m^3 G_n H_{n+1} H_{n-1}$.
2. $H_{n+m}^3 - (-s)^{3m} H_{n-m}^3 = (r^2+4s)(3G_{m+1}^2 + (r^2+s)G_m^2 - 3rG_{m+1}G_m)G_m G_n H_n^2 - (-s)^n (r^2+4s)^2 G_m^3 G_n$.
3. $H_{n+m}^3 - (-s)^{3m} H_{n-m}^3 = 3(r^2+4s)(-s)^m G_m G_n H_n^2 + (r^2+4s)^2((r^2+4s)G_n^2 + 3(-s)^n)G_n G_m^3$.

(ix)

1. $H_{n+m}^4 + s^{4m} H_{n-m}^4 = H_m^4 H_n^4 - 2((-s)^m H_n^2 + (r^2+4s)(-s)^n G_m^2)(2H_m^2 H_n^2 - ((-s)^m H_n^2 + (r^2+4s)(-s)^n G_m^2))$.
2. $H_{n+m}^4 + s^{4m} H_{n-m}^4 = H_m^4 H_n^4 - 2(sG_{m+1}G_{m-1}H_n^2 - sG_m^2 H_{n-1}H_{n+1})(2H_m^2 H_n^2 - sG_{m+1}G_{m-1}H_n^2 + sG_m^2 H_{n-1}H_{n+1})$.

(x)

1. $H_{n+m}^4 - s^{4m} H_{n-m}^4 = (r^2+4s)(2G_{m+1}^2 + r^2 G_m^2 - 2rG_m G_{m+1})G_m H_m G_n H_n^3 + 2s(r^2+4s)H_m G_m^3 G_n H_{n-1} H_n H_{n+1}$.
2. $H_{n+m}^4 - s^{4m} H_{n-m}^4 = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_m G_{m+1})(r^2+4s)G_m H_m G_n H_n^3 - 2(-s)^n (r^2+4s)^2 H_m G_m^3 G_n H_n$.

(d)

(i) $2H_{n+m} = H_m H_n + (r^2+4s)G_m G_n$.

(ii) $2(-s)^m H_{n-m} = H_m H_n - (r^2+4s)G_m G_n$.

(iii)

1. $2H_{n+m}^2 = (2G_{m+1}^2 + r^2 G_m^2 - 2rG_m G_{m+1})H_n^2 + 2sG_m^2 H_{n-1}H_{n+1} + (r^2+4s)G_m H_m G_n H_n$.
2. $2H_{n+m}^2 = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_m G_{m+1})H_n^2 - 2(-s)^n (r^2+4s)G_m^2 + (r^2+4s)G_m H_m G_n H_n$.
3. $2H_{n+m}^2 = (H_m^2 - 2(-s)^m)H_n^2 - 2(r^2+4s)(-s)^n G_m^2 + (r^2+4s)G_m H_m G_n H_n$.
4. $2H_{n+m}^2 = 2(-s)^m H_n^2 + (r^2+4s)((r^2+4s)G_n^2 + 2(-s)^n)G_m^2 + (r^2+4s)G_m H_m G_n H_n$.

(iv)

1. $2s^{2m} H_{n-m}^2 = (2G_{m+1}^2 + r^2 G_m^2 - 2rG_m G_{m+1})H_n^2 + 2sG_m^2 H_{n-1}H_{n+1} - (r^2+4s)G_m H_m G_n H_n$.
2. $2s^{2m} H_{n-m}^2 = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_m G_{m+1})H_n^2 - 2(-s)^n (r^2+4s)G_m^2 - (r^2+4s)G_m H_m G_n H_n$.
3. $2s^{2m} H_{n-m}^2 = (H_m^2 - 2(-s)^m)H_n^2 - 2(r^2+4s)(-s)^n G_m^2 - (r^2+4s)G_m H_m G_n H_n$.
4. $2s^{2m} H_{n-m}^2 = 2(-s)^m H_n^2 + (r^2+4s)((r^2+4s)G_n^2 + 2(-s)^n)G_m^2 - (r^2+4s)G_m H_m G_n H_n$.

(v) $2H_{n+m}^3 = ((H_m^2 - 3(-s)^m)H_m H_n + 3(r^2+4s)(-s)^m G_m G_n)H_n^2 + (r^2+4s)((r^2+4s)((r^2+4s)G_n^2 + 3(-s)^n)G_n G_m - 3(-s)^n H_m H_n)G_m^2$.

(vi) $2(-s)^{3m}H_{n-m}^3 = ((H_m^2 - 3(-s)^m)H_mH_n - 3(r^2+4s)(-s)^mG_mG_n)H_n^2 - (r^2+4s)((r^2+4s)((r^2+4s)G_n^2 + 3(-s)^n)G_nG_m + 3(-s)^nH_mH_n)G_m^2$.

In Corollary 2.3, we employed the following identities:

$$\begin{aligned} sG_{m-1} &= G_{m+1} - rG_m, \\ H_m &= 2G_{m+1} - rG_m, \\ (r^2s+4s^2)G_n &= -rH_{n+2} + (r^2+2s)H_{n+1}, \\ (r^2+4s)G_n &= 2H_{n+1} - rH_n, \\ (r^2+4s)G_n &= rH_n + 2sH_{n-1}, \\ G_{n+1}G_{n-1} - G_n^2 &= -(-s)^{n-1}, \\ H_{n+1}H_{n-1} - H_n^2 &= (-s)^{n-1}(r^2+4s). \end{aligned}$$

From Corollary 2.2 (d)(iii)–(vi) together with Corollary 2.3 (d)(iii)–(vi), we derive the following results.

Corollary 2.4. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $2(G_{n+m} + (-s)^mH_{n-m}) = (G_m + H_m)H_n + (H_m - (r^2+4s)G_m)G_n$.
2. $G_{n+m} + (-s)^mH_{n-m} = G_mG_{n+1} + sG_{m-1}G_n - G_mH_{n+1} + G_{m+1}H_n$.

(ii)

1. $2(G_{n+m} - (-s)^mH_{n-m}) = (G_m - H_m)H_n + (H_m + (r^2+4s)G_m)G_n$.
2. $G_{n+m} - (-s)^mH_{n-m} = G_mG_{n+1} + sG_{m-1}G_n + G_mH_{n+1} - G_{m+1}H_n$.

(iii)

1. $2(H_{n+m} + (-s)^mG_{n-m}) = (-G_m + H_m)H_n + (H_m + (r^2+4s)G_m)G_n$.
2. $H_{n+m} + (-s)^mG_{n-m} = -G_mG_{n+1} + G_{m+1}G_n + G_mH_{n+1} + sG_{m-1}H_n$.

(iv)

1. $2(H_{n+m} - (-s)^mG_{n-m}) = (G_m + H_m)H_n + (-H_m + (r^2+4s)G_m)G_n$.
2. $H_{n+m} - (-s)^mG_{n-m} = G_mG_{n+1} - G_{m+1}G_n + G_mH_{n+1} + sG_{m-1}H_n$.

(v) $2(G_{n+m} + H_{n+m}) = (G_m + H_m)H_n + (H_m + (r^2+4s)G_m)G_n$.

(vi) $2(G_{n+m} - H_{n+m}) = (G_m - H_m)H_n + (H_m - (r^2+4s)G_m)G_n$.

(vii) $2(-s)^m(G_{n-m} + H_{n-m}) = (G_n + H_n)H_m - (H_n + (r^2+4s)G_n)G_m$.

(viii) $2(-s)^m(G_{n-m} - H_{n-m}) = (G_n - H_n)H_m - (H_n - (r^2+4s)G_n)G_m$.

(b)

(i) $2(G_{n+m}^2 + s^{2m}H_{n-m}^2) = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_{m+1}G_m)(G_n^2 + H_n^2) - 2((r^2+4s)-1)(-s)^nG_m^2 - ((r^2+4s)-1)G_mH_mG_nH_n$.

(ii) $2(G_{n+m}^2 - s^{2m}H_{n-m}^2) = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_{m+1}G_m)(G_n^2 - H_n^2) + 2((r^2+4s)+1)(-s)^nG_m^2 + ((r^2+4s)+1)G_mH_mG_nH_n$.

(iii) $2(H_{n+m}^2 + s^{2m}G_{n-m}^2) = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_mG_{m+1})(G_n^2 + H_n^2) - 2((r^2+4s)-1)(-s)^nG_m^2 + ((r^2+4s)-1)G_mH_mG_nH_n$.

(iv) $2(H_{n+m}^2 - s^{2m}G_{n-m}^2) = (2G_{m+1}^2 + (r^2+2s)G_m^2 - 2rG_mG_{m+1})(-G_n^2 + H_n^2) - 2((r^2+4s)+1)(-s)^nG_m^2 + ((r^2+4s)+1)G_mH_mG_nH_n$.

(c)

(i) $2(G_{n+m}^2 + s^{2m}H_{n-m}^2) = (H_m^2 - 2(-s)^m)(G_n^2 + H_n^2) - 2((r^2+4s)-1)(-s)^nG_m^2 - ((r^2+4s)-1)G_mH_mG_nH_n$.

(ii) $2(G_{n+m}^2 - s^{2m}H_{n-m}^2) = (H_m^2 - 2(-s)^m)(G_n^2 - H_n^2) + 2((r^2+4s)+1)(-s)^nG_m^2 + ((r^2+4s)+1)G_mH_mG_nH_n$.

(iii) $2(H_{n+m}^2 + s^{2m}G_{n-m}^2) = (H_m^2 - 2(-s)^m)(G_n^2 + H_n^2) - 2((r^2+4s)-1)(-s)^n G_m^2 + ((r^2+4s)-1)H_mG_mH_nG_n$.

(iv) $2(H_{n+m}^2 - s^{2m}G_{n-m}^2) = (H_m^2 - 2(-s)^m)(-G_n^2 + H_n^2) - 2((r^2+4s)+1)(-s)^nG_m^2 + ((r^2+4s)+1)G_mH_mG_nH_n$.

(d)

(i) $2(G_{n+m}^2 + s^{2m}H_{n-m}^2) = 2(-s)^m(G_n^2 + H_n^2) + ((r^2+4s)^2G_n^2 + H_n^2 + 2((r^2+4s)-1)(-s)^n)G_m^2 - ((r^2+4s)-1)G_mH_mG_nH_n$.

(ii) $2(G_{n+m}^2 - s^{2m}H_{n-m}^2) = 2(-s)^m(G_n^2 - H_n^2) + (-(r^2+4s)^2G_n^2 + H_n^2 - 2((r^2+4s)+1)(-s)^n)G_m^2 + ((r^2+4s)+1)G_mH_mG_nH_n$.

(iii) $2(H_{n+m}^2 + s^{2m}G_{n-m}^2) = 2(-s)^m(G_n^2 + H_n^2) + ((r^2+4s)^2G_n^2 + H_n^2 + 2\left((r^2+4s)-1\right)(-s)^n)G_m^2 + ((r^2+4s)-1)H_mG_mH_nG_n$.

(iv) $2(H_{n+m}^2 - s^{2m}G_{n-m}^2) = 2(-s)^m(-G_n^2 + H_n^2) + ((r^2+4s)^2G_n^2 - H_n^2 + 2\left((r^2+4s)+1\right)(-s)^n)G_m^2 + ((r^2+4s)+1)H_mG_mH_nG_n$.

Proof.

(a) Apply Corollary 2.2 (d)(i)-(ii) together with Corollary 2.3 (d)(i)-(ii). Similarly, use Corollary 2.2 (a)(i)-(ii) and Corollary 2.3(a)(i)-(ii). Finally, employ Corollary 2.2 (d)(iii)(1), (iii)(2) and Corollary 2.3 (d)(iv)(1), (iv)(2).

(b)

(i) Use Corollary 2.2 (d) (iii) (1) with Corollary 2.3 (d) (iv) (1), or alternatively Corollary 2.2 (d) (iii) (2) with Corollary 2.3 (d) (iv) (2).

(ii) Same as (i): apply either the pair (d)(iii)(1)-(d)(iv)(1) or (d)(iii)(2)-(d)(iv)(2).

(iii) Use Corollary 2.2 (d) (iv) (1) with Corollary 2.3 (d) (iii) (1), or alternatively Corollary 2.2 (d) (iv) (2) with Corollary 2.3 (d) (iii) (2).

(iv) Again, apply either (d)(iv)(1)-(d)(iii)(1) or (d)(iv)(2)-(d)(iii)(2).

(c)

(i) Use Corollary 2.2 (d) (iii) (3) together with Corollary 2.3 (d) (iv) (3).

(ii) Same as (i).

(iii) Use Corollary 2.2 (d) (iv) (3) with Corollary 2.3 (d) (iii) (3).

(iv) Same as (iii).

(d)

(i) Use Corollary 2.2 (d) (iii) (1) with Corollary 2.3 (d) (iv) (1).

(ii) Use Corollary 2.2 (d) (iii) (4) with Corollary 2.3 (d) (iv) (4).

(iii) Use Corollary 2.2 (d) (iii) (2) with Corollary 2.3 (d) (iv) (2).

(iv) Use Corollary 2.2 (d) (iii) (2) with Corollary 2.3 (d) (iv) (2). □

From Corollary 2.2 (d) (i)-(ii), Corollary 2.3 (d) (i)-(ii) and Corollary 2.4 (a) (i)-(ii)-(ii)-(iv), we obtain the following results.

Corollary 2.5. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i) $4(G_{n+m}^2 + (-s)^{2m} H_{n-m}^2) = (r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 - 2(r^2+4s-1) G_m G_n H_m H_n.$

(ii) $4(G_{n+m}^2 - (-s)^{2m} H_{n-m}^2) = -(r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 - H_m^2 H_n^2 + 2(r^2+4s+1) G_m G_n H_m H_n.$

(iii) $4(H_{n+m}^2 + (-s)^{2m} G_{n-m}^2) = (r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 + 2(r^2+4s-1) G_m G_n H_m H_n.$

(iv) $4(H_{n+m}^2 - (-s)^{2m} G_{n-m}^2) = (r^2+4s)^2 G_m^2 G_n^2 - G_m^2 H_n^2 - G_n^2 H_m^2 + H_m^2 H_n^2 + 2(r^2+4s+1) G_m G_n H_m H_n.$

(b)

(i) $8(G_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = (-(r^2+4s) G_m G_n + G_m H_n + G_n H_m + H_m H_n)((r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 + (G_m H_n + G_n H_m)((r^2+4s) G_m G_n - H_m H_n) - 2(r^2+4s-1) G_m G_n H_m H_n).$

(ii) $8(G_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = ((r^2+4s) G_m G_n + G_m H_n + G_n H_m - H_m H_n)((r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 + (G_m H_n + G_n H_m)(-(r^2+4s) G_m G_n + H_m H_n) - 2(r^2+4s-1) G_m G_n H_m H_n).$

(iii) $8(H_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = ((r^2+4s) G_m G_n - G_m H_n + G_n H_m + H_m H_n)((r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 + (G_m H_n - G_n H_m)((r^2+4s) G_m G_n + H_m H_n) + 2(r^2+4s-1) G_m G_n H_m H_n).$

(iv) $8(H_{n+m}^3 - (-s)^{3m} G_{n-m}^3) = ((r^2+4s) G_m G_n + G_m H_n - G_n H_m + H_m H_n)((r^2+4s)^2 G_m^2 G_n^2 + G_m^2 H_n^2 + G_n^2 H_m^2 + H_m^2 H_n^2 - (G_m H_n - G_n H_m)((r^2+4s) G_m G_n + H_m H_n) + 2(r^2+4s-1) G_m G_n H_m H_n).$

Proof.

(a)

(i) Use the identity $4(a^2+b^2) = 4(a+b)^2 - 8ab$, together with Corollary 2.3 (d) (i) and Corollary 2.4 (a) (i); alternatively, apply the identity $4(a^2+b^2) = (2(a-b))^2 + 2 \times 4ab$, with Corollary 2.3 (d) (i) and Corollary 2.4 (a) (ii).

(ii) Use the identity $4(a^2-b^2) = 2(a+b) \times 2(a-b)$, together with Corollary 2.3 (d) (i) and Corollary 2.4 (a) (ii).

(iii) Apply $4(a^2+b^2) = 4(a+b)^2 - 8ab$, with Corollary 2.3 (d) (ii) and Corollary 2.4 (a) (iii); alternatively, use $4(a^2+b^2) = (2(a-b))^2 + 2 \times 4ab$, with Corollary 2.3 (d) (ii) and Corollary 2.4 (a) (iv).

(iv) Use $4(a^2-b^2) = 2(a+b) \times 2(a-b)$, together with Corollary 2.3 (d) (ii) and Corollary 2.4 (a) (iv).

(b)

(i) Use $8(a^3+b^3)=(2(a+b))^3-3\times 4ab\times 2\,(a+b)$, together with Corollary 2.3 (d) (i) and Corollary 2.4 (a) (i).

(ii) Use the identity $8(a^3-b^3)=(2(a-b))^3+3\times 4ab\times 2\,(a-b)$, together with Corollary 2.3 (d) (i) and Corollary 2.4 (a) (ii).

(iii) Apply $8(a^3+b^3)=(2(a+b))^3-3\times 4ab\times 2\,(a+b)$, with Corollary 2.3 (d) (ii) and Corollary 2.4 (a) (iii).

(iv) Use $8(a^3-b^3)=(2(a-b))^3+3\times 4ab\times 2\,(a-b)$, together with 2.3 (d) (ii) and Corollary 2.4 (a) (iv). □

By comparing Corollary 2.2 (c) with Corollary 2.3 (c), we obtain the following result, valid for all $n,m\in\mathbb{Z}$.

Corollary 2.6. *Let* $n,m\in\mathbb{Z}$. *Then*

(a)

(i) $(H_{n+m}+(-s)^m H_{n-m})-(G_{n+m}+(-s)^m G_{n-m})=H_m(H_n-G_n)$,

i.e.,

$(H_{n+m}-G_{n+m})+(-s)^m(H_{n-m}-G_{n-m})=H_m(H_n-G_n)$,

and

$G_n(H_{n+m}+(-s)^m H_{n-m})-H_n(G_{n+m}+(-s)^m G_{n-m})=0$,

i.e.,

$(G_nH_{n+m}-H_nG_{n+m})+(-s)^m(G_nH_{n-m}-H_nG_{n-m})=0$.

(ii) $(H_{n+m}-(-s)^m H_{n-m})-(G_{n+m}-(-s)^m G_{n-m})=((r^2+4s)G_n-H_n)G_m$,

i.e.,

$(H_{n+m}-G_{n+m})+(-s)^m(-H_{n-m}+G_{n-m})=((r^2+4s)G_n-H_n)G_m$,

and

$H_n(H_{n+m}-(-s)^m H_{n-m})-(r^2+4s)G_n(G_{n+m}-(-s)^m G_{n-m})=0$,

i.e.,

$(H_nH_{n+m}-(r^2+4s)G_nG_{n+m})+(-s)^m(-H_nH_{n-m}+(r^2+4s)G_nG_{n-m})=0$.

(b)

(i) $(G_{n+m}^2-s^{2m}G_{n-m}^2)+(H_{n+m}+(-s)^m H_{n-m})=(G_mG_n+1)H_mH_n$.

(ii) $(H_{n+m}^2-s^{2m}H_{n-m}^2)+(G_{n+m}+(-s)^m G_{n-m})=((r^2+4s)G_mH_n+1)H_mG_n$.

(c)

(i) $(H_{n+m}^2+s^{2m}H_{n-m}^2)-(G_{n+m}^2+s^{2m}G_{n-m}^2)=(2G_{m+1}^2+(r^2+2s)G_m^2-2rG_mG_{m+1})(H_n^2-G_n^2)-2\,(-s)^n\,((r^2+4s)+1)G_m^2$,

and

$G_n^2(H_{n+m}^2+s^{2m}H_{n-m}^2)-H_n^2(G_{n+m}^2+s^{2m}G_{n-m}^2)=-2\,(-s)^n\,((r^2+4s)G_n^2+H_n^2)G_m^2$.

(ii) $(H_{n+m}^2-s^{2m}H_{n-m}^2)=(r^2+4s)(G_{n+m}^2-s^{2m}G_{n-m}^2)$.

(iii) $(G_{n+m}^2 - s^{2m}G_{n-m}^2) - (H_{n+m} + (-s)^m H_{n-m})(G_{n+m} + (-s)^m G_{n-m}) = (G_m - H_m)G_n H_m H_n,$

and

$H_m(G_{n+m}^2 - s^{2m}G_{n-m}^2) - G_m(H_{n+m} + (-s)^m H_{n-m})(G_{n+m} + (-s)^m G_{n-m}) = 0.$

(d)

(i) $(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) - H_m^2 G_n^2(G_{n+m} + (-s)^m G_{n-m}) = -3((-s)^m G_n^2 - (-s)^n G_m^2)H_m G_n.$

(ii) $(G_{n+m}^3 - (-s)^{3m}G_{n-m}^3) - H_n^2 G_m^2(G_{n+m} - (-s)^m G_{n-m}) = 3((-s)^m G_n^2 - (-s)^n G_m^2)G_m H_n.$

(e)

(i) $2(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) - 3H_m G_n(G_{n+m}^2 + s^{2m}G_{n-m}^2) = -H_m^3 G_n^3.$

(ii) $2(G_{n+m}^3 - (-s)^{3m}G_{n-m}^3) - G_m H_n(G_{n+m}^2 + s^{2m}G_{n-m}^2) = G_m H_n H_m^2 G_n^2.$

(iii) $2(H_{n+m}^3 + (-s)^{3m}H_{n-m}^3) - 3H_m H_n(H_{n+m}^2 + s^{2m}H_{n-m}^2) = -H_m^3 H_n^3.$

(iv) $2(H_{n+m}^3 - (-s)^{3m}H_{n-m}^3) - (r^2 + 4s)G_m G_n(H_{n+m}^2 + s^{2m}H_{n-m}^2) = (r^2 + 4s)G_m G_n H_m^2 H_n^2.$

(f)

(i) $(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) + (H_{n+m}^3 + (-s)^{3m}H_{n-m}^3) = (G_n^3 H_m^2 + H_m^2 H_n^3 - 3(-s)^m (G_n^3 + H_n^3) - 3(-s)^n ((r^2 + 4s)H_n - G_n)G_m^2)H_m.$

(ii) $(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) - (H_{n+m}^3 + (-s)^{3m}H_{n-m}^3) = (G_n^3 H_m^2 - H_m^2 H_n^3 + 3(-s)^m (H_n^3 - G_n^3) + 3(-s)^n ((r^2 + 4s)H_n + G_n)G_m^2)H_m.$

Proof.

(a) **(i)** Apply Corollary 2.3 (c) (i) together with Corollary 2.2 (c) (i).

(ii) Apply Corollary 2.3 (c) (ii) together with Corollary 2.2 (c) (ii).

(b)

(i) Use Corollary 2.2 (c)(v) together with Corollary 2.3 (c) (i).

(ii) Use Corollary 2.3 (c) (vi) together with Corollary 2.2 (c) (i).

(c)

(i) Apply Corollary 2.2 (c) (v) together with Corollary 2.3 (c) (v).

(ii) Apply Corollary 2.2 (c) (vi) together with Corollary 2.3 (c) (vi).

(iii) Combine Corollary 2.2 (c) (i) with Corollary 2.2 (c)(v).

(d)

(i) Use Corollary 2.2 (c) (i) together with Corollary 2.2 (c) (vii).

(ii) Use Corollary 2.2 (c) (ii) together with Corollary 2.2 (c) (viii).

(e)

(i) Apply Corollary 2.2 (c) (v) together with Corollary 2.2 (c) (vii).

(ii) Apply Corollary 2.2 (c) (v) together with Corollary 2.2 (c) (viii).

(iii) Use Corollary 2.3 (c) (v) together with Corollary 2.3 (c) (vii).

(iv) Use Corollary 2.3 (c) (v) together with Corollary 2.3 (c) (viii).

(f)

(i) Apply Corollary 2.2 (c) (vii) together with Corollary 2.3 (c)(vii).

(ii) Apply Corollary 2.2 (c) (vii) together with Corollary 2.3 (c)(vii). □

2.1 LEMMA 2.7 and Its Consequences

By combining Corollaries 2.2 and 2.3, we obtain the following lemma.

Lemma 2.7. *Let* $n, m \in \mathbb{Z}$. *Then*

(a)

(i) $2G_{n+m} = G_m H_n + G_n H_m$.

(ii) $2(-s)^m G_{n-m} = G_n H_m - G_m H_n$.

(iii) $2H_{n+m} = H_m H_n + (r^2 + 4s) G_m G_n$.

(iv) $2(-s)^m H_{n-m} = H_m H_n - (r^2 + 4s) G_m G_n$.

(b)

(i)

1. $4(r^2 + 4s)G_{n+m}^2 = 4H_{2m}H_{2n} - (H_m H_n - (r^2 + 4s)G_m G_n)^2$.
2. $4G_{n+m}^2 = 2G_{2m}G_{2n} + G_n^2 H_m^2 + G_m^2 H_n^2$.
3. $4G_{n+m}^2 = (G_m H_n + G_n H_m)^2$.
4. $4(r^2 + 4s)G_{n+m}^2 = 4H_{2m}H_{2n} - (r^2 + 4s)^2 G_m^2 G_n^2 + 2(r^2 + 4s)G_{2m}G_{2n} - H_m^2 H_n^2$.

(ii)

1. $4(r^2 + 4s)(-s)^{2m} G_{n-m}^2 = 4H_{2m}H_{2n} - (H_m H_n + (r^2 + 4s)G_m G_n)^2$.
2. $4(-s)^{2m} G_{n-m}^2 = -2G_{2m}G_{2n} + G_m^2 H_n^2 + G_n^2 H_m^2$.
3. $4(-s)^{2m} G_{n-m}^2 = (G_n H_m - G_m H_n)^2$.
4. $4(r^2 + 4s)(-s)^{2m} G_{n-m}^2 = 4H_{2m}H_{2n} - (r^2 + 4s)^2 G_m^2 G_n^2 - 2(r^2 + 4s)G_{2m}G_{2n} - H_m^2 H_n^2$.

(iii)

1. $4H_{n+m}^2 = 4H_{2m}H_{2n} - (r^2 + 4s)(G_n H_m - G_m H_n)^2$.
2. $4H_{n+m}^2 = (r^2 + 4s)^2 G_m^2 G_n^2 + 2(r^2 + 4s)G_{2n}G_{2m} + H_m^2 H_n^2$.
3. $4H_{n+m}^2 = (H_m H_n + (r^2 + 4s)G_m G_n)^2$.
4. $4H_{n+m}^2 = 4H_{2m}H_{2n} - (r^2 + 4s)(G_m^2 H_n^2 + G_n^2 H_m^2 - 2G_{2m}G_{2n})$.

(iv)

1. $4(-s)^{2m}H_{n-m}^2 = 4H_{2m}H_{2n} - (r^2+4s)(G_mH_n + G_nH_m)^2.$
2. $4(-s)^{2m}H_{n-m}^2 = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2H_n^2.$
3. $4(-s)^{2m}H_{n-m}^2 = (H_mH_n - (r^2+4s)G_mG_n)^2.$
4. $4(-s)^{2m}H_{n-m}^2 = 4H_{2m}H_{2n} - (r^2+4s)(G_m^2H_n^2 + G_n^2H_m^2 + 2G_{2m}G_{2n}).$

(v)

1. $4(-s)^mG_{n+m}G_{n-m} = G_n^2H_m^2 - G_m^2H_n^2.$
2. $4(-s)^mG_{n+m}G_{n-m} = 4((-s)^mG_n^2 - (-s)^nG_m^2).$
3. $(-s)^mG_{n+m}G_{n-m} = ((-s)^mG_n^2 - (-s)^nG_m^2).$
4. $G_{n+m}G_{n-m} = G_n^2 - (-s)^{n-m}G_m^2.$
5. $4(-s)^mG_{n+m}G_{n-m} = (G_mH_n + G_nH_m)(G_nH_m - G_mH_n).$

(vi)

1. $4G_{2(n+m)} = 4G_{n+m}H_{n+m} = (G_mH_n + G_nH_m)\left(\left(r^2+4s\right)G_mG_n + H_mH_n\right).$
2. $4G_{n+m}H_{n+m} = H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}).$
3. $4G_{n+m}H_{n+m} = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n}.$
4. $2G_{2(n+m)} = 2G_{n+m}H_{n+m} = H_{2n}G_{2m} + H_{2m}G_{2n}.$

(vii)

1. $4(-s)^mG_{n+m}H_{n-m} = (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m}.$
2. $(-s)^mG_{n+m}H_{n-m} = (-s)^mG_{2n} + (-s)^nG_{2m}.$
3. $G_{n+m}H_{n-m} = G_{2n} + (-s)^{n-m}G_{2m}.$
4. $2(-s)^mG_{n+m}H_{n-m} = H_{2m}G_{2n} + H_{2n}G_{2m} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}).$
5. $2(-s)^mG_{n+m}H_{n-m} = (H_{2m} - (r^2+4s)G_m^2)G_{2n} + (H_{2n} - (r^2+4s)G_n^2)G_{2m}.$
6. $4(-s)^mG_{n+m}H_{n-m} = (H_mG_n + G_mH_n)(H_mH_n - (r^2+4s)G_mG_n).$

(viii)

1. $4(-s)^mG_{n-m}H_{n+m} = -(H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}.$
2. $(-s)^mG_{n-m}H_{n+m} = (-s)^mG_{2n} - (-s)^nG_{2m}.$
3. $G_{n-m}H_{n+m} = G_{2n} - (-s)^{n-m}G_{2m}.$
4. $2(-s)^mG_{n-m}H_{n+m} = H_{2n}G_{2m} + H_{2m}G_{2n} - (r^2+4s)G_m^2G_{2n} - H_n^2G_{2m}.$
5. $2(-s)^mG_{n-m}H_{n+m} = (H_{2n} - H_n^2)G_{2m} + (H_{2m} - (r^2+4s)G_m^2)G_{2n}.$
6. $4(-s)^mG_{n-m}H_{n+m} = (H_mG_n - G_mH_n)(H_mH_n + (r^2+4s)G_mG_n).$

(ix)

1. $4(-s)^{2m}G_{2(n-m)} = 4(-s)^{2m}G_{n-m}H_{n-m} = ((r^2+4s)G_m^2 + H_m^2)G_{2n} - ((r^2+4s)G_n^2 + H_n^2)G_{2m}.$
2. $2(-s)^{2m}G_{2(n-m)} = 2(-s)^{2m}G_{n-m}H_{n-m} = H_{2m}G_{2n} - H_{2n}G_{2m}.$
3. $4(-s)^{2m}G_{n-m}H_{n-m} = (G_nH_m - G_mH_n)(H_mH_n - (r^2+4s)G_mG_n).$

(x)

1. $4(-s)^mH_{n+m}H_{n-m} = H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2.$
2. $4(-s)^mH_{n+m}H_{n-m} = (H_mH_n + (r^2+4s)G_mG_n)(H_mH_n - (r^2+4s)G_mG_n).$

(c)

(i) $2G_{n+1} = H_n + rG_n$.

(ii) $2sG_{n-1} = -rG_n + H_n$.

(iii) $2H_{n+1} = rH_n + (r^2+4s)G_n$.

(iv) $2sH_{n-1} = -rH_n + (r^2+4s)G_n$.

(d)

(i)

1. $4(r^2+4s)G_{n+1}^2 = 4(r^2+2s)H_{2n} - (rH_n - (r^2+4s)G_n)^2$.
2. $4G_{n+1}^2 = 2rG_{2n} + r^2G_n^2 + H_n^2$.
3. $4G_{n+1}^2 = \left(H_n + rG_n\right)^2$.

(ii)

1. $4(r^2+4s)s^2G_{n-1}^2 = 4(r^2+2s)H_{2n} - (rH_n + (r^2+4s)G_n)^2$.
2. $4s^2G_{n-1}^2 = -2rG_{2n} + H_n^2 + r^2G_n^2$.
3. $4s^2G_{n-1}^2 = (rG_n - H_n)^2$.

(iii)

1. $4H_{n+1}^2 = 4(r^2+2s)H_{2n} - (r^2+4s)(rG_n - H_n)^2$.
2. $4H_{n+1}^2 = (r^2+4s)^2G_n^2 + 2r(r^2+4s)G_{2n} + r^2H_n^2$.
3. $4H_{n+1}^2 = (rH_n + (r^2+4s)G_n)^2$.

(iv)

1. $4s^2H_{n-1}^2 = 4(r^2+2s)H_{2n} - (r^2+4s)(H_n + rG_n)^2$.
2. $4s^2H_{n-1}^2 = (r^2+4s)^2G_n^2 + r^2H_n^2 - 2r(r^2+4s)G_{2n}$.
3. $4s^2H_{n-1}^2 = (rH_n - (r^2+4s)G_n)^2$.

(v)

1. $4sG_{n+1}G_{n-1} = H_n^2 - r^2G_n^2 = (H_n + rG_n)(H_n - rG_n)$.
2. $4sG_{n+1}G_{n-1} = 4(sG_n^2 + (-s)^n)$.
3. $sG_{n+1}G_{n-1} = sG_n^2 + (-s)^n$.

(vi)

1. $4G_{2(n+1)} = 4G_{n+1}H_{n+1} = (H_n + rG_n)\left(\left(r^2+4s\right)G_n + rH_n\right)$.
2. $4G_{n+1}H_{n+1} = r(H_n^2 + (r^2+4s)G_n^2) + 2(r^2+2s)G_{2n}$.
3. $2G_{2(n+1)} = 2G_{n+1}H_{n+1} = rH_{2n} + (r^2+2s)G_{2n}$.

(vii)

1. $4sG_{n+1}H_{n-1} = 4sG_{2n} - r(H_n^2 - (r^2+4s)G_n^2)$.
2. $sG_{n+1}H_{n-1} = sG_{2n} - r(-s)^n$.
3. $G_{n+1}H_{n-1} = G_{2n} + r(-s)^{n-1}$.
4. $2sG_{n+1}H_{n-1} = 2sG_{2n} + r((r^2+4s)G_n^2 - H_{2n})$.
5. $4sG_{n+1}H_{n-1} = (rG_n + H_n)((r^2+4s)G_n - rH_n)$.

(viii)

1. $4sG_{n-1}H_{n+1} = r(H_n^2 - (r^2+4s)G_n^2) + 4sG_{2n}$.
2. $sG_{n-1}H_{n+1} = sG_{2n} + r(-s)^n$.
3. $G_{n-1}H_{n+1} = G_{2n} - r(-s)^{n-1}$.
4. $2sG_{n-1}H_{n+1} = r(H_n^2 - H_{2n}) + 2sG_{2n}$.
5. $4sG_{n-1}H_{n+1} = (H_n - rG_n)(rH_n + (r^2+4s)G_n)$.

(ix)

1. $4s^2G_{2(n-1)} = 4s^2G_{n-1}H_{n-1} = 2(r^2+2s)G_{2n} - r((r^2+4s)G_n^2 + H_n^2)$.
2. $2s^2G_{2(n-1)} = 2s^2G_{n-1}H_{n-1} = (r^2+2s)G_{2n} - rH_{2n}$.
3. $4s^2G_{n-1}H_{n-1} = (rG_n - H_n)(rH_n - (r^2+4s)G_n)$.

(x)

1. $4sH_{n+1}H_{n-1} = (r^2+4s)^2G_n^2 - r^2H_n^2$.
2. $4sH_{n+1}H_{n-1} = (rH_n + (r^2+4s)G_n)((r^2+4s)G_n - rH_n)$.

Proof. We employ the identities

$$\begin{aligned} G_{2n} &= G_nH_n, \\ 2H_{2n} &= (r^2+4s)G_n^2 + H_n^2, \end{aligned}$$

given in Corollary 1.12, together with

$$\begin{aligned} H_n^2 &= (r^2+4s)G_n^2 + 4(-s)^n, \\ H_{2n} &= (r^2+4s)G_n^2 + 2(-s)^n, \\ H_n^2 - H_{2n} &= 2(-s)^n, \text{ i.e., } H_n^2 = H_{2n} + 2(-s)^n, \end{aligned}$$

given in Lemma 1.13 (e).

(a)

(i) From Corollary 2.2 (d) (i).

(ii) From Corollary 2.2 (d) (ii).

(iii) From Corollary 2.3 (d) (i).

(iv) From Corollary 2.3 (d) (ii).

(b) **(i)** Using $2G_{n+m} = G_mH_n + G_nH_m$ given in Corollary 2.2 (d) (i).

(ii) Using $2(-s)^mG_{n-m} = G_nH_m - G_mH_n$ given in Corollary 2.2 (d) (ii).

(iii) Using $2H_{n+m} = H_mH_n + (r^2+4s)G_mG_n$ given in Corollary 2.3 (d) (i).

(iv) Using $2(-s)^mH_{n-m} = H_mH_n - (r^2+4s)G_mG_n$ given in Corollary 2.3 (d) (ii) we get required result.

(v) Combine (i), (ii) with $2H_{2n} = (r^2+4s)G_n^2 + H_n^2$. From Lemma 1.13 (e)(vii) we have

$$G_n^2H_m^2 - G_m^2H_n^2 = 4((-s)^mG_n^2 - (-s)^nG_m^2).$$

(vi) Combine (i), (iii) with $G_{2n} = G_nH_n$ and $2H_{2n} = (r^2+4s)G_n^2 + H_n^2$.

(vii) Combine (i), (iv), Lemma 1.13 (e)(iii), and the identities $G_{2n} = G_n H_n, 2H_{2n} = (r^2+4s)G_n^2 + H_n^2$.

(viii) Combine (ii), (iii), Lemma 1.13 (e)(iii) and the identities $G_{2n} = G_n H_n, 2H_{2n} = (r^2+4s)G_n^2 + H_n^2$.

(ix) Combine (a), (iv) with the identities $G_{2n} = G_n H_n, 2H_{2n} = (r^2+4s)G_n^2 + H_n^2$.

(x) Combine (iii), (iv) with the identities $G_{2n} = G_n H_n, 2H_{2n} = (r^2+4s)G_n^2 + H_n^2$

(c) Set $m = 1$ in (a).

(d) Set $m = 1$ in (b). □

From Lemma 2.7, we obtain the following Corollary.

Corollary 2.8. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $2(r^2+4s)(G_{n+m}^2 + (-s)^{2m}G_{n-m}^2) = 4H_{2m}H_{2n} - ((r^2+4s)^2G_m^2G_n^2 + H_m^2H_n^2)$.

2. $2(G_{n+m}^2 + (-s)^{2m}G_{n-m}^2) = G_m^2H_n^2 + G_n^2H_m^2$.

3. $(r^2+4s)(G_{n+m}^2 + (-s)^{2m}G_{n-m}^2) = H_{2m}H_{2n} - 4(-s)^{n+m}$.

(ii) $G_{n+m}^2 - (-s)^{2m}G_{n-m}^2 = G_{2n}G_{2m}$.

(iii)

1. $2(H_{n+m}^2 + (-s)^{2m}H_{n-m}^2) = 4H_{2m}H_{2n} - \left(r^2+4s\right)\left(G_m^2H_n^2 + G_n^2H_m^2\right)$.

2. $2(H_{n+m}^2 + (-s)^{2m}H_{n-m}^2) = (r^2+4s)^2G_m^2G_n^2 + H_m^2H_n^2$.

3. $H_{n+m}^2 + (-s)^{2m}H_{n-m}^2 = H_{2m}H_{2n} + 4\left(-s\right)^{m+n}$.

(iv) $H_{n+m}^2 - (-s)^{2m}H_{n-m}^2 = \left(r^2+4s\right)G_{2n}G_{2m}$.

(v) $(r^2+4s)G_{n+m}^2 + (-s)^{2m}H_{n-m}^2 = H_{2m}H_{2n}$.

(vi)

1. $(r^2+4s)G_{n+m}^2 - (-s)^{2m}H_{n-m}^2 = -(H_{2n} - H_n^2)(H_{2m} - H_m^2) + (r^2+4s)G_{2n}G_{2m}$.

2. $(r^2+4s)G_{n+m}^2 - (-s)^{2m}H_{n-m}^2 = -4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m}$.

(vii) $H_{n+m}^2 + (r^2+4s)(-s)^{2m}G_{n-m}^2 = H_{2m}H_{2n}$.

(viii)

1. $H_{n+m}^2 - (r^2+4s)(-s)^{2m}G_{n-m}^2 = (H_{2n} - H_n^2)(H_{2m} - H_m^2) + (r^2+4s)G_{2n}G_{2m}$.

2. $H_{n+m}^2 - (r^2+4s)(-s)^{2m}G_{n-m}^2 = 4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m}$.

(ix)

1. $4((r^2+4s)G_{n+m}^2 + H_{n+m}^2) = 8H_{2m}H_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2 - (r^2+4s)(G_nH_m - G_mH_n)^2$.

2. $4((r^2+4s)G_{n+m}^2 + H_{n+m}^2) = \left((r^2+4s)G_n^2 + H_n^2\right)\left((r^2+4s)G_m^2 + H_m^2\right) + 4(r^2+4s)G_{2m}G_{2n}$.

3. $(r^2+4s)G_{n+m}^2 + H_{n+m}^2 = H_{2m}H_{2n} + (r^2+4s)G_{2m}G_{2n}$.

4. $4(G_{n+m}^2 + H_{n+m}^2) = G_n^2H_m^2 + G_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2 + H_m^2H_n^2 + 2(r^2+4s+1)G_{2n}G_{2m}$.

5. $2(G_{n+m}^2 + H_{n+m}^2) = (r^2+4s+1)(2\left(-s\right)^m G_n^2 + 2\left(-s\right)^n G_m^2 + (r^2+4s)G_m^2G_n^2 + G_{2n}G_{2m}) + 8\left(-s\right)^{m+n}$.

(x)

1. $4((r^2+4s)G_{n+m}^2 - H_{n+m}^2) = -(H_mH_n - (r^2+4s)G_mG_n)^2 + (r^2+4s)(G_nH_m - G_mH_n)^2$.
2. $4(G_{n+m}^2 - H_{n+m}^2) = G_n^2H_m^2 + G_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2 - H_m^2H_n^2 - 2(r^2+4s-1)G_{2n}G_{2m}$.
3. $2(G_{n+m}^2 - H_{n+m}^2) = -(r^2+4s-1)(2(-s)^m G_n^2 + 2(-s)^n G_m^2 + (r^2+4s)G_m^2G_n^2 + G_{2n}G_{2m}) - 8(-s)^{m+n}$.

(xi)

1. $4(-s)^{2m}((r^2+4s)G_{n-m}^2 + H_{n-m}^2) = 8H_{2m}H_{2n} - (H_mH_n + (r^2+4s)G_mG_n)^2 - (r^2+4s)(G_mH_n + G_nH_m)^2$.
2. $4(-s)^{2m}(G_{n-m}^2 + H_{n-m}^2) = G_m^2H_n^2 + G_n^2H_m^2 + H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s+1)G_{2n}G_{2m}$.
3. $2(-s)^{2m}(G_{n-m}^2 + H_{n-m}^2) = (r^2+4s+1)(2(-s)^m G_n^2 + 2(-s)^n G_m^2 + (r^2+4s)G_m^2G_n^2 - G_{2n}G_{2m}) + 8(-s)^{m+n}$.

(xii)

1. $4(-s)^{2m}((r^2+4s)G_{n-m}^2 - H_{n-m}^2) = -(H_mH_n + (r^2+4s)G_mG_n)^2 + (r^2+4s)(G_mH_n + G_nH_m)^2$.
2. $4(-s)^{2m}(G_{n-m}^2 - H_{n-m}^2) = G_m^2H_n^2 + G_n^2H_m^2 - H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s-1)G_{2n}G_{2m}$.
3. $2(-s)^{2m}(G_{n-m}^2 - H_{n-m}^2) = -(r^2+4s-1)(2(-s)^m G_n^2 + 2(-s)^n G_m^2 + (r^2+4s)G_m^2G_n^2 - G_{2n}G_{2m}) - 8(-s)^{m+n}$.

(b)

(i)

1. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n+m}G_{n-m}) = 4H_{2m}H_{2n} + G_n^2H_m^2 - G_m^2H_n^2 - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n+m}G_{n-m}) = 4H_{2m}H_{2n} + 4((-s)^mG_n^2 - (-s)^nG_m^2) - (H_mH_n - (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 + G_{n+m}G_{n-m}) = 4H_{2m}H_{2n} + 4G_n^2 - 4(-s)^{n-m}G_m^2 - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 + (-s)^mG_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + G_n^2H_m^2 - G_m^2H_n^2$.
5. $4(G_{n+m}^2 + (-s)^mG_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + 4((-s)^mG_n^2 - (-s)^nG_m^2)$.
6. $4(G_{n+m}^2 + G_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + 4G_n^2 - 4(-s)^{n-m}G_m^2$.

(ii)

1. $4((r^2+4s)G_{n+m}^2 - (-s)^mG_{n+m}G_{n-m}) = 4H_{2m}H_{2n} - G_n^2H_m^2 + G_m^2H_n^2 - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 - (-s)^mG_{n+m}G_{n-m}) = 4H_{2m}H_{2n} - 4((-s)^mG_n^2 - (-s)^nG_m^2) - (H_mH_n - (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 - G_{n+m}G_{n-m}) = 4H_{2m}H_{2n} - 4G_n^2 + 4(-s)^{n-m}G_m^2 - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 - (-s)^mG_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - G_n^2H_m^2 + G_m^2H_n^2$.
5. $4(G_{n+m}^2 - (-s)^mG_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - 4((-s)^mG_n^2 - (-s)^nG_m^2)$.
6. $4(G_{n+m}^2 - G_{n+m}G_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - 4G_n^2 + 4(-s)^{n-m}G_m^2$.

(iii)

1. $4((r^2+4s)G_{n+m}^2 + G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} + H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 + G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} + (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 + G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 + G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m})$.

5. $4(G_{n+m}^2 + G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2 + 4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}^2 + G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + 2H_{2n}G_{2m} + 2H_{2m}G_{2n}$.

(iv)

1. $4((r^2+4s)G_{n+m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 - G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m})$.
5. $4(G_{n+m}^2 - G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2 + 4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}^2 - G_{n+m}H_{n+m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - 2H_{2n}G_{2m} - 2H_{2m}G_{2n}$.

(v)

1. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2 + 4s)G_n^2)G_{2m} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} - (H_mH_n - (r^2 + 4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 + G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
5. $4(G_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + 4(-s)^mG_{2n} + 4(-s)^nG_{2m}$.
6. $4(G_{n+m}^2 + G_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 + 4G_{2n} + 4(-s)^{n-m}G_{2m}$.

(vi)

1. $4((r^2+4s)G_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2 + 4s)G_n^2)G_{2m} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (H_mH_n - (r^2 + 4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 - G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
4. $4(G_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
5. $4(G_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - 4(-s)^mG_{2n} - 4(-s)^nG_{2m}$.
6. $4(G_{n+m}^2 - G_{n+m}H_{n-m}) = 2G_{2m}G_{2n} + G_n^2H_m^2 + G_m^2H_n^2 - 4G_{2n} - 4(-s)^{n-m}G_{2m}$.

(vii)

1. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2 + 4s)G_m^2)G_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (H_mH_n - (r^2 + 4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2 + G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4G_{2n} - 4(-s)^{n-m}G_{2m} - (H_mH_n - (r^2+4s)G_mG_n)^2$.

4. $4((r^2+4s)G_{n+m}^2+(-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-2(r^2+4s)G_m^2G_{2n}-2H_n^2G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
5. $4(G_{n+m}^2+(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-(H_n^2-(r^2+4s)G_n^2)G_{2m}+(H_m^2-(r^2+4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}^2+(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+4(-s)^mG_{2n}-4(-s)^nG_{2m}$.
7. $4(G_{n+m}^2+G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+4G_{2n}-4(-s)^{n-m}G_{2m}$.
8. $4(G_{n+m}^2+(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+2H_{2n}G_{2m}+2H_{2m}G_{2n}-2(r^2+4s)G_m^2G_{2n}-2H_n^2G_{2m}$.

(viii)

1. $4((r^2+4s)G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-4(-s)^mG_{2n}+4(-s)^nG_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)G_{n+m}^2-G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-4G_{2n}+4(-s)^{n-m}G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
4. $4((r^2+4s)G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
5. $4(G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-4(-s)^mG_{2n}+4(-s)^nG_{2m}$.
7. $4(G_{n+m}^2-G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-4G_{2n}+4(-s)^{n-m}G_{2m}$.
8. $4(G_{n+m}^2-(-s)^m G_{n-m}H_{n+m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}$.

(ix)

1. $4((r^2+4s)G_{n+m}^2+(-s)^{2m} G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2+(-s)^{2m} G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}+2H_{2m}G_{2n}-2H_{2n}G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
3. $4(G_{n+m}^2+(-s)^{2m} G_{n-m}H_{n-m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}$.
4. $4(G_{n+m}^2+(-s)^{2m} G_{n-m}H_{n-m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+2H_{2m}G_{2n}-2H_{2n}G_{2m}$.

(x)

1. $4((r^2+4s)G_{n+m}^2-(-s)^{2m} G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)G_{n+m}^2-(-s)^{2m} G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}-(H_mH_n-(r^2+4s)G_mG_n)^2$.
3. $4(G_{n+m}^2-(-s)^{2m} G_{n-m}H_{n-m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}$.
4. $4(G_{n+m}^2-(-s)^{2m} G_{n-m}H_{n-m}) = 2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-2H_{2m}G_{2n}+2H_{2n}G_{2m}$.

(xi)

1. $4((r^2+4s)G_{n+m}^2+(-s)^m H_{n+m}H_{n-m}) = 4H_{2m}H_{2n}+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2-(H_mH_n-(r^2+4s)G_mG_n)^2$.

2. $4(G_{n+m}^2+(-s)^mH_{n+m}H_{n-m})=2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.

(xii)

1. $4((r^2+4s)G_{n+m}^2-(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2-(H_mH_n-(r^2+4s)G_mG_n)^2$.
2. $4(G_{n+m}^2-(-s)^mH_{n+m}H_{n-m})=2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2$.

(c)

(i)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}+G_n^2H_m^2-G_m^2H_n^2-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=4(H_{2m}H_{2n}+(-s)^mG_n^2-(-s)^nG_m^2)-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+G_{n+m}G_{n-m})=4(H_{2m}H_{2n}+G_n^2-(-s)^{n-m}G_m^2)-(H_mH_n+(r^2+4s)G_mG_n)^2$.
4. $2((-s)^{2m}G_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=-G_{2m}G_{2n}+G_n^2H_m^2$.
5. $4((-s)^{2m}G_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=-2G_{2m}G_{2n}+(H_n^2-4(-s)^n)G_m^2+(H_m^2+4(-s)^m)G_n^2$.
6. $4((-s)^{2m}G_{n-m}^2+G_{n+m}G_{n-m})=-2G_{2m}G_{2n}+(H_n^2-4(-s)^{n-m})G_m^2+(H_m^2+4)G_n^2$.

(ii)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-G_n^2H_m^2+G_m^2H_n^2-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=4(H_{2m}H_{2n}-(-s)^mG_n^2+(-s)^nG_m^2)-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-G_{n+m}G_{n-m})=4(H_{2m}H_{2n}-G_n^2+(-s)^{n-m}G_m^2)-(H_mH_n+(r^2+4s)G_mG_n)^2$.
4. $2((-s)^{2m}G_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=-G_{2m}G_{2n}+G_m^2H_n^2$.
5. $4((-s)^{2m}G_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=-2G_{2m}G_{2n}+(H_n^2+4(-s)^n)G_m^2+(H_m^2-4(-s)^m)G_n^2$.
6. $4((-s)^{2m}G_{n-m}^2-G_{n+m}G_{n-m})=-2G_{2m}G_{2n}+(H_n^2+4(-s)^{n-m})G_m^2+(H_m^2-4)G_n^2$.

(iii)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
4. $4((-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=-2G_{2m}G_{2n}+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}+G_m^2H_n^2+G_n^2H_m^2$.
5. $4((-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=-2G_{2m}G_{2n}+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})+G_m^2H_n^2+G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2+G_{n+m}H_{n+m})=2(-G_{2m}G_{2n}+H_{2n}G_{2m}+H_{2m}G_{2n})+G_m^2H_n^2+G_n^2H_m^2$.

(iv)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) - (H_mH_n + (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
4. $4((-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = -2G_{2m}G_{2n} - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) + G_m^2H_n^2 + G_n^2H_m^2$.
5. $4((-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = -2G_{2m}G_{2n} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} + G_m^2H_n^2 + G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n+m}) = 2(-G_{2m}G_{2n} - H_{2n}G_{2m} - H_{2m}G_{2n}) + G_m^2H_n^2 + G_n^2H_m^2$.

(v)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
4. $4((-s)^{2m}G_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = -2G_{2m}G_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.
5. $4((-s)^{2m}G_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = -2G_{2m}G_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2 + G_{n+m}H_{n-m}) = -2G_{2m}G_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.

(vi)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
4. $4((-s)^{2m}G_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = -2G_{2m}G_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.
5. $4((-s)^{2m}G_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = -2G_{2m}G_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2 - G_{n+m}H_{n-m}) = -2G_{2m}G_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m} + G_m^2H_n^2 + G_n^2H_m^2$.

(vii)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2 + G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4G_{2n} - 4(-s)^{n-m}G_{2m} - (H_mH_n + (r^2+4s)G_mG_n)^2$.

4. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+(-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-2(r^2+4s)G_m^2G_{2n}-2H_n^2G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
5. $4((-s)^{2m}G_{n-m}^2+(-s)^mG_{n-m}H_{n+m}) = -2G_{2m}G_{2n}-(H_n^2-(r^2+4s)G_n^2)G_{2m}+(H_m^2-(r^2+4s)G_m^2)G_{2n}+G_m^2H_n^2+G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2+(-s)^mG_{n-m}H_{n+m}) = -2G_{2m}G_{2n}+4((-s)^mG_{2n}-(-s)^nG_{2m})+G_m^2H_n^2+G_n^2H_m^2$.
7. $4((-s)^{2m}G_{n-m}^2+G_{n-m}H_{n+m}) = -2G_{2m}G_{2n}+4(G_{2n}-(-s)^{n-m}G_{2m})+G_m^2H_n^2+G_n^2H_m^2$.
8. $4((-s)^{2m}G_{n-m}^2+(-s)^mG_{n-m}H_{n+m}) = 2(-G_{2m}G_{2n}+H_{2n}G_{2m}+H_{2m}G_{2n}-(r^2+4s)G_m^2G_{2n}-H_n^2G_{2m})+G_m^2H_n^2+G_n^2H_m^2$.

(viii)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n}+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-4(-s)^mG_{2n}+4(-s)^nG_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-G_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-4G_{2n}+4(-s)^{n-m}G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
4. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n}-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
5. $4((-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = -2G_{2m}G_{2n}+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}+G_m^2H_n^2+G_n^2H_m^2$.
6. $4((-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = -2G_{2m}G_{2n}-4((-s)^mG_{2n}-(-s)^nG_{2m})+G_m^2H_n^2+G_n^2H_m^2$.
7. $4((-s)^{2m}G_{n-m}^2-G_{n-m}H_{n+m}) = -2G_{2m}G_{2n}-4(G_{2n}-(-s)^{n-m}G_{2m})+G_m^2H_n^2+G_n^2H_m^2$.
8. $4((-s)^{2m}G_{n-m}^2-(-s)^mG_{n-m}H_{n+m}) = -2(G_{2m}G_{2n}+H_{2n}G_{2m}+H_{2m}G_{2n}-(r^2+4s)G_m^2G_{2n}-H_n^2G_{2m})+G_m^2H_n^2+G_n^2H_m^2$.

(ix)

1. $4(-s)^{2m}((r^2+4s)G_{n-m}^2+G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4(-s)^{2m}((r^2+4s)G_{n-m}^2+G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}+2H_{2m}G_{2n}-2H_{2n}G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4(-s)^{2m}(G_{n-m}^2+G_{n-m}H_{n-m}) = -2G_{2m}G_{2n}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}+G_m^2H_n^2+G_n^2H_m^2$.
4. $4(-s)^{2m}(G_{n-m}^2+G_{n-m}H_{n-m}) = -2G_{2m}G_{2n}+2H_{2m}G_{2n}-2H_{2n}G_{2m}+G_m^2H_n^2+G_n^2H_m^2$.

(x)

1. $4(-s)^{2m}((r^2+4s)G_{n-m}^2-G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4(-s)^{2m}((r^2+4s)G_{n-m}^2-G_{n-m}H_{n-m}) = 4H_{2m}H_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}-(H_mH_n+(r^2+4s)G_mG_n)^2$.
3. $4(-s)^{2m}(G_{n-m}^2-G_{n-m}H_{n-m}) = -2G_{2m}G_{2n}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}+G_m^2H_n^2+G_n^2H_m^2$.
4. $4(-s)^{2m}(G_{n-m}^2-G_{n-m}H_{n-m}) = -2G_{2m}G_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}+G_m^2H_n^2+G_n^2H_m^2$.

(xi)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2+(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((-s)^{2m}G_{n-m}^2+(-s)^mH_{n+m}H_{n-m})=-2G_{2m}G_{2n}+G_m^2H_n^2+G_n^2H_m^2+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.

(xii)

1. $4((r^2+4s)(-s)^{2m}G_{n-m}^2-(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2-(H_mH_n+(r^2+4s)G_mG_n)^2$.
2. $4((-s)^{2m}G_{n-m}^2-(-s)^mH_{n+m}H_{n-m})=-2G_{2m}G_{2n}+G_m^2H_n^2+G_n^2H_m^2-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2$.

(d)

(i)

1. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2+G_n^2H_m^2-G_m^2H_n^2$.
2. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2+4((-s)^mG_n^2-(-s)^nG_m^2)$.
3. $4(H_{n+m}^2+G_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2+4G_n^2-4(-s)^{n-m}G_m^2$.
4. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+G_n^2H_m^2-G_m^2H_n^2$.
5. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+4((-s)^mG_n^2-(-s)^nG_m^2)$.
6. $4(H_{n+m}^2+G_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+4G_n^2-4(-s)^{n-m}G_m^2$.

(ii)

1. $4(H_{n+m}^2-(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2-G_n^2H_m^2+G_m^2H_n^2$.
2. $4(H_{n+m}^2-(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2-4((-s)^mG_n^2-(-s)^nG_m^2)$.
3. $4(H_{n+m}^2-G_{n+m}G_{n-m})=4H_{2m}H_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2-4G_n^2+4(-s)^{n-m}G_m^2$.
4. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-G_n^2H_m^2+G_m^2H_n^2$.
5. $4(H_{n+m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-4((-s)^mG_n^2-(-s)^nG_m^2)$.
6. $4(H_{n+m}^2+G_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-4G_n^2+4(-s)^{n-m}G_m^2$.

(iii)

1. $4(H_{n+m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-(r^2+4s)(G_nH_m-G_mH_n)^2$.
2. $4(H_{n+m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2$.
3. $4(H_{n+m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2$.
4. $4(H_{n+m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})+H_m^2H_n^2$.
5. $4(H_{n+m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}+H_m^2H_n^2$.
6. $4(H_{n+m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+2H_{2n}G_{2m}+2H_{2m}G_{2n}+H_m^2H_n^2$.

(iv)

1. $4(H_{n+m}^2-G_{n+m}H_{n+m})=4H_{2m}H_{2n}-H_n^2G_{2m}-H_m^2G_{2n}-(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-(r^2+4s)(G_nH_m-G_mH_n)^2$.

2. $4(H_{n+m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} - (r^2 + 4s)(G_nH_m - G_mH_n)^2$.
3. $4(H_{n+m}^2 - G_{n+m}H_{n+m}) = 4H_{2m}H_{2n} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
4. $4(H_{n+m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) + H_m^2H_n^2$.
5. $4(H_{n+m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} + H_m^2H_n^2$.
6. $4(H_{n+m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + H_m^2H_n^2$.

(v)

1. $4(H_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
2. $4(H_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
3. $4(H_{n+m}^2 + G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
4. $4(H_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} + H_m^2H_n^2$.
5. $4(H_{n+m}^2 + (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} + H_m^2H_n^2$.
6. $4(H_{n+m}^2 + G_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + 4G_{2n} + 4(-s)^{n-m}G_{2m} + H_m^2H_n^2$.

(vi)

1. $4(H_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
2. $4(H_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
3. $4(H_{n+m}^2 - G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
4. $4(H_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + H_m^2H_n^2$.
5. $4(H_{n+m}^2 - (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} + H_m^2H_n^2$.
6. $4(H_{n+m}^2 - G_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - 4G_{2n} - 4(-s)^{n-m}G_{2m} + H_m^2H_n^2$.

(vii)

1. $4(H_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
2. $4(H_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
3. $4(H_{n+m}^2 + G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4G_{2n} - 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
4. $4(H_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2G_{2n} - 2H_n^2G_{2m} - (r^2+4s)(G_nH_m - G_mH_n)^2$.
5. $4(H_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + H_m^2H_n^2$.
6. $4(H_{n+m}^2 + (-s)^mG_{n-m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + 4(-s)^mG_{2n} - 4(-s)^nG_{2m} + H_m^2H_n^2$.
7. $4(H_{n+m}^2 + G_{n-m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + 4G_{2n} - 4(-s)^{n-m}G_{2m} + H_m^2H_n^2$.

8. $4(H_{n+m}^2+(-s)^mG_{n-m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-2(r^2+4s)G_m^2G_{2n}-2H_n^2G_{2m}+H_m^2H_n^2.$

(viii)

1. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=4H_{2m}H_{2n}+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
2. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=4H_{2m}H_{2n}-4(-s)^mG_{2n}+4(-s)^nG_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
3. $4(H_{n+m}^2-G_{n-m}H_{n+m})=4H_{2m}H_{2n}-4G_{2n}+4(-s)^{n-m}G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
4. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=4H_{2m}H_{2n}-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
5. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}+H_m^2H_n^2.$
6. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}-4(-s)^mG_{2n}+4(-s)^nG_{2m}+H_m^2H_n^2.$
7. $4(H_{n+m}^2-G_{n-m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}-4G_{2n}+4(-s)^{n-m}G_{2m}+H_m^2H_n^2.$
8. $4(H_{n+m}^2-(-s)^mG_{n-m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}+H_m^2H_n^2.$

(ix)

1. $4(H_{n+m}^2+(-s)^{2m}G_{n-m}H_{n-m})=4H_{2m}H_{2n}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
2. $4(H_{n+m}^2+(-s)^{2m}G_{n-m}H_{n-m})=4H_{2m}H_{2n}+2H_{2m}G_{2n}-2H_{2n}G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
3. $4(H_{n+m}^2+(-s)^{2m}G_{n-m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}+H_m^2H_n^2.$
4. $4(H_{n+m}^2+(-s)^{2m}G_{n-m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+2H_{2m}G_{2n}-2H_{2n}G_{2m}+H_m^2H_n^2.$

(x)

1. $4(H_{n+m}^2-(-s)^{2m}G_{n-m}H_{n-m})=4H_{2m}H_{2n}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
2. $4(H_{n+m}^2-(-s)^{2m}G_{n-m}H_{n-m})=4H_{2m}H_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}-(r^2+4s)(G_nH_m-G_mH_n)^2.$
3. $4(H_{n+m}^2-(-s)^{2m}G_{n-m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}+H_m^2H_n^2.$
4. $4(H_{n+m}^2-(-s)^{2m}G_{n-m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}-2H_{2m}G_{2n}+2H_{2n}G_{2m}+H_m^2H_n^2.$

(xi)

1. $4(H_{n+m}^2+(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2-(r^2+4s)(G_nH_m-G_mH_n)^2.$
2. $4(H_{n+m}^2+(-s)^mH_{n+m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2.$

(xii)

1. $4(H_{n+m}^2-(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2-(r^2+4s)(G_nH_m-G_mH_n)^2.$
2. $4(H_{n+m}^2-(-s)^mH_{n+m}H_{n-m})=2(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2n}G_{2m}.$

(e)

(i)

1. $4((-s)^{2m}H_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}+G_n^2H_m^2-G_m^2H_n^2-(r^2+4s)(G_mH_n+G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}+4((-s)^mG_n^2-(-s)^nG_m^2)-(r^2+4s)(G_mH_n+G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2+G_{n+m}G_{n-m})=4H_{2m}H_{2n}+4G_n^2-4(-s)^{n-m}G_m^2-(r^2+4s)(G_mH_n+G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+G_n^2H_m^2-G_m^2H_n^2$.
5. $4((-s)^{2m}H_{n-m}^2+(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+4((-s)^mG_n^2-(-s)^nG_m^2)$.
6. $4((-s)^{2m}H_{n-m}^2+G_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+4G_n^2-4(-s)^{n-m}G_m^2$.

(ii)

1. $4((-s)^{2m}H_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-G_n^2H_m^2+G_m^2H_n^2-(r^2+4s)(G_mH_n+G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=4H_{2m}H_{2n}-4((-s)^mG_n^2-(-s)^nG_m^2)-(r^2+4s)(G_mH_n+G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2-G_{n+m}G_{n-m})=4H_{2m}H_{2n}-4G_n^2+4(-s)^{n-m}G_m^2-(r^2+4s)(G_mH_n+G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-G_n^2H_m^2+G_m^2H_n^2$.
5. $4((-s)^{2m}H_{n-m}^2-(-s)^mG_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-4((-s)^mG_n^2-(-s)^nG_m^2)$.
6. $4((-s)^{2m}H_{n-m}^2-G_{n+m}G_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2-4G_n^2+4(-s)^{n-m}G_m^2$.

(iii)

1. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-(r^2+4s)(G_mH_n+G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}-(r^2+4s)(G_mH_n+G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=4H_{2m}H_{2n}+2H_{2n}G_{2m}+2H_{2m}G_{2n}-(r^2+4s)(G_mH_n+G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
5. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
6. $4((-s)^{2m}H_{n-m}^2+G_{n+m}H_{n+m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+2H_{2n}G_{2m}+2H_{2m}G_{2n}$.

(iv)

1. $4((-s)^{2m}H_{n-m}^2-G_{n+m}H_{n+m})=4H_{2m}H_{2n}-H_n^2G_{2m}-H_m^2G_{2n}-(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-(r^2+4s)(G_mH_n+G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2-G_{n+m}H_{n+m})=4H_{2m}H_{2n}-(H_n^2+(r^2+4s)G_n^2)G_{2m}-(H_m^2+(r^2+4s)G_m^2)G_{2n}-(r^2+4s)(G_mH_n+G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2-G_{n+m}H_{n+m})=4H_{2m}H_{2n}-2H_{2n}G_{2m}-2H_{2m}G_{2n}-(r^2+4s)(G_mH_n+G_nH_m)^2$.

4. $4((-s)^{2m}H_{n-m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - H_n^2G_{2m} - H_m^2G_{2n} - (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) + H_m^2H_n^2$.
5. $4((-s)^{2m}H_{n-m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - (H_n^2 + (r^2+4s)G_n^2)G_{2m} - (H_m^2 + (r^2+4s)G_m^2)G_{2n} + H_m^2H_n^2$.
6. $4((-s)^{2m}H_{n-m}^2 - G_{n+m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + H_m^2H_n^2$.

(v)

1. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2 + G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} + H_m^2H_n^2$.
5. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m} + H_m^2H_n^2$.
6. $4((-s)^{2m}H_{n-m}^2 + G_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 4G_{2n} + 4(-s)^{n-m}G_{2m} + H_m^2H_n^2$.

(vi)

1. $4((-s)^{2m}H_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2 - G_{n+m}H_{n-m}) = 4H_{2m}H_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + H_m^2H_n^2$.
5. $4((-s)^{2m}H_{n-m}^2 - (-s)^mG_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m} + H_m^2H_n^2$.
6. $4((-s)^{2m}H_{n-m}^2 - G_{n+m}H_{n-m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 4G_{2n} - 4(-s)^{n-m}G_{2m} + H_m^2H_n^2$.

(vii)

1. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4(-s)^mG_{2n} - 4(-s)^nG_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2 + G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 4G_{2n} - 4(-s)^{n-m}G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
4. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2G_{2n} - 2H_n^2G_{2m} - (r^2+4s)(G_mH_n + G_nH_m)^2$.
5. $4((-s)^{2m}H_{n-m}^2 + (-s)^mG_{n-m}H_{n+m}) = (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + H_m^2H_n^2$.

6. $4((-s)^{2m}H_{n-m}^2 + (-s)^m G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 4(-s)^m G_{2n} - 4(-s)^n G_{2m} + H_m^2 H_n^2$.
7. $4((-s)^{2m}H_{n-m}^2 + G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 4G_{2n} - 4(-s)^{n-m}G_{2m} + H_m^2 H_n^2$.
8. $4((-s)^{2m}H_{n-m}^2 + (-s)^m G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2 G_{2n} - 2H_n^2 G_{2m} + H_m^2 H_n^2$.

(viii)

1. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - 4(-s)^m G_{2n} + 4(-s)^n G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
3. $4((-s)^{2m}H_{n-m}^2 - G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - (r^2+4s)(G_m H_n + G_n H_m)^2 - 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
4. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = 4H_{2m}H_{2n} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + 2(r^2+4s)G_m^2 G_{2n} + 2H_n^2 G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
5. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} + H_m^2 H_n^2$.
6. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 4(-s)^m G_{2n} + 4(-s)^n G_{2m} + H_m^2 H_n^2$.
7. $4((-s)^{2m}H_{n-m}^2 - G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 4G_{2n} + 4(-s)^{n-m}G_{2m} + H_m^2 H_n^2$.
8. $4((-s)^{2m}H_{n-m}^2 - (-s)^m G_{n-m}H_{n+m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + 2(r^2+4s)G_m^2 G_{2n} + 2H_n^2 G_{2m} + H_m^2 H_n^2$.

(ix)

1. $4(-s)^{2m}(H_{n-m}^2 + G_{n-m}H_{n-m}) = 4H_{2m}H_{2n} + ((r^2+4s)G_m^2 + H_m^2)G_{2n} - ((r^2+4s)G_n^2 + H_n^2)G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
2. $4(-s)^{2m}(H_{n-m}^2 + G_{n-m}H_{n-m}) = 4H_{2m}H_{2n} + 2H_{2m}G_{2n} - 2H_{2n}G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
3. $4(-s)^{2m}(H_{n-m}^2 + G_{n-m}H_{n-m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + ((r^2+4s)G_m^2 + H_m^2)G_{2n} - ((r^2+4s)G_n^2 + H_n^2)G_{2m} + H_m^2 H_n^2$.
4. $4(-s)^{2m}(H_{n-m}^2 + G_{n-m}H_{n-m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + 2H_{2m}G_{2n} - 2H_{2n}G_{2m} + H_m^2 H_n^2$.

(x)

1. $4(-s)^{2m}(H_{n-m}^2 - G_{n-m}H_{n-m}) = 4H_{2m}H_{2n} - ((r^2+4s)G_m^2 + H_m^2)G_{2n} + ((r^2+4s)G_n^2 + H_n^2)G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
2. $4(-s)^{2m}(H_{n-m}^2 - G_{n-m}H_{n-m}) = 4H_{2m}H_{2n} - 2H_{2m}G_{2n} + 2H_{2n}G_{2m} - (r^2+4s)(G_m H_n + G_n H_m)^2$.
3. $4(-s)^{2m}(H_{n-m}^2 - G_{n-m}H_{n-m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - ((r^2+4s)G_m^2 + H_m^2)G_{2n} + ((r^2+4s)G_n^2 + H_n^2)G_{2m} + H_m^2 H_n^2$.
4. $4(-s)^{2m}(H_{n-m}^2 - G_{n-m}H_{n-m}) = (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} - 2H_{2m}G_{2n} + 2H_{2n}G_{2m} + H_m^2 H_n^2$.

(xi)

1. $4((-s)^{2m}H_{n-m}^2 + (-s)^m H_{n+m}H_{n-m}) = 4H_{2m}H_{2n} + H_m^2 H_n^2 - (r^2+4s)^2 G_m^2 G_n^2 - (r^2+4s)(G_m H_n + G_n H_m)^2$.

2. $4((-s)^{2m}H_{n-m}^2+(-s)^mH_{n+m}H_{n-m})=(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}+H_m^2H_n^2+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.

(xii)

1. $4((-s)^{2m}H_{n-m}^2-(-s)^mH_{n+m}H_{n-m})=4H_{2m}H_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2-(r^2+4s)(G_mH_n+G_nH_m)^2$.
2. $4((-s)^{2m}H_{n-m}^2-(-s)^mH_{n+m}H_{n-m})=2(r^2+4s)^2G_m^2G_n^2-2(r^2+4s)G_{2n}G_{2m}$.

(f)

(i)

1. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
2. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
3. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2+2H_{2n}G_{2m}+2H_{2m}G_{2n}$.
4. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
5. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
6. $4((-s)^mG_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)+2H_{2n}G_{2m}+2H_{2m}G_{2n}$.
7. $4(G_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2+H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
8. $4(G_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2+(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
9. $4(G_{n+m}G_{n-m}+G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2+2H_{2n}G_{2m}+2H_{2m}G_{2n}$.

(ii)

1. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2-H_n^2G_{2m}-H_m^2G_{2n}-(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
2. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2-(H_n^2+(r^2+4s)G_n^2)G_{2m}-(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
3. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=G_n^2H_m^2-G_m^2H_n^2-2H_{2n}G_{2m}-2H_{2m}G_{2n}$.
4. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)-H_n^2G_{2m}-H_m^2G_{2n}-(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
5. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)-(H_n^2+(r^2+4s)G_n^2)G_{2m}-(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
6. $4((-s)^mG_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4((-s)^mG_n^2-(-s)^nG_m^2)-2H_{2n}G_{2m}-2H_{2m}G_{2n}$.
7. $4(G_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2-H_n^2G_{2m}-H_m^2G_{2n}-(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
8. $4(G_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2-(H_n^2+(r^2+4s)G_n^2)G_{2m}-(H_m^2+(r^2+4s)G_m^2)G_{2n}$.
9. $4(G_{n+m}G_{n-m}-G_{n+m}H_{n+m})=4G_n^2-4(-s)^{n-m}G_m^2-2H_{2n}G_{2m}-2H_{2m}G_{2n}$.

(iii)

1. $4(-s)^m(G_{n+m}G_{n-m}+G_{n+m}H_{n-m})=G_n^2H_m^2-G_m^2H_n^2+(H_m^2-(r^2+4s)G_m^2)G_{2n}+(H_n^2-(r^2+4s)G_n^2)G_{2m}$.

2. $4(-s)^m(G_{n+m}G_{n-m} + G_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 + 4(-s)^mG_{2n} + 4(-s)^nG_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m} + G_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 + 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
4. $4(-s)^m(G_{n+m}G_{n-m} + G_{n+m}H_{n-m}) = 4((-s)^mG_n^2 - (-s)^nG_m^2) + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
5. $(-s)^m(G_{n+m}G_{n-m} + G_{n+m}H_{n-m}) = (-s)^m(G_{2n} + G_n^2) + (-s)^n(G_{2m} - G_m^2)$.
6. $(-s)^mG_{n+m}G_{n-m} + G_{n+m}H_{n-m} = G_{2n} + (-s)^{n-m}G_{2m} + (-s)^mG_n^2 - (-s)^nG_m^2$.
7. $4(G_{n+m}G_{n-m} + (-s)^mG_{n+m}H_{n-m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
8. $G_{n+m}G_{n-m} + (-s)^mG_{n+m}H_{n-m} = G_n^2 - (-s)^{n-m}G_m^2 + (-s)^mG_{2n} + (-s)^nG_{2m}$.
9. $G_{n+m}G_{n-m} + G_{n+m}H_{n-m} = G_n^2 + G_{2n} + (-s)^{n-m}(G_{2m} - G_m^2)$.

(iv)

1. $4(-s)^m(G_{n+m}G_{n-m} - G_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
2. $4(-s)^m(G_{n+m}G_{n-m} - G_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 - 4(-s)^mG_{2n} - 4(-s)^nG_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m} - G_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 - 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
4. $4(-s)^m(G_{n+m}G_{n-m} - G_{n+m}H_{n-m}) = 4((-s)^mG_n^2 - (-s)^nG_m^2) - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
5. $(-s)^m(G_{n+m}G_{n-m} - G_{n+m}H_{n-m}) = (-s)^m(-G_{2n} + G_n^2) - (-s)^n(G_{2m} + G_m^2)$.
6. $(-s)^mG_{n+m}G_{n-m} - G_{n+m}H_{n-m} = ((-s)^mG_n^2 - (-s)^nG_m^2) - G_{2n} - (-s)^{n-m}G_{2m}$.
7. $4(G_{n+m}G_{n-m} - (-s)^mG_{n+m}H_{n-m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
8. $G_{n+m}G_{n-m} - (-s)^mG_{n+m}H_{n-m} = G_n^2 - (-s)^{n-m}G_m^2 - (-s)^mG_{2n} - (-s)^nG_{2m}$.
9. $G_{n+m}G_{n-m} - G_{n+m}H_{n-m} = G_n^2 - G_{2n} - (-s)^{n-m}(G_{2m} + G_m^2)$.

(v)

1. $4(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = G_n^2H_m^2 - G_m^2H_n^2 - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
2. $4(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = G_n^2H_m^2 - G_m^2H_n^2 + 4(-s)^mG_{2n} - 4(-s)^nG_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = G_n^2H_m^2 - G_m^2H_n^2 + 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
4. $4(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = G_n^2H_m^2 - G_m^2H_n^2 + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2G_{2n} - 2H_n^2G_{2m}$.
5. $4(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = 4((-s)^mG_n^2 - (-s)^nG_m^2) - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
6. $(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = ((-s)^mG_n^2 - (-s)^nG_m^2) + (-s)^mG_{2n} - (-s)^nG_{2m}$.
7. $(-s)^mG_{n+m}G_{n-m} + G_{n-m}H_{n+m} = ((-s)^mG_n^2 - (-s)^nG_m^2) + G_{2n} - (-s)^{n-m}G_{2m}$.
8. $2(-s)^m(G_{n+m}G_{n-m} + G_{n-m}H_{n+m}) = 2((-s)^mG_n^2 - (-s)^nG_m^2) + H_{2n}G_{2m} + H_{2m}G_{2n} - (r^2+4s)G_m^2G_{2n} - H_n^2G_{2m}$.
9. $4(G_{n+m}G_{n-m} + (-s)^mG_{n-m}H_{n+m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
10. $G_{n+m}G_{n-m} + (-s)^mG_{n-m}H_{n+m} = G_n^2 - (-s)^{n-m}G_m^2 + (-s)^mG_{2n} - (-s)^nG_{2m}$.
11. $G_{n+m}G_{n-m} + G_{n-m}H_{n+m} = G_n^2 - (-s)^{n-m}G_m^2 + G_{2n} - (-s)^{n-m}G_{2m}$.

12. $2(G_{n+m}G_{n-m}+(-s)^m G_{n-m}H_{n+m}) = 2G_n^2-2(-s)^{n-m}G_m^2+H_{2n}G_{2m}+H_{2m}G_{2n}-(r^2+4s)G_m^2G_{2n}-H_n^2G_{2m}$.

(vi)

1. $4(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = G_n^2H_m^2-G_m^2H_n^2+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}$.
2. $4(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = G_n^2H_m^2-G_m^2H_n^2-4(-s)^mG_{2n}+4(-s)^nG_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = G_n^2H_m^2-G_m^2H_n^2-4G_{2n}+4(-s)^{n-m}G_{2m}$.
4. $4(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = G_n^2H_m^2-G_m^2H_n^2-2H_{2n}G_{2m}-2H_{2m}G_{2n}+2(r^2+4s)G_m^2G_{2n}+2H_n^2G_{2m}$.
5. $4(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = 4((-s)^mG_n^2-(-s)^nG_m^2)+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}$.
6. $(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = ((-s)^mG_n^2-(-s)^nG_m^2)-(-s)^mG_{2n}+(-s)^nG_{2m}$.
7. $(-s)^mG_{n+m}G_{n-m}-G_{n-m}H_{n+m} = ((-s)^mG_n^2-(-s)^nG_m^2)-G_{2n}+(-s)^{n-m}G_{2m}$.
8. $2(-s)^m(G_{n+m}G_{n-m}-G_{n-m}H_{n+m}) = 2((-s)^mG_n^2-(-s)^nG_m^2)-H_{2n}G_{2m}-H_{2m}G_{2n}+(r^2+4s)G_m^2G_{2n}+H_n^2G_{2m}$.
9. $4(G_{n+m}G_{n-m}-(-s)^mG_{n-m}H_{n+m}) = 4G_n^2-4(-s)^{n-m}G_m^2+(H_n^2-(r^2+4s)G_n^2)G_{2m}-(H_m^2-(r^2+4s)G_m^2)G_{2n}$.
10. $G_{n+m}G_{n-m}-(-s)^mG_{n-m}H_{n+m} = G_n^2-(-s)^{n-m}G_m^2-(-s)^mG_{2n}+(-s)^nG_{2m}$.
11. $G_{n+m}G_{n-m}-G_{n-m}H_{n+m} = G_n^2-(-s)^{n-m}G_m^2-G_{2n}+(-s)^{n-m}G_{2m}$.
12. $2(G_{n+m}G_{n-m}-(-s)^mG_{n-m}H_{n+m}) = 2G_n^2-2(-s)^{n-m}G_m^2-H_{2n}G_{2m}-H_{2m}G_{2n}+(r^2+4s)G_m^2G_{2n}+H_n^2G_{2m}$.

(vii)

1. $4((-s)^mG_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = G_n^2H_m^2-G_m^2H_n^2+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}$.
2. $4((-s)^mG_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = G_n^2H_m^2-G_m^2H_n^2+2H_{2m}G_{2n}-2H_{2n}G_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = 4((-s)^mG_n^2-(-s)^nG_m^2)+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}$.
4. $4((-s)^mG_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = 4((-s)^mG_n^2-(-s)^nG_m^2)+2H_{2m}G_{2n}-2H_{2n}G_{2m}$.
5. $4(G_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = 4G_n^2-4(-s)^{n-m}G_m^2+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}$.
6. $4(G_{n+m}G_{n-m}+(-s)^{2m}G_{n-m}H_{n-m}) = 4G_n^2-4(-s)^{n-m}G_m^2+2H_{2m}G_{2n}-2H_{2n}G_{2m}$.

(viii)

1. $4((-s)^mG_{n+m}G_{n-m}-(-s)^{2m}G_{n-m}H_{n-m}) = G_n^2H_m^2-G_m^2H_n^2-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}$.
2. $4((-s)^mG_{n+m}G_{n-m}-(-s)^{2m}G_{n-m}H_{n-m}) = G_n^2H_m^2-G_m^2H_n^2-2H_{2m}G_{2n}+2H_{2n}G_{2m}$.
3. $4((-s)^mG_{n+m}G_{n-m}-(-s)^{2m}G_{n-m}H_{n-m}) = 4((-s)^mG_n^2-(-s)^nG_m^2)-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}$.
4. $4((-s)^mG_{n+m}G_{n-m}-(-s)^{2m}G_{n-m}H_{n-m}) = 4((-s)^mG_n^2-(-s)^nG_m^2)-2H_{2m}G_{2n}+2H_{2n}G_{2m}$.
5. $4(G_{n+m}G_{n-m}-(-s)^{2m}G_{n-m}H_{n-m}) = 4G_n^2-4(-s)^{n-m}G_m^2-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}$.

6. $4(G_{n+m}G_{n-m} - (-s)^{2m}G_{n-m}H_{n-m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 - 2H_{2m}G_{2n} + 2H_{2n}G_{2m}$.

(ix)

1. $4(-s)^m(G_{n+m}G_{n-m} + H_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.
2. $4(-s)^m(G_{n+m}G_{n-m} + H_{n+m}H_{n-m}) = 4((-s)^mG_n^2 - (-s)^nG_m^2) + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}G_{n-m} + (-s)^mH_{n+m}H_{n-m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.

(x)

1. $4(-s)^m(G_{n+m}G_{n-m} - H_{n+m}H_{n-m}) = G_n^2H_m^2 - G_m^2H_n^2 - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.
2. $4(-s)^m(G_{n+m}G_{n-m} - H_{n+m}H_{n-m}) = 4((-s)^mG_n^2 - (-s)^nG_m^2) - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}G_{n-m} - (-s)^mH_{n+m}H_{n-m}) = 4G_n^2 - 4(-s)^{n-m}G_m^2 - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.

(g)

(i)

1. $2(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = (G_{2n}H_m^2 + G_{2m}H_n^2)$.
2. $4(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) + 4(-s)^mG_{2n} + 4(-s)^nG_{2m}$.
3. $4(G_{n+m}H_{n+m} + G_{n+m}H_{n-m}) = H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) + 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
4. $2(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = (G_{2n}H_m^2 + G_{2m}H_n^2)$.
5. $4(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + 4(-s)^mG_{2n} + 4(-s)^nG_{2m}$.
6. $4(G_{n+m}H_{n+m} + G_{n+m}H_{n-m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
7. $4(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = 2H_{2n}G_{2m} + 2H_{2m}G_{2n} + (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
8. $2(G_{n+m}H_{n+m} + (-s)^mG_{n+m}H_{n-m}) = H_{2n}G_{2m} + H_{2m}G_{2n} + 2(-s)^mG_{2n} + 2(-s)^nG_{2m}$.
9. $2(G_{n+m}H_{n+m} + G_{n+m}H_{n-m}) = H_{2n}G_{2m} + H_{2m}G_{2n} + 2G_{2n} + 2(-s)^{n-m}G_{2m}$.

(ii)

1. $2(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = \left(r^2+4s\right)\left(G_{2n}G_m^2 + G_{2m}G_n^2\right)$.
2. $4(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) - 4(-s)^mG_{2n} - 4(-s)^nG_{2m}$.
3. $4(G_{n+m}H_{n+m} - G_{n+m}H_{n-m}) = H_n^2G_{2m} + H_m^2G_{2n} + (r^2+4s)(G_m^2G_{2n} + G_n^2G_{2m}) - 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
4. $2(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = \left(r^2+4s\right)\left(G_{2n}G_m^2 + G_{2m}G_n^2\right)$.
5. $4(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - 4(-s)^mG_{2n} - 4(-s)^nG_{2m}$.
6. $4(G_{n+m}H_{n+m} - G_{n+m}H_{n-m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
7. $4(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - (H_m^2 - (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m}$.
8. $2(G_{n+m}H_{n+m} - (-s)^mG_{n+m}H_{n-m}) = H_{2n}G_{2m} + H_{2m}G_{2n} - 2(-s)^mG_{2n} - 2(-s)^nG_{2m}$.

9. $2(G_{n+m}H_{n+m} - G_{n+m}H_{n-m}) = H_{2n}G_{2m} + H_{2m}G_{2n} - 2G_{2n} - 2(-s)^{n-m}G_{2m}$.

(iii)

1. $2(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = H_m^2 G_{2n} + (r^2+4s)G_n^2 G_{2m}$.
2. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) + 4(-s)^m G_{2n} - 4(-s)^n G_{2m}$.
3. $4(G_{n+m}H_{n+m} + G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) + 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
4. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2 G_{2n} - 2H_n^2 G_{2m}$.
5. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + 4(-s)^m G_{2n} - 4(-s)^n G_{2m}$.
7. $4(G_{n+m}H_{n+m} + G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + 4G_{2n} - 4(-s)^{n-m}G_{2m}$.
8. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - 2(r^2+4s)G_m^2 G_{2n} - 2H_n^2 G_{2m}$.
9. $4(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = 2H_{2n}G_{2m} + 2H_{2m}G_{2n} - (H_n^2 - (r^2+4s)G_n^2)G_{2m} + (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
10. $2(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = H_{2n}G_{2m} + H_{2m}G_{2n} + 2(-s)^m G_{2n} - 2(-s)^n G_{2m}$.
11. $2(G_{n+m}H_{n+m} + G_{n-m}H_{n+m}) = H_{2n}G_{2m} + H_{2m}G_{2n} + 2G_{2n} - 2(-s)^{n-m}G_{2m}$.
12. $2(G_{n+m}H_{n+m} + (-s)^m G_{n-m}H_{n+m}) = H_{2n}G_{2m} + H_{2m}G_{2n} + H_{2n}G_{2m} + H_{2m}G_{2n} - (r^2+4s)G_m^2 G_{2n} - H_n^2 G_{2m}$.

(iv)

1. $2(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = H_n^2 G_{2m} + (r^2+4s)G_m^2 G_{2n}$.
2. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) - 4(-s)^m G_{2n} + 4(-s)^n G_{2m}$.
3. $4(G_{n+m}H_{n+m} - G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) - 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
4. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = H_n^2 G_{2m} + H_m^2 G_{2n} + (r^2+4s)(G_m^2 G_{2n} + G_n^2 G_{2m}) - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + 2(r^2+4s)G_m^2 G_{2n} + 2H_n^2 G_{2m}$.
5. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.
6. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - 4(-s)^m G_{2n} + 4(-s)^n G_{2m}$.
7. $4(G_{n+m}H_{n+m} - G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - 4G_{2n} + 4(-s)^{n-m}G_{2m}$.
8. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = (H_n^2 + (r^2+4s)G_n^2)G_{2m} + (H_m^2 + (r^2+4s)G_m^2)G_{2n} - 2H_{2n}G_{2m} - 2H_{2m}G_{2n} + 2(r^2+4s)G_m^2 G_{2n} + 2H_n^2 G_{2m}$.
9. $4(G_{n+m}H_{n+m} - (-s)^m G_{n-m}H_{n+m}) = 2H_{2n}G_{2m} + 2H_{2m}G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - (H_m^2 - (r^2+4s)G_m^2)G_{2n}$.

10. $2(G_{n+m}H_{n+m}-(-s)^mG_{n-m}H_{n+m})=H_{2n}G_{2m}+H_{2m}G_{2n}-2(-s)^mG_{2n}+2(-s)^nG_{2m}$.
11. $2(G_{n+m}H_{n+m}-G_{n-m}H_{n+m})=H_{2n}G_{2m}+H_{2m}G_{2n}-2G_{2n}+2(-s)^{n-m}G_{2m}$.
12. $2(G_{n+m}H_{n+m}-(-s)^mG_{n-m}H_{n+m})=(r^2+4s)G_m^2G_{2n}+H_n^2G_{2m}$.

(v)

1. $2(G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m})=\left(H_m^2+(r^2+4s)G_m^2\right)G_{2n}$.
2. $4(G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m})=H_n^2G_{2m}-2H_{2n}G_{2m}+H_m^2G_{2n}+2H_{2m}G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
3. $4(G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m})=2\left(H_m^2+(r^2+4s)G_m^2\right)G_{2n}$.
4. $4(G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m})=(H_n^2+(r^2+4s)G_n^2-2H_{2n})G_{2m}+(H_m^2+(r^2+4s)G_m^2+2H_{2m})G_{2n}$.
5. $4(G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m})=2H_{2n}G_{2m}+2H_{2m}G_{2n}+((r^2+4s)G_m^2+H_m^2)G_{2n}-((r^2+4s)G_n^2+H_n^2)G_{2m}$.
6. $G_{n+m}H_{n+m}+(-s)^{2m}G_{n-m}H_{n-m}=G_{2n}H_{2m}$.

(vi)

1. $2(G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=\left(H_n^2+(r^2+4s)G_n^2\right)G_{2m}$.
2. $4(G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=H_n^2G_{2m}+H_m^2G_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})$.
3. $4(G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=2\left(H_n^2+(r^2+4s)G_n^2\right)G_{2m}$.
4. $4(G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}-2H_{2m}G_{2n}+2H_{2n}G_{2m}$.
5. $4(G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=2H_{2n}G_{2m}+2H_{2m}G_{2n}-((r^2+4s)G_m^2+H_m^2)G_{2n}+((r^2+4s)G_n^2+H_n^2)G_{2m}$.
6. $G_{n+m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m}=G_{2m}H_{2n}$.

(vii)

1. $4(G_{n+m}H_{n+m}+(-s)^mH_{n+m}H_{n-m})=H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.
2. $4(G_{n+m}H_{n+m}+(-s)^mH_{n+m}H_{n-m})=(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}H_{n+m}+(-s)^mH_{n+m}H_{n-m})=2H_{2n}G_{2m}+2H_{2m}G_{2n}+H_m^2H_n^2-(r^2+4s)^2G_m^2G_n^2$.

(viii)

1. $4(G_{n+m}H_{n+m}-(-s)^mH_{n+m}H_{n-m})=H_n^2G_{2m}+H_m^2G_{2n}+(r^2+4s)(G_m^2G_{2n}+G_n^2G_{2m})-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2$.
2. $4(G_{n+m}H_{n+m}-(-s)^mH_{n+m}H_{n-m})=(H_n^2+(r^2+4s)G_n^2)G_{2m}+(H_m^2+(r^2+4s)G_m^2)G_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}H_{n+m}-(-s)^mH_{n+m}H_{n-m})=2H_{2n}G_{2m}+2H_{2m}G_{2n}-H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2$.

(h) (i) 1. $2(-s)^m(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2)G_{2n}$.

2. $4(-s)^m(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2+4(-s)^m)G_{2n}+(H_n^2-(r^2+4s)G_n^2-4(-s)^n)G_{2m}$.
3. $4((-s)^mG_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2+4)G_{2n}+(H_n^2-(r^2+4s)G_n^2-4(-s)^{n-m})G_{2m}$.

4. $4(-s)^m(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(H_m^2+2H_{2m}-3(r^2+4s)G_m^2)G_{2n}+(-H_n^2+2H_{2n}-(r^2+4s)G_n^2)G_{2m}$.
5. $4(-s)^m(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(-H_n^2+(r^2+4s)G_n^2+4(-s)^n)G_{2m}+(H_m^2-(r^2+4s)G_m^2+4(-s)^m)G_{2n}$.
6. $G_{n+m}H_{n-m}+G_{n-m}H_{n+m}=2G_{2n}$.
7. $(-s)^mG_{n+m}H_{n-m}+G_{n-m}H_{n+m}=(1+(-s)^m)G_{2n}+((-s)^n-(-s)^{n-m})G_{2m}$.
8. $2(-s)^m(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=(2(-s)^m-(r^2+4s)G_m^2+H_{2m})G_{2n}+(2(-s)^n+H_{2n}-H_n^2)G_{2m}$.
9. $4(G_{n+m}H_{n-m}+(-s)^mG_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2+4)G_{2n}-(H_n^2-(r^2+4s)G_n^2-4(-s)^{n-m})G_{2m}$.
10. $G_{n+m}H_{n-m}+(-s)^mG_{n-m}H_{n+m}=(1+(-s)^m)G_{2n}+((-s)^{n-m}-(-s)^n)G_{2m}$.
11. $2(G_{n+m}H_{n-m}+G_{n-m}H_{n+m})=4G_{2n}$.
12. $2(G_{n+m}H_{n-m}+(-s)^mG_{n-m}H_{n+m})=(H_{2m}+2-(r^2+4s)G_m^2)G_{2n}+(H_{2n}+2(-s)^{n-m}-H_n^2)G_{2m}$.

(ii)
1. $2(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(H_n^2-(r^2+4s)G_n^2)G_{2m}$.
2. $4(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2-4(-s)^m)G_{2n}+(H_n^2-(r^2+4s)G_n^2+4(-s)^n)G_{2m}$.
3. $4((-s)^mG_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(H_m^2-(r^2+4s)G_m^2-4)G_{2n}+(H_n^2-(r^2+4s)G_n^2+4(-s)^{n-m})G_{2m}$.
4. $4(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(H_m^2-2H_{2m}+(r^2+4s)G_m^2)G_{2n}+(3H_n^2-2H_{2n}-(r^2+4s)G_n^2)G_{2m}$.
5. $4(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(-H_m^2+(r^2+4s)G_m^2+4(-s)^m)G_{2n}+(H_n^2-(r^2+4s)G_n^2+4(-s)^n)G_{2m}$.
6. $(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=2(-s)^nG_{2m}$.
7. $(-s)^mG_{n+m}H_{n-m}-G_{n-m}H_{n+m}=((-s)^m-1)G_{2n}+((-s)^n+(-s)^{n-m})G_{2m}$.
8. $2(-s)^m(G_{n+m}H_{n-m}-G_{n-m}H_{n+m})=(-H_{2m}+(r^2+4s)G_m^2+2(-s)^m)G_{2n}+(-H_{2n}+H_n^2+2(-s)^n)G_{2m}$.
9. $4(G_{n+m}H_{n-m}-(-s)^mG_{n-m}H_{n+m})=(-H_m^2+(r^2+4s)G_m^2+4)G_{2n}+(H_n^2-(r^2+4s)G_n^2+4(-s)^{n-m})G_{2m}$.
10. $G_{n+m}H_{n-m}-(-s)^mG_{n-m}H_{n+m}=(1-(-s)^m)G_{2n}+((-s)^{n-m}+(-s)^n)G_{2m}$.
11. $G_{n+m}H_{n-m}-G_{n-m}H_{n+m}=2(-s)^{n-m}G_{2m}$.
12. $2(G_{n+m}H_{n-m}-(-s)^mG_{n-m}H_{n+m})=(-H_{2m}+(r^2+4s)G_m^2+2)G_{2n}+(-H_{2n}+H_n^2+2(-s)^{n-m})G_{2m}$.

(iii)
1. $2((-s)^mG_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=H_m^2G_{2n}-(r^2+4s)G_n^2G_{2m}$.
2. $4((-s)^mG_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=(H_m^2-(r^2+4s)G_m^2+2H_{2m})G_{2n}+(H_n^2-(r^2+4s)G_n^2-2H_{2n})G_{2m}$.
3. $4((-s)^mG_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=((r^2+4s)G_m^2+H_m^2+4(-s)^m)G_{2n}-((r^2+4s)G_n^2+H_n^2-4(-s)^n)G_{2m}$.
4. $2((-s)^mG_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=(2(-s)^m+H_{2m})G_{2n}+(2(-s)^n-H_{2n})G_{2m}$.
5. $4(G_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=((r^2+4s)G_m^2+H_m^2+4)G_{2n}-((r^2+4s)G_n^2+H_n^2-4(-s)^{n-m})G_{2m}$.
6. $2(G_{n+m}H_{n-m}+(-s)^{2m}G_{n-m}H_{n-m})=(2+H_{2m})G_{2n}+(2(-s)^{n-m}-H_{2n})G_{2m}$.

(iv)

1. $2((-s)^mG_{n+m}H_{n-m}-(-s)^{2m}G_{n-m}H_{n-m})=H_n^2G_{2m}-(r^2+4s)G_m^2G_{2n}$.

2. $4((-s)^m G_{n+m}H_{n-m} - (-s)^{2m} G_{n-m}H_{n-m}) = (H_m^2 - (r^2+4s)G_m^2 - 2H_{2m})G_{2n} + (H_n^2 - (r^2+4s)G_n^2 + 2H_{2n})G_{2m}$.
3. $4((-s)^m G_{n+m}H_{n-m} - (-s)^{2m} G_{n-m}H_{n-m}) = -((r^2+4s)G_m^2 + H_m^2 - 4(-s)^m)G_{2n} + ((r^2+4s)G_n^2 + H_n^2 + 4(-s)^n)G_{2m}$.
4. $2((-s)^m G_{n+m}H_{n-m} - (-s)^{2m} G_{n-m}H_{n-m}) = (2(-s)^m - H_{2m})G_{2n} + (2(-s)^n + H_{2n})G_{2m}$.
5. $4(G_{n+m}H_{n-m} - (-s)^{2m} G_{n-m}H_{n-m}) = -((r^2+4s)G_m^2 + H_m^2 - 4)G_{2n} + ((r^2+4s)G_n^2 + H_n^2 + 4(-s)^{n-m})G_{2m}$.
6. $2(G_{n+m}H_{n-m} - (-s)^{2m} G_{n-m}H_{n-m}) = (2 - H_{2m})G_{2n} + (2(-s)^{n-m} + H_{2n})G_{2m}$.

(v)

1. $4(-s)^m(G_{n+m}H_{n-m} + H_{n+m}H_{n-m}) = (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.
2. $4(-s)^m(G_{n+m}H_{n-m} + H_{n+m}H_{n-m}) = 4(-s)^m G_{2n} + 4(-s)^n G_{2m} + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}H_{n-m} + (-s)^m H_{n+m}H_{n-m}) = 4G_{2n} + 4(-s)^{n-m}G_{2m} + H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2$.

(vi)

1. $4(-s)^m(G_{n+m}H_{n-m} - H_{n+m}H_{n-m}) = (H_m^2 - (r^2+4s)G_m^2)G_{2n} + (H_n^2 - (r^2+4s)G_n^2)G_{2m} - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.
2. $4(-s)^m(G_{n+m}H_{n-m} - H_{n+m}H_{n-m}) = 4(-s)^m G_{2n} + 4(-s)^n G_{2m} - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.
3. $4(G_{n+m}H_{n-m} - (-s)^m H_{n+m}H_{n-m}) = 4G_{2n} + 4(-s)^{n-m}G_{2m} - H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2$.

(i) (i)

1. $2((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = -H_n^2G_{2m} + H_m^2G_{2n}$.
2. $4((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = -(H_n^2 - (r^2+4s)G_n^2 + 2H_{2n})G_{2m} + (H_m^2 - (r^2+4s)G_m^2 + 2H_{2m})G_{2n}$.
3. $4((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = ((r^2+4s)G_m^2 + H_m^2 + 4(-s)^m)G_{2n} - ((r^2+4s)G_n^2 + H_n^2 + 4(-s)^n)G_{2m}$.
4. $2((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = (2(-s)^m + H_{2m})G_{2n} - (2(-s)^n + H_{2n})G_{2m}$.
5. $4(G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = ((r^2+4s)G_m^2 + H_m^2 + 4)G_{2n} - ((r^2+4s)G_n^2 + H_n^2 + 4(-s)^{n-m})G_{2m}$.
6. $2(G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = (2 + H_{2m})G_{2n} - (2(-s)^{n-m} + H_{2n})G_{2m}$.
7. $4((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = (H_m^2 + 2H_{2m} - (r^2+4s)G_m^2)G_{2n} - (3H_n^2 - 2H_{2n} + (r^2+4s)G_n^2)G_{2m}$.
8. $2((-s)^m G_{n-m}H_{n+m} + (-s)^{2m} G_{n-m}H_{n-m}) = (2H_{2m} - (r^2+4s)G_m^2)G_{2n} - H_n^2G_{2m}$.

(ii)

1. $2((-s)^m G_{n-m}H_{n+m} - (-s)^{2m} G_{n-m}H_{n-m}) = (r^2+4s)\left(G_n^2G_{2m} - G_m^2G_{2n}\right)$.
2. $4((-s)^m G_{n-m}H_{n+m} - (-s)^{2m} G_{n-m}H_{n-m}) = -(H_n^2 - (r^2+4s)G_n^2 - 2H_{2n})G_{2m} + (H_m^2 - (r^2+4s)G_m^2 - 2H_{2m})G_{2n}$.
3. $4((-s)^m G_{n-m}H_{n+m} - (-s)^{2m} G_{n-m}H_{n-m}) = -((r^2+4s)G_m^2 + H_m^2 - 4(-s)^m)G_{2n} + ((r^2+4s)G_n^2 + H_n^2 - 4(-s)^n)G_{2m}$.
4. $2((-s)^m G_{n-m}H_{n+m} - (-s)^{2m} G_{n-m}H_{n-m}) = (2(-s)^m - H_{2m})G_{2n} - (2(-s)^n - H_{2n})G_{2m}$.
5. $4(G_{n-m}H_{n+m} - (-s)^{2m} G_{n-m}H_{n-m}) = -((r^2+4s)G_m^2 + H_m^2 - 4)G_{2n} + ((r^2+4s)G_n^2 + H_n^2 - 4(-s)^{n-m})G_{2m}$.

6. $2(G_{n-m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=(2-H_{2m})G_{2n}-(2(-s)^{n-m}-H_{2n})G_{2m}$.
7. $4((-s)^{m}G_{n-m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=-(H_{m}^{2}-2H_{2m}+3(r^{2}+4s)G_{m}^{2})G_{2n}+(-H_{n}^{2}+2H_{2n}+(r^{2}+4s)G_{n}^{2})G_{2m}$.
8. $2((-s)^{m}G_{n-m}H_{n+m}-(-s)^{2m}G_{n-m}H_{n-m})=-(r^{2}+4s)G_{m}^{2}G_{2n}-\left(H_{n}^{2}-2H_{2n}\right)G_{2m}$.

(iii)

1. $4(-s)^{m}(G_{n-m}H_{n+m}+H_{n+m}H_{n-m})=-(H_{n}^{2}-(r^{2}+4s)G_{n}^{2})G_{2m}+(H_{m}^{2}-(r^{2}+4s)G_{m}^{2})G_{2n}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
2. $4(-s)^{m}(G_{n-m}H_{n+m}+H_{n+m}H_{n-m})=4(-s)^{m}G_{2n}-4(-s)^{n}G_{2m}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
3. $4(G_{n-m}H_{n+m}+(-s)^{m}H_{n+m}H_{n-m})=4G_{2n}-4(-s)^{n-m}G_{2m}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
4. $4(-s)^{m}(G_{n-m}H_{n+m}+H_{n+m}H_{n-m})=(2H_{2m}-2(r^{2}+4s)G_{m}^{2})G_{2n}+(2H_{2n}-2H_{n}^{2})G_{2m}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.

(iv)

1. $4(-s)^{m}(G_{n-m}H_{n+m}-H_{n+m}H_{n-m})=-(H_{n}^{2}-(r^{2}+4s)G_{n}^{2})G_{2m}+(H_{m}^{2}-(r^{2}+4s)G_{m}^{2})G_{2n}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
2. $4(-s)^{m}(G_{n-m}H_{n+m}-H_{n+m}H_{n-m})=4(-s)^{m}G_{2n}-4(-s)^{n}G_{2m}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
3. $4(G_{n-m}H_{n+m}-(-s)^{m}H_{n+m}H_{n-m})=4G_{2n}-4(-s)^{n-m}G_{2m}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.
4. $4(-s)^{m}(G_{n-m}H_{n+m}-H_{n+m}H_{n-m})=(2H_{2m}-2(r^{2}+4s)G_{m}^{2})G_{2n}+(2H_{2n}-2H_{n}^{2})G_{2m}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$.

(j) (i)

1. $4((-s)^{2m}G_{n-m}H_{n-m}+(-s)^{m}H_{n+m}H_{n-m})=((r^{2}+4s)G_{m}^{2}+H_{m}^{2})G_{2n}-((r^{2}+4s)G_{n}^{2}+H_{n}^{2})G_{2m}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$
2. $4((-s)^{2m}G_{n-m}H_{n-m}+(-s)^{m}H_{n+m}H_{n-m})=2H_{2m}G_{2n}-2H_{2n}G_{2m}+H_{m}^{2}H_{n}^{2}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$

(ii)

1. $4((-s)^{2m}G_{n-m}H_{n-m}-(-s)^{m}H_{n+m}H_{n-m})=((r^{2}+4s)G_{m}^{2}+H_{m}^{2})G_{2n}-((r^{2}+4s)G_{n}^{2}+H_{n}^{2})G_{2m}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$
2. $4((-s)^{2m}G_{n-m}H_{n-m}-(-s)^{m}H_{n+m}H_{n-m})=2H_{2m}G_{2n}-2H_{2n}G_{2m}-H_{m}^{2}H_{n}^{2}+(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}$

Proof. We make use of the identities

$$\begin{aligned} G_{2n} &= G_nH_n, \\ 2H_{2n} &= (r^2+4s)G_n^2+H_n^2, \end{aligned}$$

as given in Corollary 1.12, together with

$$\begin{aligned} H_n^2 &= (r^2+4s)G_n^2+4(-s)^n, \\ H_{2n} &= (r^2+4s)G_n^2+2(-s)^n, \\ H_n^2-H_{2n} &= 2(-s)^n, \text{ i.e., } H_n^2=H_{2n}+2(-s)^n, \end{aligned}$$

as established in Lemma 1.13 (e)

(a)

- **(i)** Add Lemma 2.7 (b) (i) (1) and Lemma 2.7 (b) (ii) (1) side by side, then apply the identities

$$2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2, \qquad H_n^2 = H_{2n} + 2(-s)^n.$$

- **(ii)** Subtract Lemma 2.7 (b) (i) (1) from Lemma 2.7 (b) (ii) (1) and use the identity $G_{2n} = G_n H_n$.
- **(iii)** Add Lemma 2.7 (b) (iii) (1) and Lemma 2.7 (b) (iv) (1) side by side, then apply

$$2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2, \qquad H_n^2 = H_{2n} + 2(-s)^n.$$

- **(iv)** Subtract Lemma 2.7 (b) (iii) (1) from Lemma 2.7 (b) (iv) (1) and use the identity $G_{2n} = G_n H_n$.
- **(v)** Add Lemma 2.7 (b) (i) (1) and Lemma 2.7 (b) (iv) (1) side by side, then apply $2H_{2n} = ((r^2+4s)G_n^2+H_n^2)$.
- **(vi)** Subtract Lemma 2.7 (b) (i) (1) from Lemma 2.7 (b) (iv) (1) and use the identities

$$2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2, \quad G_{2n} = G_n H_n, \quad H_n^2 - H_{2n} = 2(-s)^n.$$

- **(vii)** Add Lemma 2.7 (b) (iii) (1) and Lemma 2.7 (b) (ii) (1) side by side, then apply $2H_{2n} = ((r^2 + 4s)G_n^2 + H_n^2)$.
- **(viii)** Subtract Lemma 2.7 (b) (1) (iii) from Lemma 2.7 (b) (ii) (1) and use the identities

$$2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2, \quad G_{2n} = G_n H_n, \quad H_n^2 - H_{2n} = 2(-s)^n.$$

- **(ix)** Add Lemma 2.7 (b)(i)(1) with Lemma 2.7 (b)(iii)(1), and Lemma 2.7 (b)(i)(2) with Lemma 2.7 (b)(iii)(2).
- **(x)** Subtract Lemma 2.7 (b)(i)(1) from Lemma 2.7 (b)(iii)(1), and Lemma 2.7 (b)(i)(2) from Lemma 2.7 (b)(iii)(2).
- **(xi)** Add Lemma 2.7 (b)(ii)(1) with Lemma 2.7 (b)(iv)(1), and Lemma 2.7 (b)(ii)(2) with Lemma 2.7 (b)(iv)(2).
- **(xii)** Subtract Lemma 2.7 (b)(ii)(1) from Lemma 2.7 (b)(iv)(1), and Lemma 2.7 (b)(ii)(2) from Lemma 2.7 (b)(iv)(2).

(b)

- **(i)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (v).
- **(ii)** Same as (i).
- **(iii)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (vi).
- **(iv)** Same as (iii).
- **(v)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (vii).
- **(vi)** Same as (v).
- **(vii)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (viii).
- **(viii)** Same as (vii).
- **(ix)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (ix).
- **(x)** Same as (ix).
- **(xi)** Combine Lemma 2.7 (b) (i) with Lemma 2.7 (b) (x).
- **(xii)** Same as (xi).

(c)

(i) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (v).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (vi).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (vii).

(vi) Same as (v).

(vii) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (viii).

(viii) Same as (vii).

(ix) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (ix).

(x) Same as (ix).

(xi) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (b) (x).

(xii) Same as (xi).

(d)

(i) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (v).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (vi).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (vii).

(vi) Same as (v).

(vii) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (viii).

(viii) Same as (vii).

(ix) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (ix).

(x) Same as (ix).

(xi) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (b) (x).

(xii) Same as (xi).

(e)

(i) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (v).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (vi).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (vii).

(vi) Same as (v).

(vii) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (viii).

(viii) Same as (vii).

(ix) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (ix).

(x) Same as (ix).

(xi) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (b) (x).

(xii) Same as (xi).

(f)

(i) Combine Lemma 2.7 (b) (v) with Lemma 2.7 (b) (vi).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (v) with Lemma 2.7 (b) (vii).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (v) with Lemma 2.7 (b) (viii).

(vi) Same as (v).

(vii) Combine Lemma 2.7 (b) (v) with Lemma 2.7 (b) (ix).

(viii) Same as (vii).

(ix) Combine Lemma 2.7 (b) (v) with Lemma 2.7 (b) (x).

(x) Same as (ix).

(g)

(i) Combine Lemma 2.7 (b) (vi) with Lemma 2.7 (b) (vii).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (vi) with Lemma 2.7 (b) (viii).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (vi) with Lemma 2.7 (b) (ix).

(vi) Same as (v).

(vii) Combine Lemma 2.7 (b) (vi) with Lemma 2.7 (b) (x).

(viii) Same as (vii).

(h)

(i) Combine Lemma 2.7 (b) (vii) with Lemma 2.7 (b) (viii).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (vii) with Lemma 2.7 (b) (ix).

(iv) Same as (iii).

(v) Combine Lemma 2.7 (b) (vii) with Lemma 2.7 (b) (x).

(vi) Same as (v).

(i)

(i) Combine Lemma 2.7 (b) (viii) with Lemma 2.7 (b) (ix).

(ii) Same as (i).

(iii) Combine Lemma 2.7 (b) (viii) with Lemma 2.7 (b) (x).

(iv) Same as (iii).

(j)

(i) Combine Lemma 2.7 (b) (ix) with Lemma 2.7 (b) (x).

(ii) Same as (i). □

Note that

$$\begin{aligned} G_n^2 H_m^2 &= G_m^2 H_n^2 + 4(-s)^m G_n^2 - 4(-s)^n G_m^2 \\ &\Rightarrow \\ G_n^2 H_m^2 - G_m^2 H_n^2 &= 4(-s)^m G_n^2 - 4(-s)^n G_m^2 \end{aligned}$$

From Lemma 2.7, we get the following Corollary.

Corollary 2.9. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $4(G_{n+m}^4 + (-s)^{4m} G_{n-m}^4) = (G_m^2 H_n^2 + G_n^2 H_m^2)^2 - 8((-s)^m G_n^2 - (-s)^n G_m^2)^2.$
2. $(r^2+4s)^2 (G_{n+m}^4 + (-s)^{4m} G_{n-m}^4) = (H_{2m} H_{2n} - 4(-s)^{n+m})^2 - 2(r^2+4s)^2((-s)^m G_n^2 - (-s)^n G_m^2)^2.$
3. $8(G_{n+m}^4 + (-s)^{4m} G_{n-m}^4) = (G_m^2 H_n^2 + G_n^2 H_m^2)^2 + 4G_m^2 G_n^2 H_m^2 H_n^2.$
4. $8(r^2+4s)^2 (G_{n+m}^4 + (-s)^{4m} G_{n-m}^4) = 8(H_{2m} H_{2n} - 4(-s)^{n+m})^2 - (r^2+4s)^2 (H_m^2 G_n^2 - G_m^2 H_n^2)^2.$
5. $G_{n+m}^4 + (-s)^{4m} G_{n-m}^4 = G_{2n}^2 G_{2m}^2 + 2((-s)^m G_n^2 - (-s)^n G_m^2)^2.$
6. $8(G_{n+m}^4 + (-s)^{4m} G_{n-m}^4) = 8G_{2n}^2 G_{2m}^2 + (H_m^2 G_n^2 - G_m^2 H_n^2)^2.$

(ii)

1. $2(G_{n+m}^4 - (-s)^{4m} G_{n-m}^4) = (G_m^2 H_n^2 + G_n^2 H_m^2) G_{2n} G_{2m}.$
2. $(r^2+4s)(G_{n+m}^4 - (-s)^{4m} G_{n-m}^4) = (H_{2m} H_{2n} - 4(-s)^{n+m}) G_{2n} G_{2m}.$

(iii)

1. $4(H_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = ((r^2+4s)^2 G_m^2 G_n^2 + H_m^2 H_n^2)^2 - 8((-s)^m H_n^2 + (r^2+4s)(-s)^n G_m^2)^2.$
2. $H_{n+m}^4 + (-s)^{4m} H_{n-m}^4 = (H_{2m} H_{2n} + 4(-s)^{m+n})^2 - 2((-s)^m H_n^2 + (r^2+4s)(-s)^n G_m^2)^2.$
3. $8(H_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = ((r^2+4s)^2 G_m^2 G_n^2 + H_m^2 H_n^2)^2 + 4(r^2+4s)^2 G_m^2 G_n^2 H_m^2 H_n^2.$
4. $8(H_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = 8(H_{2m} H_{2n} + 4(-s)^{m+n})^2 - (H_m^2 H_n^2 - (r^2+4s)^2 G_m^2 G_n^2)^2.$
5. $H_{n+m}^4 + (-s)^{4m} H_{n-m}^4 = \left(r^2+4s\right)^2 G_{2n}^2 G_{2m}^2 + 2((-s)^m H_n^2 + (r^2+4s)(-s)^n G_m^2)^2.$
6. $8(H_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = 8\left(r^2+4s\right)^2 G_{2n}^2 G_{2m}^2 + (H_m^2 H_n^2 - (r^2+4s)^2 G_m^2 G_n^2)^2.$

(iv)

1. $2(H_{n+m}^4 - (-s)^{4m} H_{n-m}^4) = \left(r^2+4s\right)((r^2+4s)^2 G_m^2 G_n^2 + H_m^2 H_n^2) G_{2n} G_{2m}.$

2. $H_{n+m}^4 - (-s)^{4m} H_{n-m}^4 = \left(r^2+4s\right)(H_{2m}H_{2n} + 4\,(-s)^{m+n})G_{2n}G_{2m}$.

(v)

1. $8((r^2+4s)^2 G_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = 8H_{2m}^2 H_{2n}^2 - (r^2+4s)(H_mG_n + G_mH_n)^2(H_mH_n - (r^2+4s)G_mG_n)^2$.
2. $8((r^2+4s)^2 G_{n+m}^4 + (-s)^{4m} H_{n-m}^4) = 8(-4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m})^2 + (r^2+4s)(H_mG_n + G_mH_n)^2(H_mH_n - (r^2+4s)G_mG_n)^2$

(vi) $(r^2+4s)^2 G_{n+m}^4 - (-s)^{4m} H_{n-m}^4 = (-4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m})H_{2m}H_{2n}$.

(vii)

1. $8(H_{n+m}^4 + (r^2+4s)^2(-s)^{4m} G_{n-m}^4) = 8H_{2m}^2 H_{2n}^2 - (r^2+4s)(H_mH_n + (r^2+4s)G_mG_n)^2(H_mG_n - G_mH_n)^2$.
2. $8(H_{n+m}^4 + (r^2+4s)^2(-s)^{4m} G_{n-m}^4) = 8(4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m})^2 + (r^2+4s)(H_mH_n + (r^2+4s)G_mG_n)^2(H_mG_n - G_mH_n)^2$.

(viii) $H_{n+m}^4 - (r^2+4s)^2(-s)^{4m} G_{n-m}^4 = (4(-s)^{m+n} + (r^2+4s)G_{2n}G_{2m})H_{2m}H_{2n}$.

(b)

(i)

1. $4(G_{n+1}^4 + (-s)^4 G_{n-1}^4) = (H_n^2 + r^2G_n^2)^2 - 8(sG_n^2 + (-s)^n)^2$.
2. $(r^2+4s)^2(G_{n+1}^4 + (-s)^4 G_{n-1}^4) = ((r^2+2s)H_{2n} - 4(-s)^{n+1})^2 - 2(r^2+4s)^2(sG_n^2 + (-s)^n)^2$.
3. $8(G_{n+1}^4 + (-s)^4 G_{n-1}^4) = (H_n^2 + r^2G_n^2)^2 + 4r^2G_n^2H_n^2$.
4. $8(r^2+4s)^2(G_{n+1}^4 + (-s)^4 G_{n-1}^4) = 8((r^2+2s)H_{2n} - 4(-s)^{n+1})^2 - (r^2+4s)^2(r^2G_n^2 - H_n^2)^2$.
5. $G_{n+1}^4 + (-s)^4 G_{n-1}^4 = r^2G_{2n}^2 + 2(sG_n^2 + (-s)^n)^2$.
6. $8(G_{n+1}^4 + (-s)^4 G_{n-1}^4) = 8r^2G_{2n}^2 + (r^2G_n^2 - H_n^2)^2$.

(ii)

1. $2(G_{n+1}^4 - (-s)^4 G_{n-1}^4) = r(H_n^2 + r^2G_n^2)G_{2n}$.
2. $(r^2+4s)(G_{n+1}^4 - (-s)^4 G_{n-1}^4) = r((r^2+2s)H_{2n} - 4(-s)^{n+1})G_{2n}$.

(iii)

1. $4(H_{n+1}^4 + (-s)^4 H_{n-1}^4) = ((r^2+4s)^2G_n^2 + r^2H_n^2)^2 - 8(-sH_n^2 + (r^2+4s)(-s)^n)^2$.
2. $H_{n+1}^4 + (-s)^4 H_{n-1}^4 = ((r^2+2s)H_{2n} + 4\,(-s)^{1+n})^2 - 2(-sH_n^2 + (r^2+4s)(-s)^n)^2$.
3. $8(H_{n+1}^4 + (-s)^4 H_{n-1}^4) = ((r^2+4s)^2G_n^2 + r^2H_n^2)^2 + 4r^2(r^2+4s)^2G_n^2H_n^2$.
4. $8(H_{n+1}^4 + (-s)^4 H_{n-1}^4) = 8((r^2+2s)H_{2n} + 4\,(-s)^{1+n})^2 - (r^2H_n^2 - (r^2+4s)^2G_n^2)^2$.
5. $H_{n+1}^4 + (-s)^4 H_{n-1}^4 = r^2\left(r^2+4s\right)^2 G_{2n}^2 + 2(-sH_n^2 + (r^2+4s)(-s)^n)^2$.
6. $8(H_{n+1}^4 + (-s)^4 H_{n-1}^4) = 8r^2\left(r^2+4s\right)^2 G_{2n}^2 + (r^2H_n^2 - (r^2+4s)^2G_n^2)^2$.

(iv)

1. $2(H_{n+1}^4 - (-s)^4 H_{n-1}^4) = r\left(r^2+4s\right)((r^2+4s)^2G_n^2 + r^2H_n^2)G_{2n}$.
2. $H_{n+1}^4 - (-s)^4 H_{n-1}^4 = r\left(r^2+4s\right)((r^2+2s)H_{2n} + 4\,(-s)^{1+n})G_{2n}$.

(v)

1. $8((r^2+4s)^2G_{n+1}^4 + (-s)^4 H_{n-1}^4) = 8(r^2+2s)^2H_{2n}^2 - (r^2+4s)(rG_n + H_n)^2(rH_n - (r^2+4s)G_n)^2$.

2. $8((r^2+4s)^2G_{n+1}^4+(-s)^4H_{n-1}^4) = 8(-4(-s)^{1+n}+r(r^2+4s)G_{2n})^2+(r^2+4s)(rG_n+H_n)^2(rH_n-(r^2+4s)G_n)^2$

(vi) $(r^2+4s)^2G_{n+1}^4-(-s)^4H_{n-1}^4 = (r^2+2s)(-4(-s)^{1+n}+r(r^2+4s)G_{2n})H_{2n}$.

(vii)

1. $8(H_{n+1}^4+(r^2+4s)^2(-s)^4G_{n-1}^4) = 8(r^2+2s)^2H_{2n}^2-(r^2+4s)(rH_n+(r^2+4s)G_n)^2(rG_n-H_n)^2$.
2. $8(H_{n+1}^4+(r^2+4s)^2(-s)^4G_{n-1}^4) = 8(4(-s)^{1+n}+r(r^2+4s)G_{2n})^2+(r^2+4s)(rH_n+(r^2+4s)G_n)^2(rG_n-H_n)^2$.

(viii) $H_{n+1}^4-(r^2+4s)^2(-s)^4G_{n-1}^4 = (r^2+2s)(4(-s)^{1+n}+r(r^2+4s)G_{2n})H_{2n}$.

Proof. We employ the identities

$$\begin{aligned} G_{2n} &= G_nH_n, \\ 2H_{2n} &= (r^2+4s)G_n^2+H_n^2, \end{aligned}$$

as given in Corollary 1.12, together with

$$\begin{aligned} H_n^2 &= (r^2+4s)G_n^2+4(-s)^n, \\ H_{2n} &= (r^2+4s)G_n^2+2(-s)^n, \\ H_n^2-H_{2n} &= 2(-s)^n, \text{ i.e., } H_n^2=H_{2n}+2(-s)^n, \end{aligned}$$

as established in Lemma 1.13 (e).

(a)

(i) (1)-(2). Apply Corollary 2.8 (a) (i), Theorem 1.16 (b) (ii) and the identity

$$a^2+b^2=(a+b)^2-2ab$$

with $a=2G_{n+m}^2,\ b=2(-s)^{2m}G_{n-m}^2$ and $a=(r^2+4s)G_{n+m}^2,\ b=(r^2+4s)(-s)^{2m}G_{n-m}^2$, respectively.

(3)-(4). Use Corollary 2.8 (a) (i), Corollary 2.5 (a) (i), and the same identity with the same substitutions.

(5)-(6). Apply Corollary 2.8 (a) (ii), Theorem 1.16 (b) (ii), Corollary 2.5 (a) (i) and the identity

$$a^2+b^2=(a-b)^2+2ab$$

with $a=G_{n+m}^2,\ b=(-s)^{2m}G_{n-m}^2$.

(ii) (1)-(2). Use Corollary 2.8 (a) (i), Corollary 2.8 (a) (ii) and the identity

$$a^2-b^2=(a+b)(a-b).$$

(iii) (1)-(2). Apply Corollary 2.8 (a) (iii) (2), Corollary 2.8 (a) (iii) (3), Theorem 1.16 (b) (iv) and the identity

$$a^2+b^2=(a+b)^2-2ab$$

with $a=2H_{n+m}^2,\ b=2(-s)^{2m}H_{n-m}^2$ and $a=H_{n+m}^2,\ b=(-s)^{2m}H_{n-m}^2$, respectively.

(3)-(4). Use Corollary 2.8 (a) (iii) (2), Corollary 2.8 (a) (iii) (3), Corollary 2.5 (a)(ii), and the same identity with the same substitutions.

(5)-(6). Apply Corollary 2.8 (a) (iv), Theorem 1.16 (b) (iv), Corollary 2.5 (a)(ii) and the identity

$$a^2+b^2=(a-b)^2+2ab$$

with $a=2H_{n+m}^2,\ b=2(-s)^{2m}H_{n-m}^2$.

(iv) (1)-(2). Use Corollary 2.8 (a) (iii) (2), Corollary 2.8 (a) (iii) (3), Corollary 2.8 (a) (iv) and the identity $a^2 - b^2 = (a+b)(a-b)$.

(v) (1). Apply Corollary 2.8 (a) (v), Corollary 2.5 (a) (iii) and the identity $a^2 + b^2 = (a+b)^2 - 2ab$ with $a = (r^2+4s)G_{n+m}^2$, $b = (-s)^{2m}G_{n-m}^2$.

(2). Use Corollary 2.8 (a) (vi), Corollary 2.5 (a) (iii) and the identity $a^2 + b^2 = (a-b)^2 + 2ab$ with $a = (r^2+4s)G_{n+m}^2$, $b = (-s)^{2m}G_{n-m}^2$.

(vi) Apply Corollary 2.8 (a) (v), Corollary 2.8 (a) (vi) and the identity $a^2 - b^2 = (a+b)(a-b)$.

(vii) (1). Use Corollary 2.8 (a) (vii), Corollary 2.5 (a) (iv) and the identity $a^2 + b^2 = (a+b)^2 - 2ab$ with $a = H_{n+m}^2$, $b = (r^2+4s)(-s)^{2m}H_{n-m}^2$.

(2). Apply Corollary 2.8 (a) (viii), Corollary 2.5 (a) (iv) and the identity $a^2 + b^2 = (a-b)^2 + 2ab$ with $a = H_{n+m}^2$, $b = (r^2+4s)(-s)^{2m}H_{n-m}^2$.

(viii) Use Corollary 2.8 (a) (vii), Corollary 2.8 (a) (viii) and the identity $a^2 - b^2 = (a+b)(a-b)$.

(b) Set $m = 1$ in (a). □

By combining Corollary 2.8 with Corollary 2.10, we obtain the following results.

Corollary 2.10. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i) $(r^2+4s)(G_{n+m}^2 + (-s)^{2m}G_{n-m}^2) - (H_{n+m}^2 + (-s)^{2m}H_{n-m}^2) = -2(H_{2m} - H_m^2)(H_{2n} - H_n^2) = -8(-s)^{m+n}$,

i.e.,

$((r^2+4s)G_{n+m}^2 - (-s)^{2m}H_{n-m}^2) - (H_{n+m}^2 - (r^2+4s)(-s)^{2m}G_{n-m}^2) = -2(H_{2n} - H_n^2)(H_{2m} - H_m^2) = -8(-s)^{m+n}$.

(ii) $(G_{n+m}^2 - (-s)^{2m}G_{n-m}^2) - (H_{n+m}^2 - (-s)^{2m}H_{n-m}^2) = (1 - (r^2+4s))G_{2n}G_{2m}$.

(iii) $(r^2+4s)G_{n+m}^2 + (-s)^{2m}H_{n-m}^2 = H_{n+m}^2 + (r^2+4s)(-s)^{2m}G_{n-m}^2$.

(iv) $(G_{n+m}^2 - (-s)^{2m}G_{n-m}^2) - (H_{n+m}^2 + (r^2+4s)(-s)^{2m}G_{n-m}^2) = G_{2n}G_{2m} - H_{2m}H_{2n}$,

i.e.,

$H_{2m}H_{2n}(G_{n+m}^2 - (-s)^{2m}G_{n-m}^2) = G_{2n}G_{2m}(H_{n+m}^2 + (r^2+4s)(-s)^{2m}G_{n-m}^2)$.

(v) $((r^2+4s)G_{n+m}^2 - (-s)^{2m}H_{n-m}^2) + (H_{n+m}^2 - (r^2+4s)(-s)^{2m}G_{n-m}^2) = 2\left(r^2+4s\right)G_{2m}G_{2n}$.

(b)

(i) $(r^2+4s)^2(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) + (H_{n+m}^4 - (-s)^{4m}H_{n-m}^4) = 2(r^2+4s)G_{2n}G_{2m}H_{2m}H_{2n} = 2(r^2+4s)G_{4n}G_{4m}$.

(ii) $(r^2+4s)^2(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) - (H_{n+m}^4 - (-s)^{4m}H_{n-m}^4) = -8(r^2+4s)(-s)^{n+m}G_{2n}G_{2m}$.

(iii) $8(G_{n+m}^4 + (-s)^{4m}G_{n-m}^4) - 2(G_m^2H_n^2 + G_n^2H_m^2)(G_{n+m}^2 + (-s)^{2m}G_{n-m}^2) = 4G_{2m}^2G_{2n}^2$.

(iv) $2\left(r^2+4s\right)(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) + 2G_{2n}G_{2m}(H_{n+m}^2 + (-s)^{2m}H_{n-m}^2) = 4G_{2n}G_{2m}H_{2m}H_{2n} = 4G_{4n}G_{4m}$.

Proof.

(a)

- **(i)** Apply Corollary 2.8 (a) (i), Corollary 2.8 (a) (iii), together with the identity $2H_{2n} = ((r^2+4s)G_n^2 + H_n^2)$.Alternatively, use Corollary 2.8 (a)(vi) and (a)(viii).
- **(ii)** Use Corollary 2.8 (a) (ii) and Corollary 2.8 (a) (iv).
- **(iii)** Apply Corollary 2.8 (s) (v) and Corollary 2.8 (a) (vii).
- **(iv)** Use Corollary 2.8 (a) (ii) and Corollary 2.8 (a) (vii).
- **v)** Apply Corollary 2.8 (a) (vi) and Corollary 2.8 (a) (viii).

(b)

- **(i)** Multiply Corollary 2.10 (a) (ii) (2) by (r^2+4s) and then add Corollary 2.10 (a) (iv) (2) side by side.
- **(ii)** Multiply Corollary 2.10 (a) (ii) (2) by (r^2+4s) and then subtract it from Corollary 2.10 (a) (iv) (2).
- **(iii)** Multiply Corollary 2.8 (a) (i) (1) by $(G_m^2H_n^2 + G_n^2H_m^2)$ and then subtract it from Corollary 2.10 (a) (i) (6).
- **(iv)** Multiply Corollary 2.10 (a) (ii) (1) by $\left(r^2+4s\right)$ and Corollary 2.8 (a) (iii) (1) by $G_{2n}G_{2m}$ then add both side by side. □

By applying Lemma 2.7, we obtain the following corollary.

Corollary 2.11. *Let $n, m \in \mathbb{Z}$. Then*

(a)

- **(i)** $4(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) = (3G_m^2H_n^2 + G_n^2H_m^2)G_nH_m$.
- **(ii)** $4(G_{n+m}^3 - (-s)^{3m}G_{n-m}^3) = (3G_n^2H_m^2 + G_m^2H_n^2)G_mH_n$.
- **(iii)** $4(H_{n+m}^3 + (-s)^{3m}H_{n-m}^3) = (H_m^2H_n^2 + 3(r^2+4s)^2G_m^2G_n^2)H_mH_n$.
- **(iv)** $4(H_{n+m}^3 - (-s)^{3m}H_{n-m}^3) = (r^2+4s)(3H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2)G_mG_n$.

(b)

- **(i)** $8(G_{n+m}^4 + (-s)^{4m}G_{n-m}^4) = G_m^4H_n^4 + G_n^4H_m^4 + 6G_{2m}^2G_{2n}^2$.
- **(ii)** $2(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) = (G_m^2H_n^2 + G_n^2H_m^2)G_{2m}G_{2n}$.
- **(iii)** $8(H_{n+m}^4 + (-s)^{4m}H_{n-m}^4) = H_m^4H_n^4 + (r^2+4s)^4G_m^4G_n^4 + 6(r^2+4s)^2G_{2m}^2G_{2n}^2$.
- **(iv)** $2(H_{n+m}^4 - (-s)^{4m}H_{n-m}^4) = (r^2+4s)\left(H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2\right)G_{2m}G_{2n}$.

(c)

- **(i)** $16(G_{n+m}^5 + (-s)^{5m}G_{n-m}^5) = (G_n^4H_m^4 + 5G_m^4H_n^4 + 10G_m^2H_m^2G_n^2H_n^2)G_nH_m = (5G_m^4H_n^4 + G_n^4H_m^4 + 10G_{2m}^2G_{2n}^2)G_nH_m$.
- **(ii)** $16(G_{n+m}^5 - (-s)^{5m}G_{n-m}^5) = (5G_n^4H_m^4 + G_m^4H_n^4 + 10G_{2m}^2G_{2n}^2)G_mH_n$.
- **(iii)** $16(H_{n+m}^5 + (-s)^{5m}H_{n-m}^5) = (H_m^4H_n^4 + 5(r^2+4s)^4G_m^4G_n^4 + 10(r^2+4s)^2G_{2m}^2G_{2n}^2)H_mH_n$.
- **(iv)** $16(H_{n+m}^5 - (-s)^{5m}H_{n-m}^5) = (r^2+4s)(5H_m^4H_n^4 + (r^2+4s)^4G_m^4G_n^4 + 10(r^2+4s)^2G_{2m}^2G_{2n}^2)G_mG_n$.

(d)

(i) $32(G_{n+m}^6 + (-s)^{6m}G_{n-m}^6) = (G_m^2H_n^2 + G_n^2H_m^2)(G_m^4H_n^4 + G_n^4H_m^4 + 14G_m^2H_m^2G_n^2H_n^2) = (G_m^2H_n^2 + G_n^2H_m^2)(G_m^4H_n^4 + G_n^4H_m^4 + 14G_{2m}^2G_{2n}^2)$.

(ii) $16(G_{n+m}^6 - (-s)^{6m}G_{n-m}^6) = (3G_m^2H_n^2 + G_n^2H_m^2)(G_m^2H_n^2 + 3G_n^2H_m^2)G_{2m}G_{2n}$.

(iii) $32(H_{n+m}^6 + (-s)^{6m}H_{n-m}^6) = (H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2)(H_m^4H_n^4 + (r^2+4s)^4G_m^4G_n^4 + 14(r^2+4s)^2G_{2m}^2G_{2n}^2)$.

(iv) $16(H_{n+m}^6 - (-s)^{6m}H_{n-m}^6) = (r^2+4s)(H_m^2H_n^2 + 3(r^2+4s)^2G_m^2G_n^2)(3H_m^2H_n^2 + (r^2+4s)^2G_m^2G_n^2)G_{2m}G_{2n}$.

(e)

(i) $64(G_{n+m}^7 + (-s)^{7m}G_{n-m}^7) = (7G_m^6H_n^6 + G_n^6H_m^6 + 7(5G_m^2H_n^2 + 3G_n^2H_m^2)G_{2m}^2G_{2n}^2)G_nH_m$.

(ii) $64(G_{n+m}^7 - (-s)^{7m}G_{n-m}^7) = (7G_n^6H_m^6 + G_m^6H_n^6 + 7(3G_m^2H_n^2 + 5G_n^2H_m^2)G_{2m}^2G_{2n}^2)G_mH_n$.

(iii) $64(H_{n+m}^7 + (-s)^{7m}H_{n-m}^7) = (7(r^2 + 4s)^6G_m^6G_n^6 + H_m^6H_n^6 + 7(r^2 + 4s)^2(3H_m^2H_n^2 + 5(r^2 + 4s)^2G_m^2G_n^2)G_{2m}^2G_{2n}^2)H_mH_n$.

(iv) $64(H_{n+m}^7 - (-s)^{7m}H_{n-m}^7) = (r^2 + 4s)((r^2 + 4s)^6G_m^6G_n^6 + 7H_m^6H_n^6 + 7(r^2 + 4s)^2(5H_m^2H_n^2 + 3(r^2 + 4s)^2G_m^2G_n^2)G_{2m}^2G_{2n}^2)G_mG_n$.

(f)

(i) $128(G_{n+m}^8 + (-s)^{8m}G_{n-m}^8) = G_m^8H_n^8 + G_n^8H_m^8 + 14(2G_m^2H_n^2 + G_n^2H_m^2)(G_m^2H_n^2 + 2G_n^2H_m^2)G_{2m}^2G_{2n}^2$.

(ii) $16(G_{n+m}^8 - (-s)^{8m}G_{n-m}^8) = (G_m^2H_n^2 + G_n^2H_m^2)(G_m^4H_n^4 + G_n^4H_m^4 + 6G_{2m}^2G_{2n}^2)G_{2m}G_{2n}$.

(iii) $128(H_{n+m}^8 + (-s)^{8m}H_{n-m}^8) = H_m^8H_n^8 + (r^2 + 4s)^8G_m^8G_n^8 + 14(r^2 + 4s)^2(H_m^2H_n^2 + 2(r^2 + 4s)^2G_m^2G_n^2)(2H_m^2H_n^2 + (r^2 + 4s)^2G_m^2G_n^2)G_{2m}^2G_{2n}^2$.

(iv) $16(H_{n+m}^8 - (-s)^{8m}H_{n-m}^8) = (r^2 + 4s)(H_m^2H_n^2 + (r^2 + 4s)^2G_m^2G_n^2)(H_m^4H_n^4 + (r^2 + 4s)^4G_m^4G_n^4 + 6(r^2 + 4s)^2G_{2m}^2G_{2n}^2)G_{2m}G_{2n}$.

(g)

(i) $256(G_{n+m}^9 + (-s)^{9m}G_{n-m}^9) = (3G_m^2H_n^2 + G_n^2H_m^2)(3G_m^6H_n^6 + G_n^6H_m^6 + 3(9G_m^2H_n^2 + 11G_n^2H_m^2)G_{2m}^2G_{2n}^2)G_nH_m$.

(ii) $256(G_{n+m}^9 - (-s)^{9m}G_{n-m}^9) = (G_m^2H_n^2 + 3G_n^2H_m^2)(G_m^6H_n^6 + 3G_n^6H_m^6 + 3(11G_m^2H_n^2 + 9G_n^2H_m^2)G_{2m}^2G_{2n}^2)G_mH_n$.

(iii) $256(H_{n+m}^9 + (-s)^{9m}H_{n-m}^9) = (H_m^2H_n^2 + 3(r^2 + 4s)^2G_m^2G_n^2)(H_m^6H_n^6 + 3(r^2 + 4s)^6G_m^6G_n^6 + 3(r^2 + 4s)^2(11H_m^2H_n^2 + 9(r^2 + 4s)^2G_m^2G_n^2)G_{2m}^2G_{2n}^2)H_mH_n$.

(iv) $256(H_{n+m}^9 - (-s)^{9m}H_{n-m}^9) = (r^2 + 4s)(3H_m^2H_n^2 + (r^2 + 4s)^2G_m^2G_n^2)(3H_m^6H_n^6 + (r^2 + 4s)^6G_m^6G_n^6 + 3(r^2 + 4s)^2(9H_m^2H_n^2 + 11(r^2 + 4s)^2G_m^2G_n^2)G_{2m}^2G_{2n}^2)G_mG_n$.

(h)

(i) $512(G_{n+m}^{10} + (-s)^{10m}G_{n-m}^{10}) = (G_m^2H_n^2 + G_n^2H_m^2)(G_m^8H_n^8 + G_n^8H_m^8 + 2(22G_m^4H_n^4 + 22G_n^4H_m^4 + 83G_{2m}^2G_{2n}^2)G_{2m}^2G_{2n}^2)$.

(ii) $256(G_{n+m}^{10} - (-s)^{10m}G_{n-m}^{10}) = (G_m^4H_n^4 + 5G_n^4H_m^4 + 10G_{2m}^2G_{2n}^2)(5G_m^4H_n^4 + G_n^4H_m^4 + 10G_{2m}^2G_{2n}^2)G_{2m}G_{2n}$.

(iii) $512(H_{n+m}^{10} + (-s)^{10m}H_{n-m}^{10}) = (H_m^2H_n^2 + (r^2 + 4s)^2G_m^2G_n^2)(H_m^8H_n^8 + (r^2 + 4s)^8G_m^8G_n^8 + 2(r^2 + 4s)^2(22H_m^4H_n^4 + 22(r^2 + 4s)^4G_m^4G_n^4 + 83(r^2 + 4s)^2G_{2m}^2G_{2n}^2)G_{2m}^2G_{2n}^2)$.

(iv) $256(H_{n+m}^{10} - (-s)^{10m}H_{n-m}^{10}) = (r^2 + 4s)(H_m^4H_n^4 + 5(r^2 + 4s)^4G_m^4G_n^4 + 10(r^2 + 4s)^2G_{2m}^2G_{2n}^2)(5H_m^4H_n^4 + (r^2 + 4s)^4G_m^4G_n^4 + 10(r^2 + 4s)^2G_{2m}^2G_{2n}^2)G_{2m}G_{2n}$.

Proof. We employ the identities

$$\begin{aligned} 2G_{n+m} &= G_mH_n + G_nH_m && \text{Lemma 2.7 (a) (i)}, \\ 2(-s)^mG_{n-m} &= G_nH_m - G_mH_n && \text{Lemma 2.7 (a) (ii)}, \\ 2H_{n+m} &= H_mH_n + (r^2+4s)G_mG_n && \text{Lemma 2.7 (a) (iii)}, \\ 2(-s)^mH_{n-m} &= H_mH_n - (r^2+4s)G_mG_n && \text{Lemma 2.7 (a) (iv)}. \ \square \end{aligned}$$

By applying Corollary 2.11, we obtain the following corollary.

Corollary 2.12. *Let $n \in \mathbb{Z}$. Then*

(a)

- **(i)** $4(G_{n+1}^3 + (-s)^3G_{n-1}^3) = r(3H_n^2 + r^2G_n^2)G_n$.
- **(ii)** $4(G_{n+1}^3 - (-s)^3G_{n-1}^3) = (3r^2G_n^2 + H_n^2)H_n$.
- **(iii)** $4(H_{n+1}^3 + (-s)^3H_{n-1}^3) = r(r^2H_n^2 + 3(r^2+4s)^2G_n^2)H_n$.
- **(iv)** $4(H_{n+1}^3 - (-s)^3H_{n-1}^3) = (r^2+4s)(3r^2H_n^2 + (r^2+4s)^2G_n^2)G_n$.

(b)

- **(i)** $8(G_{n+1}^4 + (-s)^4G_{n-1}^4) = H_n^4 + r^4G_n^4 + 6r^2G_{2n}^2$.
- **(ii)** $2(G_{n+1}^4 - (-s)^4G_{n-1}^4) = r(H_n^2 + r^2G_n^2)G_{2n}$.
- **(iii)** $8(H_{n+1}^4 + (-s)^4H_{n-1}^4) = r^4H_n^4 + (r^2+4s)^4G_n^4 + 6r^2(r^2+4s)^2G_{2n}^2$.
- **(iv)** $2(H_{n+1}^4 - (-s)^4H_{n-1}^4) = r(r^2+4s)\left(r^2H_n^2 + (r^2+4s)^2G_n^2\right)G_{2n}$.

(c)

- **(i)** $16(G_{n+1}^5 + (-s)^5G_{n-1}^5) = r(r^4G_n^4 + 5H_n^4 + 10r^2G_n^2H_n^2)G_n = r(r^4G_n^4 + 5H_n^4 + 10r^2G_{2n}^2)G_n$.
- **(ii)** $16(G_{n+1}^5 - (-s)^5G_{n-1}^5) = (5r^4G_n^4 + H_n^4 + 10r^2G_{2n}^2)H_n$.
- **(iii)** $16(H_{n+1}^5 + (-s)^5H_{n-1}^5) = r(r^4H_n^4 + 5(r^2+4s)^4G_n^4 + 10r^2(r^2+4s)^2G_{2n}^2)H_n$.
- **(iv)** $16(H_{n+1}^5 - (-s)^5H_{n-1}^5) = (r^2+4s)(5r^4H_n^4 + (r^2+4s)^4G_n^4 + 10r^2(r^2+4s)^2G_{2n}^2)G_n$.

(d)

- **(i)** $32(G_{n+1}^6 + (-s)^6G_{n-1}^6) = (H_n^2 + r^2G_n^2)(r^4G_n^4 + H_n^4 + 14r^2G_n^2H_n^2) = (H_n^2 + r^2G_n^2)(H_n^4 + r^4G_n^4 + 14r^2G_{2n}^2)$.
- **(ii)** $16(G_{n+1}^6 - (-s)^6G_{n-1}^6) = r(3H_n^2 + r^2G_n^2)(H_n^2 + 3r^2G_n^2)G_{2n}$.
- **(iii)** $32(H_{n+1}^6 + (-s)^6H_{n-1}^6) = (r^2H_n^2 + (r^2+4s)^2G_n^2)(r^4H_n^4 + (r^2+4s)^4G_n^4 + 14r^2(r^2+4s)^2G_{2n}^2)$.
- **(iv)** $16(H_{n+1}^6 - (-s)^6H_{n-1}^6) = r(r^2+4s)(r^2H_n^2 + 3(r^2+4s)^2G_n^2)(3r^2H_n^2 + (r^2+4s)^2G_n^2)G_{2n}$.

(e)

- **(i)** $64(G_{n+1}^7 + (-s)^7G_{n-1}^7) = r(7H_n^6 + r^6G_n^6 + 7(5H_n^2 + 3r^2G_n^2)r^2G_{2n}^2)G_n$.
- **(ii)** $64(G_{n+1}^7 - (-s)^7G_{n-1}^7) = (7r^6G_n^6 + H_n^6 + 7r^2(3H_n^2 + 5r^2G_n^2)G_{2n}^2)H_n$.
- **(iii)** $64(H_{n+1}^7 + (-s)^7H_{n-1}^7) = r(7(r^2+4s)^6G_n^6 + r^6H_n^6 + 7r^2(r^2+4s)^2(3r^2H_n^2 + 5(r^2+4s)^2G_n^2)G_{2n}^2)H_n$.

(iv) $64(H_{n+1}^7-(-s)^7H_{n-1}^7)=(r^2+4s)((r^2+4s)^6G_n^6+7r^6H_n^6+7(r^2+4s)^2(5r^2H_n^2+3(r^2+4s)^2G_n^2)r^2G_{2n}^2)G_n$.

(f)

(i) $128(G_{n+1}^8+(-s)^8G_{n-1}^8)=H_n^8+r^8G_n^8+14r^2(2H_n^2+r^2G_n^2)(H_n^2+2r^2G_n^2)G_{2n}^2$.

(ii) $16(G_{n+1}^8-(-s)^8G_{n-1}^8)=r(H_n^2+r^2G_n^2)(H_n^4+r^4G_n^4+6r^2G_{2n}^2)G_{2n}$.

(iii) $128(H_{n+1}^8+(-s)^8H_{n-1}^8)=r^8H_n^8+(r^2+4s)^8G_n^8+14r^2(r^2+4s)^2(r^2H_n^2+2(r^2+4s)^2G_n^2)(2r^2H_n^2+(r^2+4s)^2G_n^2)G_{2n}^2$.

(iv) $16(H_{n+1}^8-(-s)^8H_{n-1}^8)=r(r^2+4s)(r^2H_n^2+(r^2+4s)^2G_n^2)(r^4H_n^4+(r^2+4s)^4G_n^4+6r^2(r^2+4s)^2G_{2n}^2)G_{2n}$.

(g)

(i) $256(G_{n+1}^9+(-s)^9G_{n-1}^9)=r(3H_n^2+r^2G_n^2)(3H_n^6+r^6G_n^6+3r^2(9H_n^2+11r^2G_n^2)G_{2n}^2)G_n$.

(ii) $256(G_{n+1}^9-(-s)^9G_{n-1}^9)=(H_n^2+3r^2G_n^2)(H_n^6+3r^6G_n^6+3r^2(11H_n^2+9r^2G_n^2)G_{2n}^2)H_n$.

(iii) $256(H_{n+1}^9+(-s)^9H_{n-1}^9)=r(r^2H_n^2+3(r^2+4s)^2G_n^2)(r^6H_n^6+3(r^2+4s)^6G_n^6+3r^2(r^2+4s)^2(11r^2H_n^2+9(r^2+4s)^2G_n^2)G_{2n}^2)H_n$.

(iv) $256(H_{n+1}^9-(-s)^9H_{n-1}^9)=(r^2+4s)(3r^2H_n^2+(r^2+4s)^2G_n^2)(3r^6H_n^6+(r^2+4s)^6G_n^6+3r^2(r^2+4s)^2(9r^2H_n^2+11(r^2+4s)^2G_n^2)G_{2n}^2)G_n$.

(h)

(i) $512(G_{n+1}^{10}+(-s)^{10}G_{n-1}^{10})=(H_n^2+r^2G_n^2)(H_n^8+r^8G_n^8+2r^2(22H_n^4+22r^4G_n^4+83r^2G_{2n}^2)G_{2n}^2)$.

(ii) $256(G_{n+1}^{10}-(-s)^{10}G_{n-1}^{10})=r(H_n^4+5r^4G_n^4+10r^2G_{2n}^2)(5H_n^4+r^4G_n^4+10r^2G_{2n}^2)G_{2n}$.

(iii) $512(H_{n+1}^{10}+(-s)^{10}H_{n-1}^{10})=(r^2H_n^2+(r^2+4s)^2G_n^2)(r^8H_n^8+(r^2+4s)^8G_n^8+2r^2(r^2+4s)^2(22r^4H_n^4+22(r^2+4s)^4G_n^4+83r^2(r^2+4s)^2G_{2n}^2)G_{2n}^2)$.

(iv) $256(H_{n+1}^{10}-(-s)^{10}H_{n-1}^{10})=r(r^2+4s)(r^4H_n^4+5(r^2+4s)^4G_n^4+10r^2(r^2+4s)^2G_{2n}^2)(5r^4H_n^4+(r^2+4s)^4G_n^4+10r^2(r^2+4s)^2G_{2n}^2)G_{2n}$.

Proof. Set $m=1$ in Corollar 2.11. □

2.2 LEMMA 2.13 and Its Consequences

By applying Lemma 2.7, we obtain the following lemma.

Lemma 2.13. *Let $n,m\in\mathbb{Z}$. Then*

(a)

(i) *1.* $8G_{n+m}^3=G_m^3H_n^3+G_n^3H_m^3+3\left(G_mH_n+G_nH_m\right)G_{2m}G_{2n}$.

2. $8G_{n+m}^3=(2G_{2m}G_{2n}+G_n^2H_m^2+G_m^2H_n^2)(G_mH_n+G_nH_m)$.

3. $8G_{n+m}^3=\left(G_mH_n+G_nH_m\right)^3$.

4. $8(r^2+4s)G_{n+m}^3=(4H_{2m}H_{2n}-(H_mH_n-(r^2+4s)G_mG_n)^2)(G_mH_n+G_nH_m)$.

5. $8(r^2+4s)G_{n+m}^3=(4H_{2m}H_{2n}-(r^2+4s)^2G_m^2G_n^2+2(r^2+4s)G_{2m}G_{2n}-H_m^2H_n^2)(G_mH_n+G_nH_m)$.

(ii) *1.* $8(-s)^{3m}G_{n-m}^3=G_n^3H_m^3-G_m^3H_n^3+3\left(G_mH_n-G_nH_m\right)G_{2m}G_{2n}$.

2. $8\left(-s\right)^{3m}G_{n-m}^{3}=(-2G_{2m}G_{2n}+G_{m}^{2}H_{n}^{2}+G_{n}^{2}H_{m}^{2})(G_{n}H_{m}-G_{m}H_{n})$.
3. $8\left(-s\right)^{3m}G_{n-m}^{3}=(G_{n}H_{m}-G_{m}H_{n})^{3}$.
4. $8(r^{2}+4s)(-s)^{3m}G_{n-m}^{3}=(4H_{2m}H_{2n}-(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})^{2})(G_{n}H_{m}-G_{m}H_{n})$.
5. $8(r^{2}+4s)(-s)^{3m}G_{n-m}^{3}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}-2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(G_{n}H_{m}-G_{m}H_{n})$.

(iii) 1. $8H_{n+m}^{3}=H_{m}^{3}H_{n}^{3}+(r^{2}+4s)^{3}G_{m}^{3}G_{n}^{3}+3(r^{2}+4s)(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})G_{2m}G_{2n}$.
2. $8H_{n+m}^{3}=((r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}+2(r^{2}+4s)G_{2n}G_{2m}+H_{m}^{2}H_{n}^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
3. $8H_{n+m}^{3}=(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})^{3}$.
4. $8H_{n+m}^{3}=(4H_{2m}H_{2n}-(r^{2}+4s)(G_{n}H_{m}-G_{m}H_{n})^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
5. $8H_{n+m}^{3}=(4H_{2m}H_{2n}-(r^{2}+4s)(G_{m}^{2}H_{n}^{2}+G_{n}^{2}H_{m}^{2}-2G_{2m}G_{2n}))(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.

(iv) 1. $8(-s)^{3m}H_{n-m}^{3}=H_{m}^{3}H_{n}^{3}-(r^{2}+4s)^{3}G_{m}^{3}G_{n}^{3}-3(r^{2}+4s)(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})G_{2m}G_{2n}$.
2. $8(-s)^{3m}H_{n-m}^{3}=((r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}-2(r^{2}+4s)G_{2n}G_{2m}+H_{m}^{2}H_{n}^{2})(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.
3. $8(-s)^{3m}H_{n-m}^{3}=(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})^{3}$.
4. $8(-s)^{3m}H_{n-m}^{3}=(4H_{2m}H_{2n}-(r^{2}+4s)(G_{m}H_{n}+G_{n}H_{m})^{2})(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.
5. $8(-s)^{3m}H_{n-m}^{3}=(4H_{2m}H_{2n}-(r^{2}+4s)(G_{m}^{2}H_{n}^{2}+G_{n}^{2}H_{m}^{2}+2G_{2m}G_{2n}))(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.

(v) 1. $8(-s)^{m}G_{n+m}^{2}G_{n-m}=G_{n}^{3}H_{m}^{3}-G_{m}^{3}H_{n}^{3}+(G_{n}H_{m}-G_{m}H_{n})G_{2m}G_{2n}$.
2. $8(-s)^{m}G_{n+m}^{2}G_{n-m}=(2G_{2m}G_{2n}+G_{n}^{2}H_{m}^{2}+G_{m}^{2}H_{n}^{2})(G_{n}H_{m}-G_{m}H_{n})$.
3. $8(r^{2}+4s)(-s)^{m}G_{n+m}^{2}G_{n-m}=(4H_{2m}H_{2n}-(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})^{2})(G_{n}H_{m}-G_{m}H_{n})$.
4. $8(r^{2}+4s)(-s)^{m}G_{n+m}^{2}G_{n-m}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}+2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(G_{n}H_{m}-G_{m}H_{n})$.

(vi) 1. $8(-s)^{2m}G_{n-m}^{2}G_{n+m}=G_{m}^{3}H_{n}^{3}+G_{n}^{3}H_{m}^{3}-(G_{n}H_{m}+G_{m}H_{n})G_{2m}G_{2n}$.
2. $8(-s)^{2m}G_{n-m}^{2}G_{n+m}=(-2G_{2m}G_{2n}+G_{m}^{2}H_{n}^{2}+G_{n}^{2}H_{m}^{2})(G_{m}H_{n}+G_{n}H_{m})$.
3. $8(r^{2}+4s)(-s)^{2m}G_{n-m}^{2}G_{n+m}=(4H_{2m}H_{2n}-(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})^{2})(G_{m}H_{n}+G_{n}H_{m})$.
4. $8(r^{2}+4s)(-s)^{2m}G_{n-m}^{2}G_{n+m}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}-2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(G_{m}H_{n}+G_{n}H_{m})$.

(vii) 1. $8G_{n+m}^{2}H_{n+m}=\left(G_{m}H_{n}+G_{n}H_{m}\right)^{2}\left(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n}\right)$.
2. $8G_{n+m}^{2}H_{n+m}=(2G_{2m}G_{2n}+G_{n}^{2}H_{m}^{2}+G_{m}^{2}H_{n}^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
3. $8(r^{2}+4s)G_{n+m}^{2}H_{n+m}=(4H_{2m}H_{2n}-(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
4. $8(r^{2}+4s)G_{n+m}^{2}H_{n+m}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}+2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.

(viii) 1. $8(-s)^{m}G_{n+m}^{2}H_{n-m}=(2G_{2m}G_{2n}+G_{n}^{2}H_{m}^{2}+G_{m}^{2}H_{n}^{2})(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.
2. $8(-s)^{m}G_{n+m}^{2}H_{n-m}=\left(G_{m}H_{n}+G_{n}H_{m}\right)^{2}\left(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n}\right)$.
3. $8(r^{2}+4s)(-s)^{m}G_{n+m}^{2}H_{n-m}=(4H_{2m}H_{2n}-(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})^{2})(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.
4. $8(r^{2}+4s)(-s)^{m}G_{n+m}^{2}H_{n-m}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}+2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(H_{m}H_{n}-(r^{2}+4s)G_{m}G_{n})$.

(ix) 1. $8(-s)^{2m}G_{n-m}^{2}H_{n+m}=(-2G_{2m}G_{2n}+G_{m}^{2}H_{n}^{2}+G_{n}^{2}H_{m}^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
2. $8(-s)^{2m}G_{n-m}^{2}H_{n+m}=(G_{n}H_{m}-G_{m}H_{n})^{2}(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
3. $8(r^{2}+4s)(-s)^{2m}G_{n-m}^{2}H_{n+m}=(4H_{2m}H_{2n}-(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.
4. $8(r^{2}+4s)(-s)^{2m}G_{n-m}^{2}H_{n+m}=(4H_{2m}H_{2n}-(r^{2}+4s)^{2}G_{m}^{2}G_{n}^{2}-2(r^{2}+4s)G_{2m}G_{2n}-H_{m}^{2}H_{n}^{2})(H_{m}H_{n}+(r^{2}+4s)G_{m}G_{n})$.

(x)

1. $8\left(-s\right)^{3m} G_{n-m}^2 H_{n-m} = (-2G_{2m}G_{2n} + G_m^2 H_n^2 + G_n^2 H_m^2)(H_m H_n - (r^2+4s)G_m G_n)$.
2. $8\left(-s\right)^{3m} G_{n-m}^2 H_{n-m} = (G_n H_m - G_m H_n)^2 (H_m H_n - (r^2+4s)G_m G_n)$.
3. $8(r^2+4s)\left(-s\right)^{3m} G_{n-m}^2 H_{n-m} = (4H_{2m}H_{2n} - (H_m H_n + (r^2+4s)G_m G_n)^2)(H_m H_n - (r^2+4s)G_m G_n)$.
4. $8(r^2+4s)(-s)^{3m} G_{n-m}^2 H_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2m}G_{2n} - H_m^2 H_n^2)(H_m H_n - (r^2+4s)G_m G_n)$.

(xi)

1. $8(-s)^m H_{n+m}^2 H_{n-m} = ((r^2+4s)^2 G_m^2 G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(H_m H_n - (r^2+4s)G_m G_n)$.
2. $8(-s)^m H_{n+m}^2 H_{n-m} = (H_m H_n + (r^2+4s)G_m G_n)^2 (H_m H_n - (r^2+4s)G_m G_n)$.
3. $8(-s)^m H_{n+m}^2 H_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_n H_m - G_m H_n)^2)(H_m H_n - (r^2+4s)G_m G_n)$.
4. $8(-s)^m H_{n+m}^2 H_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 - 2G_{2m}G_{2n}))(H_m H_n - (r^2+4s)G_m G_n)$.

(xii)

1. $8(-s)^{2m} H_{n-m}^2 H_{n+m} = ((r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(H_m H_n + (r^2+4s)G_m G_n)$.
2. $8(-s)^{2m} H_{n-m}^2 H_{n+m} = (H_m H_n - (r^2+4s)G_m G_n)^2 (H_m H_n + (r^2+4s)G_m G_n)$.
3. $8(-s)^{2m} H_{n-m}^2 H_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m H_n + G_n H_m)^2)(H_m H_n + (r^2+4s)G_m G_n)$.
4. $8(-s)^{2m} H_{n-m}^2 H_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 + 2G_{2m}G_{2n}))(H_m H_n + (r^2+4s)G_m G_n)$.

(xiii)

1. $8H_{n+m}^2 G_{n+m} = ((r^2+4s)^2 G_m^2 G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(G_m H_n + G_n H_m)$.
2. $8H_{n+m}^2 G_{n+m} = (H_m H_n + (r^2+4s)G_m G_n)^2 (G_m H_n + G_n H_m)$.
3. $8H_{n+m}^2 G_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_n H_m - G_m H_n)^2)(G_m H_n + G_n H_m)$.
4. $8H_{n+m}^2 G_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 - 2G_{2m}G_{2n}))(G_m H_n + G_n H_m)$.

(xiv)

1. $8(-s)^m H_{n+m}^2 G_{n-m} = ((r^2+4s)^2 G_m^2 G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(G_n H_m - G_m H_n)$.
2. $8(-s)^m H_{n+m}^2 G_{n-m} = (H_m H_n + (r^2+4s)G_m G_n)^2 (G_n H_m - G_m H_n)$.
3. $8(-s)^m H_{n+m}^2 G_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_n H_m - G_m H_n)^2)(G_n H_m - G_m H_n)$.
4. $8(-s)^m H_{n+m}^2 G_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 - 2G_{2m}G_{2n}))(G_n H_m - G_m H_n)$.

(xv)

1. $8(-s)^{2m} H_{n-m}^2 G_{n+m} = ((r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(G_m H_n + G_n H_m)$.
2. $8(-s)^{2m} H_{n-m}^2 G_{n+m} = (H_m H_n - (r^2+4s)G_m G_n)^2 (G_m H_n + G_n H_m)$.
3. $8(-s)^{2m} H_{n-m}^2 G_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m H_n + G_n H_m)^2)(G_m H_n + G_n H_m)$.
4. $8(-s)^{2m} H_{n-m}^2 G_{n+m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 + 2G_{2m}G_{2n}))(G_m H_n + G_n H_m)$.

(xvi)

1. $8(-s)^{3m} H_{n-m}^2 G_{n-m} = ((r^2+4s)^2 G_m^2 G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2 H_n^2)(G_n H_m - G_m H_n)$.
2. $8(-s)^{3m} H_{n-m}^2 G_{n-m} = (H_m H_n - (r^2+4s)G_m G_n)^2 (G_n H_m - G_m H_n)$.
3. $8(-s)^{3m} H_{n-m}^2 G_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m H_n + G_n H_m)^2)(G_n H_m - G_m H_n)$.
4. $8(-s)^{3m} H_{n-m}^2 G_{n-m} = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2 + 2G_{2m}G_{2n}))(G_n H_m - G_m H_n)$.

(b)

(i) 1. $8G_{n+1}^3 = H_n^3 + r^3 G_n^3 + 3r\left(H_n + rG_n\right)G_{2n}$.

2. $8G_{n+1}^3 = (2rG_{2n} + r^2G_n^2 + H_n^2)(H_n + rG_n)$.
3. $8G_{n+1}^3 = (H_n + rG_n)^3$.
4. $8(r^2 + 4s)G_{n+1}^3 = (4(r^2 + 2s)H_{2n} - (rH_n - (r^2 + 4s)G_n)^2)(H_n + rG_n)$.
5. $8(r^2 + 4s)G_{n+1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)^2G_n^2 + 2r(r^2 + 4s)G_{2n} - r^2H_n^2)(H_n + rG_n)$.

(ii) 1. $8(-s)^3G_{n-1}^3 = r^3G_n^3 - H_n^3 + 3r(H_n - rG_n)G_{2n}$.
2. $8(-s)^3G_{n-1}^3 = (-2rG_{2n} + H_n^2 + r^2G_n^2)(rG_n - H_n)$.
3. $8(-s)^3G_{n-1}^3 = (rG_n - H_n)^3$.
4. $8(r^2 + 4s)(-s)^3G_{n-1}^3 = (4(r^2 + 2s)H_{2n} - (rH_n + (r^2 + 4s)G_n)^2)(rG_n - H_n)$.
5. $8(r^2 + 4s)(-s)^3G_{n-1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)^2G_n^2 - 2r(r^2 + 4s)G_{2n} - r^2H_n^2)(rG_n - H_n)$.

(iii) 1. $8H_{n+1}^3 = r^3H_n^3 + (r^2 + 4s)^3G_n^3 + 3r(r^2 + 4s)(rH_n + (r^2 + 4s)G_n)G_{2n}$.
2. $8H_{n+1}^3 = ((r^2 + 4s)^2G_n^2 + 2r(r^2 + 4s)G_{2n} + r^2H_n^2)(rH_n + (r^2 + 4s)G_n)$.
3. $8H_{n+1}^3 = (rH_n + (r^2 + 4s)G_n)^3$.
4. $8H_{n+1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)(rG_n - H_n)^2)(rH_n + (r^2 + 4s)G_n)$.
5. $8H_{n+1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)(H_n^2 + r^2G_n^2 - 2rG_{2n}))(rH_n + (r^2 + 4s)G_n)$.

(iv) 1. $8(-s)^3H_{n-1}^3 = r^3H_n^3 - (r^2 + 4s)^3G_n^3 - 3r(r^2 + 4s)(rH_n - (r^2 + 4s)G_n)G_{2n}$.
2. $8(-s)^3H_{n-1}^3 = ((r^2 + 4s)^2G_n^2 - 2r(r^2 + 4s)G_{2n} + r^2H_n^2)(rH_n - (r^2 + 4s)G_n)$.
3. $8(-s)^3H_{n-1}^3 = (rH_n - (r^2 + 4s)G_n)^3$.
4. $8(-s)^3H_{n-1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)(H_n + rG_n)^2)(rH_n - (r^2 + 4s)G_n)$.
5. $8(-s)^3H_{n-1}^3 = (4(r^2 + 2s)H_{2n} - (r^2 + 4s)(H_n^2 + r^2G_n^2 + 2rG_{2n}))(rH_n - (r^2 + 4s)G_n)$.

Proof.

(a)

(i) Apply Lemma 2.7 (a) (i) together with Lemma 2.7 (b) (i).

(ii) Apply Lemma 2.7 (a) (ii) together with Lemma 2.7 (b) (ii).

(iii) Apply Lemma 2.7 (a) (iii) together with Lemma 2.7 (b) (iii).

(iv) Apply Lemma 2.7 (a) (iv) together with Lemma 2.7 (b) (iv).

(v) Combine Lemma 2.7 (b) (i) with Lemma 2.7 (a) (ii).

(vi) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (a) (i).

(vii) Combine Lemma 2.7 (b) (i) with Lemma 2.7 (a) (iii).

(viii) Combine Lemma 2.7 (b) (i) with Lemma 2.7 (a) (iv).

(ix) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (a) (iii).

(x) Combine Lemma 2.7 (b) (ii) with Lemma 2.7 (a) (iv).

(xi) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (a) (iv).

(xii) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (a) (iii).

(xiii) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (a) (i).

(xiv) Combine Lemma 2.7 (b) (iii) with Lemma 2.7 (a) (ii).

(xv) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (a) (i).

(xvi) Combine Lemma 2.7 (b) (iv) with Lemma 2.7 (a) (ii).

(b) Set $m = 1$ in (a)(i)-(iv). □

By applying Lemma 2.13, we obtain the following Corollary.

Corollary 2.14. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $4(G_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = G_n^3 H_m^3 + 3G_m H_n G_{2m} G_{2n}$.
2. $4(G_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = (3G_m^2 H_n^2 + G_n^2 H_m^2) G_n H_m$.
3. $4(r^2+4s)(G_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = (-(r^2+4s)^2 G_m^2 G_n^2 - H_m^2 H_n^2 + 2(r^2+4s) G_m^2 H_n^2 + 4H_{2m} H_{2n}) G_n H_m$.
4. $4(r^2 + 4s)(G_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = -G_n H_m^3 H_n^2 - (r^2 + 4s)^2 G_m^2 G_n^3 H_m + 4G_n H_m H_{2m} H_{2n} + 2(r^2 + 4s) G_m H_n G_{2m} G_{2n}$.

(ii)

1. $4(G_{n+m}^3 - (-s)^{3m} G_{n-m}^3) = G_m^3 H_n^3 + 3G_n H_m G_{2m} G_{2n}$.
2. $4(G_{n+m}^3 - (-s)^{3m} G_{n-m}^3) = (G_m^2 H_n^2 + 3G_n^2 H_m^2) G_m H_n$.
3. $4(r^2+4s)(G_{n+m}^3 - (-s)^{3m} G_{n-m}^3) = (-H_m^2 H_n^2 - (r^2+4s)^2 G_m^2 G_n^2 + 2(r^2+4s) G_n^2 H_m^2 + 4H_{2m} H_{2n}) G_m H_n$.
4. $4(r^2 + 4s)(G_{n+m}^3 - (-s)^{3m} G_{n-m}^3) = -(H_m^2 H_n^2 + (r^2 + 4s)^2 G_m^2 G_n^2 - 4H_{2m} H_{2n}) G_m H_n + 2(r^2 + 4s) G_n H_m G_{2m} G_{2n}$.

(iii)

1. $4(H_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = H_m^3 H_n^3 + 3(r^2 + 4s)^2 G_m G_n G_{2m} G_{2n}$.
2. $4(H_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = (H_m^2 H_n^2 + 3(r^2 + 4s)^2 G_m^2 G_n^2) H_m H_n$.
3. $4(H_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = (4H_{2m} H_{2n} - (r^2 + 4s)(G_m^2 H_n^2 + G_n^2 H_m^2) + 2(r^2 + 4s)^2 G_m^2 G_n^2) H_m H_n$.
4. $4(H_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = (4H_{2m} H_{2n} - (r^2 + 4s)(G_m^2 H_n^2 + G_n^2 H_m^2)) H_m H_n + 2(r^2 + 4s)^2 G_m G_n G_{2m} G_{2n}$.

(iv)

1. $4(H_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = (r^2 + 4s)((r^2 + 4s)^2 G_m^3 G_n^3 + 3H_m H_n G_{2m} G_{2n})$.
2. $4(H_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = (r^2 + 4s)(3H_m^2 H_n^2 + (r^2 + 4s)^2 G_m^2 G_n^2) G_m G_n$.
3. $4(H_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = (r^2 + 4s)(2H_m^2 H_n^2 - (r^2 + 4s)(G_m^2 H_n^2 + G_n^2 H_m^2) + 4H_{2m} H_{2n}) G_m G_n$.
4. $4(H_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = (r^2 + 4s)(4G_m G_n H_{2m} H_{2n} + 2H_m H_n G_{2m} G_{2n} - (r^2 + 4s)(G_m^2 H_n^2 + G_n^2 H_m^2) G_m G_n)$.

(v) $8(G_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = G_m^3 H_n^3 + G_n^3 H_m^3 + H_m^3 H_n^3 - (r^2 + 4s)^3 G_m^3 G_n^3 + 3(G_m H_n + G_n H_m - (r^2 + 4s)(H_m H_n - (r^2 + 4s) G_m G_n)) G_{2m} G_{2n}$.

(vi) $8(G_{n+m}^3 - (-s)^{3m} H_{n-m}^3) = G_m^3 H_n^3 + G_n^3 H_m^3 - H_m^3 H_n^3 + (r^2 + 4s)^3 G_m^3 G_n^3 + 3(G_m H_n + G_n H_m + (r^2 + 4s)(H_m H_n - (r^2 + 4s) G_m G_n)) G_{2m} G_{2n}$.

(vii) $8(H_{n+m}^3 + (-s)^{3m} G_{n-m}^3) = H_m^3 H_n^3 + (r^2 + 4s)^3 G_m^3 G_n^3 + 3(r^2 + 4s)(H_m H_n + (r^2 + 4s) G_m G_n) G_{2m} G_{2n} + G_n^3 H_m^3 - G_m^3 H_n^3 + 3\left(G_m H_n - G_n H_m\right) G_{2m} G_{2n}$.

(viii) $8(H_{n+m}^3-(-s)^{3m}G_{n-m}^3)=-G_n^3H_m^3+G_m^3H_n^3+H_m^3H_n^3+(r^2+4s)^3G_m^3G_n^3+3(r^2+4s)(H_mH_n+(r^2+4s)G_mG_n)G_{2m}G_{2n}-3(G_mH_n-G_nH_m)G_{2m}G_{2n}$.

(ix) $8(G_{n+m}^3+H_{n+m}^3)=G_m^3H_n^3+G_n^3H_m^3+H_m^3H_n^3+(r^2+4s)^3G_m^3G_n^3+(3\left(G_mH_n+G_nH_m\right)+3(r^2+4s)(H_mH_n+(r^2+4s)G_mG_n))G_{2m}G_{2n}$.

(x) $8(G_{n+m}^3-H_{n+m}^3)=G_m^3H_n^3+G_n^3H_m^3-H_m^3H_n^3-(r^2+4s)^3G_m^3G_n^3+(3\left(G_mH_n+G_nH_m\right)-3(r^2+4s)(H_mH_n+(r^2+4s)G_mG_n))G_{2m}G_{2n}$.

(xi) $8(-s)^{3m}(G_{n-m}^3+H_{n-m}^3)=G_n^3H_m^3-G_m^3H_n^3+H_m^3H_n^3-(r^2+4s)^3G_m^3G_n^3+3(G_mH_n-G_nH_m-(r^2+4s)(H_mH_n-(r^2+4s)G_mG_n))G_{2m}G_{2n}$.

(xii) $8(-s)^{3m}(G_{n-m}^3-H_{n-m}^3)=G_n^3H_m^3-G_m^3H_n^3-H_m^3H_n^3+(r^2+4s)^3G_m^3G_n^3+3\left(G_mH_n-G_nH_m\right)G_{2m}G_{2n}+3(r^2+4s)(H_mH_n-(r^2+4s)G_mG_n)G_{2m}G_{2n}$.

(b)

(i)

1. $4(G_{n+1}^3+(-s)^3G_{n-1}^3)=r^3G_n^3+3rH_nG_{2n}$.
2. $4(G_{n+1}^3+(-s)^3G_{n-1}^3)=r(3H_n^2+G_n^2r^2)G_n$.
3. $4(r^2+4s)(G_{n+1}^3+(-s)^3G_{n-1}^3)=r(-(r^2+4s)^2G_n^2-r^2H_n^2+2(r^2+4s)H_n^2+4(r^2+2s)H_{2n})G_n$.
4. $4(r^2+4s)(G_{n+1}^3+(-s)^3G_{n-1}^3)=-r^3G_nH_n^2-r(r^2+4s)^2G_n^3+4r(r^2+2s)G_nH_{2n}+2r(r^2+4s)H_nG_{2n}$.

(ii)

1. $4(G_{n+1}^3-(-s)^3G_{n-1}^3)=H_n^3+3r^2G_nG_{2n}$.
2. $4(G_{n+1}^3-(-s)^3G_{n-1}^3)=(H_n^2+3r^2G_n^2)H_n$.
3. $4(r^2+4s)(G_{n+1}^3-(-s)^3G_{n-1}^3)=(-r^2H_n^2-(r^2+4s)^2G_n^2+2r^2(r^2+4s)G_n^2+4(r^2+2s)H_{2n})H_n$.
4. $4(r^2+4s)(G_{n+1}^3-(-s)^3G_{n-1}^3)=-(r^2H_n^2+(r^2+4s)^2G_n^2-4(r^2+2s)H_{2n})H_n+2r^2(r^2+4s)G_nG_{2n}$.

(iii)

1. $4(H_{n+1}^3+(-s)^3H_{n-1}^3)=r^3H_n^3+3r(r^2+4s)^2G_nG_{2n}$.
2. $4(H_{n+1}^3+(-s)^3H_{n-1}^3)=r(r^2H_n^2+3(r^2+4s)^2G_n^2)H_n$.
3. $4(H_{n+1}^3+(-s)^3H_{n-1}^3)=(4(r^2+2s)H_{2n}-(r^2+4s)(H_n^2+r^2G_n^2)+2(r^2+4s)^2G_n^2)rH_n$.
4. $4(H_{n+1}^3+(-s)^3H_{n-1}^3)=(4(r^2+2s)H_{2n}-(r^2+4s)(H_n^2+G_n^2r^2))rH_n+2r(r^2+4s)^2G_nG_{2n}$.

(iv)

1. $4(H_{n+1}^3-(-s)^3H_{n-1}^3)=(r^2+4s)((r^2+4s)^2G_n^3+3r^2H_nG_{2n})$.
2. $4(H_{n+1}^3-(-s)^3H_{n-1}^3)=(r^2+4s)(3r^2H_n^2+(r^2+4s)^2G_n^2)G_n$.
3. $4(H_{n+1}^3-(-s)^3H_{n-1}^3)=(r^2+4s)(2r^2H_n^2-(r^2+4s)(H_n^2+G_n^2r^2)+4(r^2+2s)H_{2n})G_n$.
4. $4(H_{n+1}^3-(-s)^3H_{n-1}^3)=(r^2+4s)(4(r^2+2s)G_nH_{2n}+2r^2H_nG_{2n}-(r^2+4s)(H_n^2+r^2G_n^2)G_n)$.

(v) $8(G_{n+1}^3+(-s)^3H_{n-1}^3)=H_n^3+r^3G_n^3+r^3H_n^3-(r^2+4s)^3G_n^3+3(H_n+rG_n-(r^2+4s)(rH_n-(r^2+4s)G_n))rG_{2n}$.

(vi) $8(G_{n+1}^3-(-s)^3H_{n-1}^3)=H_n^3+r^3G_n^3-r^3H_n^3+(r^2+4s)^3G_n^3+3(H_n+rG_n+(r^2+4s)(rH_n-(r^2+4s)G_n))rG_{2n}$.

(vii) $8(H_{n+1}^3+(-s)^3G_{n-1}^3)=r^3H_n^3+(r^2+4s)^3G_n^3+3(r^2+4s)(rH_n+(r^2+4s)G_n)rG_{2n}+r^3G_n^3-H_n^3+3\left(H_n-G_nr\right)rG_{2n}$.

(viii) $8(H_{n+1}^3-(-s)^3G_{n-1}^3)=-r^3G_n^3+H_n^3+r^3H_n^3+(r^2+4s)^3G_n^3+3(r^2+4s)(rH_n+(r^2+4s)G_n)rG_{2n}-3(H_n-G_nr)rG_{2n}$.

(ix) $8(G_{n+1}^3+H_{n+1}^3) = H_n^3+r^3G_n^3+r^3H_n^3+(r^2+4s)^3G_n^3+(3\,(H_n+rG_n)+3(r^2+4s)(rH_n+(r^2+4s)G_n))rG_{2n}$.

(x) $8(G_{n+1}^3-H_{n+1}^3) = H_n^3+G_n^3r^3-r^3H_n^3-(r^2+4s)^3G_n^3+(3\,(H_n+rG_n)-3(r^2+4s)(rH_n+(r^2+4s)G_n))rG_{2n}$.

(xi) $8(-s)^3(G_{n-1}^3+H_{n-1}^3) = G_n^3r^3-H_n^3+r^3H_n^3-(r^2+4s)^3G_n^3+3(H_n-rG_n-(r^2+4s)(rH_n-(r^2+4s)G_n))rG_{2n}$.

(xii) $8(-s)^3(G_{n-1}^3-H_{n-1}^3) = r^3G_n^3-H_n^3-r^3H_n^3+(r^2+4s)^3G_n^3+3\,(H_n-rG_n)\,rG_{2n}+3(r^2+4s)(rH_n-(r^2+4s)G_n)rG_{2n}$.

Proof. The proof proceeds by systematically pairing and combining the corresponding parts of Lemma 2.13, either through direct application, addition, or subtraction, as outlined below.

(a)

(i) Apply Lemma 2.13 (a) (i) together with Lemma 2.13 (a) (ii).

(ii) Apply Lemma 2.13 (a) (i) together with Lemma 2.13 (a) (ii).

(iii) Apply Lemma 2.13 (a) (iii) together with Lemma 2.13 (a) (iv).

(iv) Apply Lemma 2.13 (a) (iii) together with Lemma 2.13 (a) (iv).

(v) Add Lemma 2.13 (i) and Lemma 2.13 (iv) side by side.

(vi) Subtract Lemma 2.13 (i) from Lemma 2.13 (iv).

(vii) Add Lemma 2.13 (ii) and Lemma 2.13 (iii) side by side.

(viii) Subtract Lemma 2.13 (ii) from Lemma 2.13 (iii).

(ix) Add Lemma 2.13 (i) and Lemma 2.13 (iii) side by side.

(x) Subtract Lemma 2.13 (i) from Lemma 2.13 (iii).

(xi) Add Lemma 2.13 (ii) and Lemma 2.13 (iv) side by side.

(xii) Subtract Lemma 2.13 (ii) from Lemma 2.13 (iv).

(b) Set $m = 1$ in (a). □

2.3 LEMMA 2.15 and Its Consequences

By applying Lemma 2.7 together with Lemma 2.13, we obtain the following lemma.

Lemma 2.15. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $16G_{n+m}^4 = G_m^4H_n^4 + G_n^4H_m^4 + 2(2G_m^2H_n^2 + 2G_n^2H_m^2 + 3G_{2m}G_{2n})G_{2n}G_{2m}$.
2. $16G_{n+m}^4 = (G_m^3H_n^3 + G_n^3H_m^3 + 3\,(G_mH_n + G_nH_m)\,G_{2m}G_{2n})(G_mH_n + G_nH_m)$.
3. $16G_{n+m}^4 = (G_mH_n + G_nH_m)^4$.
4. $16(r^2+4s)G_{n+m}^4 = (4H_{2m}H_{2n} - (H_mH_n - (r^2+4s)G_mG_n)^2)(G_m^2H_n^2 + G_n^2H_m^2 + 2G_{2m}G_{2n})$.
5. $16(r^2+4s)G_{n+m}^4 = (4H_{2m}H_{2n} - (r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2m}G_{2n} - H_m^2H_n^2)(G_m^2H_n^2 + G_n^2H_m^2 + 2G_{2m}G_{2n})$.

(ii)

1. $16(-s)^{4m}G_{n-m}^4 = G_m^4H_n^4 + G_n^4H_m^4 - 2(2G_m^2H_n^2 + 2G_n^2H_m^2 - 3G_{2m}G_{2n})G_{2n}G_{2m}$.
2. $16(-s)^{4m}G_{n-m}^4 = (G_n^3H_m^3 - G_m^3H_n^3 + 3(G_mH_n - G_nH_m)G_{2m}G_{2n})(G_nH_m - G_mH_n)$.
3. $16(-s)^{4m}G_{n-m}^4 = (G_nH_m - G_mH_n)^4$.
4. $16(r^2+4s)(-s)^{4m}G_{n-m}^4 = (4H_{2m}H_{2n} - (H_mH_n + (r^2+4s)G_mG_n)^2)(-2G_{2m}G_{2n} + G_m^2H_n^2 + G_n^2H_m^2)$.
5. $16(r^2+4s)(-s)^{4m}G_{n-m}^4 = (4H_{2m}H_{2n} - (r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2m}G_{2n} - H_m^2H_n^2)(-2G_{2m}G_{2n} + G_m^2H_n^2 + G_n^2H_m^2)$.

(iii)

1. $16H_{n+m}^4 = (r^2+4s)^4G_m^4G_n^4 + H_m^4H_n^4 + 2(r^2+4s)(2H_m^2H_n^2 + 3(r^2+4s)G_{2m}G_{2n} + 2(r^2+4s)^2G_m^2G_n^2)G_{2n}G_{2m}$.
2. $16H_{n+m}^4 = (H_m^3H_n^3 + (r^2+4s)^3G_m^3G_n^3 + 3(r^2+4s)(H_mH_n + (r^2+4s)G_mG_n)G_{2m}G_{2n})(H_mH_n + (r^2+4s)G_mG_n)$.
3. $16H_{n+m}^4 = (H_mH_n + (r^2+4s)G_mG_n)^4$.
4. $16H_{n+m}^4 = (4H_{2m}H_{2n} - (r^2+4s)(G_nH_m - G_mH_n)^2)((r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + H_m^2H_n^2)$.
5. $16H_{n+m}^4 = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2H_n^2 + G_n^2H_m^2 - 2G_{2m}G_{2n}))((r^2+4s)^2G_m^2G_n^2 + 2(r^2+4s)G_{2n}G_{2m} + H_m^2H_n^2)$.

(iv)

1. $16(-s)^{4m}H_{n-m}^4 = (r^2+4s)^4G_m^4G_n^4 + H_m^4H_n^4 - 2(r^2+4s)(2H_m^2H_n^2 - 3(r^2+4s)G_{2m}G_{2n} + 2(r^2+4s)^2G_m^2G_n^2)G_{2n}G_{2m}$.
2. $16(-s)^{4m}H_{n-m}^4 = (H_m^3H_n^3 - (r^2+4s)^3G_m^3G_n^3 - 3(r^2+4s)(H_mH_n - (r^2+4s)G_mG_n)G_{2m}G_{2n})(H_mH_n - (r^2+4s)G_mG_n)$.
3. $16(-s)^{4m}H_{n-m}^4 = (H_mH_n - (r^2+4s)G_mG_n)^4$.
4. $16(-s)^{4m}H_{n-m}^4 = (4H_{2m}H_{2n} - (r^2+4s)(G_mH_n + G_nH_m)^2)((r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2H_n^2)$.
5. $16(-s)^{4m}H_{n-m}^4 = (4H_{2m}H_{2n} - (r^2+4s)(G_m^2H_n^2 + G_n^2H_m^2 + 2G_{2m}G_{2n}))((r^2+4s)^2G_m^2G_n^2 - 2(r^2+4s)G_{2n}G_{2m} + H_m^2H_n^2)$.

(b)

(i)

1. $16G_{n+1}^4 = H_n^4 + r^4G_n^4 + 2r(2H_n^2 + 2G_n^2r^2 + 3rG_{2n})G_{2n}$.
2. $16G_{n+1}^4 = (H_n^3 + r^3G_n^3 + 3\,(H_n + G_nr)\,rG_{2n})(H_n + rG_n)$.
3. $16G_{n+1}^4 = (H_n + rG_n)^4$.
4. $16(r^2+4s)G_{n+1}^4 = (4(r^2+2s)H_{2n} - (rH_n - (r^2+4s)G_n)^2)(H_n^2 + r^2G_n^2 + 2rG_{2n})$.
5. $16(r^2+4s)G_{n+1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)^2G_n^2 + 2r(r^2+4s)G_{2n} - r^2H_n^2)(H_n^2 + r^2G_n^2 + 2rG_{2n})$.

(ii)

1. $16(-s)^4G_{n-1}^4 = H_n^4 + r^4G_n^4 - 2(2H_n^2 + 2r^2G_n^2 - 3rG_{2n})G_{2n}r$.
2. $16(-s)^4G_{n-1}^4 = (G_n^3r^3 - H_n^3 + 3r(H_n - rG_n)G_{2n})(rG_n - H_n)$.
3. $16(-s)^4G_{n-1}^4 = (G_nr - H_n)^4$.
4. $16(r^2+4s)(-s)^4G_{n-1}^4 = (4(r^2+2s)H_{2n} - (rH_n + (r^2+4s)G_n)^2)(-2rG_{2n} + H_n^2 + r^2G_n^2)$.
5. $16(r^2+4s)(-s)^4G_{n-1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)^2G_n^2 - 2r(r^2+4s)G_{2n} - r^2H_n^2)(-2rG_{2n} + H_n^2 + r^2G_n^2)$.

(iii)

1. $16H_{n+1}^4 = (r^2+4s)^4G_n^4 + r^4H_n^4 + 2(r^2+4s)(2r^2H_n^2 + 3r(r^2+4s)G_{2n} + 2(r^2+4s)^2G_n^2)rG_{2n}$.

2. $16H_{n+1}^4 = (r^3H_n^3 + (r^2+4s)^3G_n^3 + 3(r^2+4s)(rH_n + (r^2+4s)G_n)rG_{2n})(rH_n + (r^2+4s)G_n)$.
3. $16H_{n+1}^4 = (rH_n + (r^2+4s)G_n)^4$.
4. $16H_{n+1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)(rG_n - H_n)^2)((r^2+4s)^2G_n^2 + 2r(r^2+4s)G_{2n} + r^2H_n^2)$.
5. $16H_{n+1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)(H_n^2 + r^2G_n^2 - 2rG_{2n}))((r^2+4s)^2G_n^2 + 2r(r^2+4s)G_{2n} + r^2H_n^2)$.

(iv)

1. $16(-s)^4H_{n-1}^4 = (r^2+4s)^4G_n^4 + r^4H_n^4 - 2(r^2+4s)(2r^2H_n^2 - 3r(r^2+4s)G_{2n} + 2(r^2+4s)^2G_n^2)rG_{2n}$.
2. $16(-s)^4H_{n-1}^4 = (r^3H_n^3 - (r^2+4s)^3G_n^3 - 3(r^2+4s)(rH_n - (r^2+4s)G_n)rG_{2n})(rH_n - (r^2+4s)G_n)$.
3. $16(-s)^4H_{n-1}^4 = (rH_n - (r^2+4s)G_n)^4$.
4. $16(-s)^4H_{n-1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)(H_n + rG_n)^2)((r^2+4s)^2G_n^2 - 2r(r^2+4s)G_{2n} + r^2H_n^2)$.
5. $16(-s)^4H_{n-1}^4 = (4(r^2+2s)H_{2n} - (r^2+4s)(H_n^2 + G_n^2r^2 + 2rG_{2n}))((r^2+4s)^2G_n^2 - 2r(r^2+4s)G_{2n} + r^2H_n^2)$.

Proof.

(a) The proof proceeds by combining corresponding parts of Lemma 2.7 and Lemma 2.13, as outlined below.

- **(i)** Apply Lemma 2.7 (a) (i) together with Lemma 2.13 (a) (i).
- **(ii)** Apply Lemma 2.7 (a) (ii) together with Lemma 2.13 (a) (ii).
- **(iii)** Apply Lemma 2.7 (a) (iii) together with Lemma 2.13 (a) (iii).
- **(iv)** Apply Lemma 2.7 (a) (iv) together with Lemma 2.13 (a) (iv).

(b) Take $m = 1$ in (a). ☐

By applying Lemma 2.15, we obtain the following corollary.

Corollary 2.16. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i)

1. $8(G_{n+m}^4 + (-s)^{4m}G_{n-m}^4) = G_m^4H_n^4 + G_n^4H_m^4 + 6G_{2m}^2G_{2n}^2$.
2. $8(r^2+4s)(G_{n+m}^4 + (-s)^{4m}G_{n-m}^4) = -H_m^4G_{2n}^2 - H_n^4G_{2m}^2 - (r^2+4s)^2(G_n^4G_{2m}^2 + G_m^4G_{2n}^2) + 4(r^2+4s)G_{2m}^2G_{2n}^2 + 4(G_m^2H_n^2 + G_n^2H_m^2)H_{2n}H_{2m}$.

(ii)

1. $2(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) = (G_m^2H_n^2 + G_n^2H_m^2)G_{2n}G_{2m}$.
2. $4(r^2+4s)(G_{n+m}^4 - (-s)^{4m}G_{n-m}^4) = (4H_{2m}H_{2n} - H_m^2H_n^2 - (r^2+4s)^2G_m^2G_n^2 + (r^2+4s)(G_m^2H_n^2 + G_n^2H_m^2))G_{2n}G_{2m}$.

(iii)

1. $8(H_{n+m}^4 + (-s)^{4m}H_{n-m}^4) = H_m^4H_n^4 + (r^2+4s)^4G_m^4G_n^4 + 6(r^2+4s)^2G_{2m}^2G_{2n}^2$
2. $8(H_{n+m}^4 + (-s)^{4m}H_{n-m}^4) = -(r^2+4s)^3(G_n^4G_{2m}^2 + G_m^4G_{2n}^2) + 4(r^2+4s)^2(G_{2m}^2G_{2n}^2 + G_m^2G_n^2H_{2m}H_{2n}) - (r^2+4s)(H_n^4G_{2m}^2 + H_m^4G_{2n}^2) + 4H_{2m}H_{2n}H_m^2H_n^2$

(iv)

1. $2(H_{n+m}^4-(-s)^{4m}H_{n-m}^4)=(r^2+4s)(H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2)G_{2n}G_{2m}$.
2. $4(H_{n+m}^4-(-s)^{4m}H_{n-m}^4)=(r^2+4s)(H_m^2H_n^2+(r^2+4s)^2G_m^2G_n^2-(r^2+4s)(G_m^2H_n^2+G_n^2H_m^2)+4H_{2m}H_{2n})G_{2n}G_{2m}$.

(b)

(i)

1. $8(G_{n+1}^4+(-s)^4G_{n-1}^4)=H_n^4+r^4G_n^4+6r^2G_{2n}^2$.
2. $8(r^2+4s)(G_{n+1}^4+(-s)^4G_{n-1}^4)=-r^2(r^2+4s)^2G_n^4-r^2H_n^4+2(r^4+4r^2s-8s^2)G_{2n}^2+4(r^2+2s)\left(r^2G_n^2+H_n^2\right)H_{2n}$.

(ii)

1. $2(G_{n+1}^4-(-s)^4G_{n-1}^4)=r(H_n^2+r^2G_n^2)G_{2n}$.
2. $4(r^2+4s)(G_{n+1}^4-(-s)^4G_{n-1}^4)=r(-4s(r^2+4s)G_n^2+4sH_n^2+4(r^2+2s)H_{2n})G_{2n}$.

(iii)

1. $8(H_{n+1}^4+(-s)^4H_{n-1}^4)=r^4H_n^4+(r^2+4s)^4G_n^4+6r^2(r^2+4s)^2G_{2n}^2$
2. $8(H_{n+1}^4+(-s)^4H_{n-1}^4)=-r^2(r^2+4s)H_n^4-r^2(r^2+4s)^3G_n^4+4(r^2+2s)(r^2H_n^2+(r^2+4s)^2G_n^2)H_{2n}+2(r^2+4s)(r^4+4r^2s-8s^2)G_{2n}^2$.

(iv)

1. $2(H_{n+1}^4-(-s)^4H_{n-1}^4)=r(r^2+4s)(r^2H_n^2+(r^2+4s)^2G_n^2)G_{2n}$.
2. $4(H_{n+1}^4-(-s)^4H_{n-1}^4)=r(r^2+4s)(4s(r^2+4s)G_n^2-4sH_n^2+4(r^2+2s)H_{2n})G_{2n}$.

Proof.

(a) Use Lemma 2.15 (a).

(b) Set $m=1$ in (a). □

Remark 2.17. Using Lemma 2.7 (a), namely

$$
\begin{aligned}
2G_{n+m} &= G_mH_n+G_nH_m && \text{Lemma 2.7 (a) (i)},\\
2(-s)^mG_{n-m} &= G_nH_m-G_mH_n && \text{Lemma 2.7 (a) (ii)},\\
2H_{n+m} &= H_mH_n+(r^2+4s)G_mG_n && \text{Lemma 2.7 (a) (iii)},\\
2(-s)^mH_{n-m} &= H_mH_n-(r^2+4s)G_mG_n && \text{Lemma 2.7 (a) (iv)},
\end{aligned}
$$

we can derive several identities. For example, we obtain

$$16(G_{n+m}^4+H_{n+m}^4)=(G_mH_n+G_nH_m)^4+(H_mH_n+(r^2+4s)G_mG_n)^4$$

and

$$
\begin{aligned}
\frac{1}{2G_{n+m}}-\frac{1}{2(-s)^mG_{n-m}} &= 2^{-1}((G_{n+m})^{-1}-(-s)^{-m}(G_{n-m})^{-1})\\
&= (G_mH_n+G_nH_m)^{-1}-(G_nH_m-G_mH_n)^{-1}\\
&= \frac{2G_mH_n}{G_m^2H_n^2-G_n^2H_m^2},
\end{aligned}
$$

and

$$\begin{aligned} 2^{-2}((G_{n+m})^{-2}+(-s)^{-2m}(G_{n-m})^{-2}) &= (G_mH_n+G_nH_m)^{-2}+(G_nH_m-G_mH_n)^{-2} \\ &= \frac{2(G_m^2H_n^2+G_n^2H_m^2)}{(G_m^2H_n^2-G_n^2H_m^2)^2}, \end{aligned}$$

where the denominators are assumed to be nonzero.

3 Identities: Group II

From Corollary 1.11, we know that

$$G_{n+m} = G_nG_{m+1}+sG_{n-1}G_m, \tag{3.1}$$

$$H_{n+m} = H_nG_{m+1}+sH_{n-1}G_m \tag{3.2}$$

$$(r^2+4s)H_{n+m} = (2H_{m+2}-rH_{m+1})H_n+s(2H_{m+1}-rH_m)H_{n-1}. \tag{3.3}$$

Note that (3.2) and (3.3) can also be written in the symmetric form

$$H_{n+m}=G_{n+1}H_m+sG_nH_{m-1},$$

and

$$(r^2+4s)H_{n+m}=(2H_{n+2}-rH_{n+1})H_m+s(2H_{n+1}-rH_n)H_{m-1},$$

simply by interchanging n and m.

Next, we establish several related identities involving G_n and H_n.

Theorem 3.1. *For all integers m,n, we have the following identities.*

(a)

- **(i)** $(r^2+4s)G_{n+m}=H_{n+1}H_m+sH_nH_{m-1}$.
- **(ii)** $(r^2+4s)(-s)^mG_{n-m}=(r^2+4s)G_nH_m-H_{n+1}H_m-sH_nH_{m-1}$.
- **(iii)** $(r^2+4s)G_{n+m}=H_{n+1}H_m+H_n(H_{m+1}-rH_m)$.
- **(iv)** $(r^2+4s)(-s)^mG_{n-m}=(r^2+4s)G_nH_m-H_{n+1}H_m-H_n(H_{m+1}-rH_m)$.
- **(v)** $(r^2+4s)G_{n+m}=(r^2+4s)G_nH_m+H_nH_{m+1}-H_mH_{n+1}$.
- **(vi)** $(r^2+4s)(-s)^mG_{n-m}=H_{n+1}H_m-H_nH_{m+1}$.

(b)

- **(i)** $(r^2+4s)H_{n+m}=(2H_{m+1}-rH_m)H_{n+1}+(-rH_{m+1}+(r^2+2s)H_m)H_n$
- **(ii)** $(r^2+4s)(-s)^mH_{n-m}=(-2H_{m+1}+rH_m)H_{n+1}+(rH_{m+1}+2sH_m)H_n$

(c)

- **(i)** $(-W_1^2+sW_0^2+rW_0W_1)(W_0W_{n+m+1}-W_1W_{n+m})=(W_0(-2W_1+rW_0)W_{m+1}+(W_1^2+sW_0^2)W_m)W_{n+1}+((W_1^2+sW_0^2)W_{m+1}-W_1(rW_1+2sW_0)W_m)W_n$.

(ii) $(-s)^m(W_0W_{n-m+1} - W_1W_{n-m}) = W_mW_{n+1} - W_{m+1}W_n$.

(iii) $2H_{n+m+1} - rH_{n+m} = H_mH_{n+1} + (H_{m+1} - rH_m)H_n$.

(iv) $(-s)^m(2H_{n-m+1} - rH_{n-m}) = H_mH_{n+1} - H_{m+1}H_n$.

(d) $(r^2 + 4s)(G_{n+1}G_m + sG_nG_{m-1}) - H_{n+1}H_m - sH_nH_{m-1} = 0$.

Proof. Note that

$$H_{n+1} = rH_n + sH_{n-1} \Rightarrow H_{n+1} - rH_n = sH_{n-1} \Rightarrow H_{n-1} = \frac{1}{s}(H_{n+1} - rH_n).$$

(a) **(i)** We prove $(r^2+4s)G_{n+m} = H_{n+1}H_m + sH_nH_{m-1}$ using Binet's formulas of G_n and H_n. If the roots α and β of characteristic equation (1.2) are distinct ($\alpha \neq \beta$, i.e., $(r^2 + 4s) \neq 0$), then

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ H_n = \alpha^n + \beta^n.$$

Hence

$$\begin{aligned}(r^2 + 4s)G_{n+m} &= (\alpha - \beta)^2 \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} \\ &= (\alpha^{n+1} + \beta^{n+1})(\alpha^m + \beta^m) - \alpha\beta(\alpha^n + \beta^n)(\alpha^{m-1} + \beta^{m-1}) \\ &= H_{n+1}H_m + sH_nH_{m-1}\end{aligned}$$

which simplifies to

$$(r^2 + 4s)G_{n+m} = H_{n+1}H_m + sH_nH_{m-1}.$$

If $\alpha = \beta$, (i.e., $(r^2 + 4s) = 0$), then $G_n = n\alpha^{n-1}$, $H_n = 2\alpha^n$, $r^2 + 4s = 0$, $s = -\alpha^2$ and so we obtain

$$\begin{aligned}(r^2 + 4s)G_{n+m} &= 0 \\ &= ((2\alpha)^2 - 4\alpha^2)(n + 1)\alpha^{n+1-1} \\ &= 2\alpha^{n+1} \times 2\alpha^m + (-\alpha^2) \times 2\alpha^n \times 2\alpha^{m-1} \\ &= H_{n+1}H_m + sH_nH_{m-1},\end{aligned}$$

i.e., the same identity holds.

(ii) Similarly, using Binet's formulas, if $\alpha \neq \beta$ we obtain

$$\begin{aligned}(r^2 + 4s)(-s)^mG_{n-m} &= ((\alpha + \beta)^2 - 4\alpha\beta)(\alpha\beta)^m \frac{\alpha^{n-m} - \beta^{n-m}}{\alpha - \beta} \\ &= ((\alpha + \beta)^2 - 4\alpha\beta)\frac{\alpha^n - \beta^n}{\alpha - \beta}(\alpha^m + \beta^m) \\ &\quad -(\alpha^{n+1} + \beta^{n+1})(\alpha^m + \beta^m) + \alpha\beta(\alpha^n + \beta^n)(\alpha^{m-1} + \beta^{m-1}) \\ &= (r^2 + 4s)G_nH_m - H_{n+1}H_m - sH_nH_{m-1}\end{aligned}$$

which simplifies to

$$(r^2 + 4s)(-s)^mG_{n-m} = (r^2 + 4s)G_nH_m - H_{n+1}H_m - sH_nH_{m-1}$$

If $\alpha = \beta$, the formula reduces consistently to the same identity:

$$\begin{aligned}(r^2 + 4s)(-s)^mG_{n-m} &= 0 \\ &= ((2\alpha)^2 - 4\alpha^2)(\alpha^2)^m(n + 1)\alpha^{n+1-1} \\ &= ((2\alpha)^2 - 4\alpha^2) \times n\alpha^{n-1} \times 2\alpha^m - 2\alpha^{n+1} \times 2\alpha^m + \alpha^2 \times 2\alpha^n \times 2\alpha^{m-1} \\ &= (r^2 + 4s)G_nH_m - H_{n+1}H_m - sH_nH_{m-1}.\end{aligned}$$

(iii) Substituting $sH_{m-1} = H_{m+1} - rH_m$ into (i) yields

$$(r^2+4s)G_{n+m} = H_{n+1}H_m + H_n(H_{m+1} - rH_m).$$

(iv) Substituting $sH_{m-1} = H_{m+1} - rH_m$ into (ii) yields

$$(r^2+4s)(-s)^m G_{n-m} = (r^2+4s)G_nH_m - H_{n+1}H_m - H_n(H_{m+1} - rH_m).$$

(v) *First proof.* Apply Binet's formulas for G_n and H_n.

Second proof. Replace m by $-m$ in (ii) and use the identities

$$H_{-n} = (-s)^{-n}H_n, \qquad G_{-n} = -(-s)^{-n}G_n,$$

which are special cases of Theorem 1.7.Then

$$(r^2+4s)(-s)^m G_{n-m} = (r^2+4s)G_nH_m - H_{n+1}H_m - sH_nH_{m-1},$$

and substituting $m \mapsto -m$ yields

$$(r^2+4s)G_{n+m} = (r^2+4s)G_nH_m + H_nH_{m+1} - H_mH_{n+1}.$$

(vi) *First proof.* Apply Binet's formulas for G_n and H_n. *Second proof.* Replace m by $-m$ in (i) and use the identities given in (v).

(b) **(i)** Substitute $H_{m+2} = rH_{m+1} + sH_m$ and $H_{n-1} = \dfrac{1}{s}(H_{n+1} - rH_n)$ into

$$(r^2+4s)H_{n+m} = (2H_{m+2} - rH_{m+1})H_n + s(2H_{m+1} - rH_m)H_{n-1}$$

which is given in Corollary 1.11.

(ii) Replace m by $-m$ in (i) and use the identities

$$H_{-m} = (-s)^{-m}H_m, \qquad H_{-(m-1)} = (-s)^{-m+1}H_{m-1}, \qquad H_{m-1} = \frac{H_{m+1} - rH_m}{s}.$$

(c) **(i)** *First proof.* Using (a)(i), (1.14), (1.15) and the relations $W_{n+2} = rW_{n+1} + sW_n$, $W_{m-1} = \frac{1}{s}(W_{m+1} - rW_m)$, we get required identity.

Second proof. Apply Binet's formula for W_n.

Third proof. After proving (ii) via Binet's formula, replace m by $-m$ in

$$(-s)^m(W_0W_{n-m+1} - W_1W_{n-m}) = W_mW_{n+1} - W_{m+1}W_n$$

and use the identity

$$(-W_1^2 + sW_0^2 + rW_0W_1)W_{-n} = (-1)^{n+1}s^{-n}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n)$$

which is given in Theorem 1.9.

Fourth proof. Using (a) (iii), (1.14),(1.15) and the relations $W_{n+2} = rW_{n+1} + sW_n$, $W_{m-1} = \frac{1}{s}(W_{m+1} - rW_m)$, we obtain required identity.

(ii) *First proof.* Using (a)(i), (1.14), (1.15) and the relations $W_{n+2} = rW_{n+1} + sW_n$, $W_{m-1} = \frac{W_{m+1} - rW_m}{s}$, we obtain the required identity.

Second proof. Apply Binet's formula for W_n

Third proof. After proving (i) via Binet's formula, replace m by $-m$ in

$$\begin{aligned} & (-W_1^2 + sW_0^2 + rW_0W_1)(W_0W_{n+m+1} - W_1W_{n+m}) \\ = \; & (W_0(-2W_1 + rW_0)W_{m+1} + (W_1^2 + sW_0^2)W_m)W_{n+1} \\ & +((W_1^2 + sW_0^2)W_{m+1} - W_1(rW_1 + 2sW_0)W_m)W_n \end{aligned}$$

and use the identity

$$(-W_1^2 + sW_0^2 + rW_0W_1)W_{-n} = (-1)^{n+1}s^{-n}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n)$$

which is given in Theorem 1.9.

Fourth proof. Using (a)(iv), (1.14) and (1.15) and the relations $W_{n+2} = rW_{n+1} + sW_n$, $W_{m-1} = \frac{1}{s}(W_{m+1} - rW_m)$, we obtain required identity.

(iii)-(iv) Setting $W_n = H_n$ in (i) and (ii), respectively, yields the desired identities.

(d) We know that

$$G_{n+m} = G_{n+1}G_m + sG_nG_{m-1}.$$

From (a)(i), it follows that

$$\begin{aligned} H_{n+1}H_m + sH_nH_{m-1} &= (r^2 + 4s)G_{n+m} \\ &= (r^2 + 4s)(G_{n+1}G_m + sG_nG_{m-1}), \end{aligned}$$

i.e.,

$$(r^2 + 4s)(G_{n+1}G_m + sG_nG_{m-1}) - H_{n+1}H_m - sH_nH_{m-1} = 0. \square$$

Note that by using Theorem 3.1(a)(i), Theorem 3.1(a)(ii), and the identity

$$(r^2 + 4s)G_n = 2H_{n+1} - rH_n,$$

we obtain the following results (already derived as corollaries of Theorem 2.1, given in Corollary 2.2(c)(i)–(ii) and Corollary 2.3(c)(i)–(ii)): for all $n, m \in \mathbb{Z}$,

$$G_{n+m} + (-s)^m G_{n-m} = H_mG_n, \qquad G_{n+m} - (-s)^m G_{n-m} = G_mH_n.$$

4 Identities: Group III

Next, we present several identities involving W_m, G_n and H_n. It is worth noting that the proofs of parts (d)(iii) and (d)(iv) in the following theorem have already been established by mathematical induction in Theorem 2.1.

Theorem 4.1. *For all integers m, n, we have the following identities.*

(a)

(i) $2W_{n+m} = (2W_{m+1} - rW_m)G_n + W_mH_n = 2G_nW_{m+1} + (H_n - rG_n)W_m$.

(ii) $2(-s)^mW_{n-m} = -(2W_{m+1} - rW_m)G_n + 2W_nH_m - W_mH_n$.

(iii) $2W_{n+m} = 2H_mW_n + (-2W_{m+1} + rW_m + W_0(2H_{m+1} - rH_m))G_n + (W_m - W_0H_m)H_n$.

(iv) $2\,(-s)^m\,W_{n-m} = (2W_{m+1} - rW_m + W_0(-2H_{m+1} + rH_m))G_n + (-W_m + W_0H_m)H_n$.

(b)

(i) $2W_{n+m} = W_nH_m + (W_1H_n + sW_0H_{n-1})G_m$.

(ii) $2(-s)^mW_{n-m} = W_nH_m - (W_1H_n + sW_0H_{n-1})G_m$.

(iii) $2W_{n+m} = W_nH_m + (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m$.

(iv) $2(-s)^mW_{n-m} = W_nH_m - (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m$.

(v) $W_nH_m - W_mH_n = (2W_{m+1} - rW_m)G_n - (W_1H_n + sW_0H_{n-1})G_m$.

(c)

(i) $W_{n+m} = W_{n+1}G_m + sW_nG_{m-1} = W_{n+1}G_m + W_n(G_{m+1} - rG_m)$.

(ii) $(-s)^mW_{n-m} = -W_{n+1}G_m + W_nG_{m+1}$.

(d)

(i) $(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} = ((-W_1 + rW_0)W_{m+1} + sW_0W_m)W_n + s(W_0W_{m+1} - W_1W_m)W_{n-1}$.

(ii) $(-W_1^2+sW_0^2+rW_0W_1)(-s)^mW_{n-m} = (-W_1W_{m+1}+W_m(rW_1+sW_0))W_n - s(W_0W_{m+1}-W_1W_m)W_{n-1}$.

(iii) $(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} = (W_0W_{m+1} - W_1W_m)W_{n+1} + (-W_1W_{m+1} + (rW_1 + sW_0)W_m)W_n$.

(iv) $(-W_1^2+sW_0^2+rW_0W_1)(-s)^mW_{n-m} = (-W_0W_{m+1}+W_1W_m)W_{n+1}+((-W_1+rW_0)W_{m+1}+sW_0W_m)W_n$.

(e)

(i) $W_{n+m} + (-s)^mW_{n-m} = W_nH_m$.

(ii) $W_{n+m} - (-s)^mW_{n-m} = (W_1H_n + sW_0H_{n-1})G_m = (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m$.

(iii) $W_{n+m} + (-s)^mW_{n-m} = W_n\,(G_{m+1} + sG_{m-1}) = W_nH_m$.

(iv) $W_{n+m} - (-s)^mW_{n-m} = (2W_{n+1} - rW_n)G_m$.

(f)

(i) $2W_{n+m} = 2G_mW_{n+1} + (H_m - rG_m)W_n$.

(ii) $2(-s)^mW_{n-m} = -2G_mW_{n+1} + (H_m + rG_m)W_n$.

Proof.

(a)

(i) *First proof.* From Lemma 1.2, we know that

$$\begin{aligned} W_n &= W_0G_{n+1} + (W_1 - rW_0)G_n, \\ W_n &= W_1G_n + sW_0G_{n-1}. \end{aligned}$$

Adding these two equations side by side gives

$$2W_n = (2W_1 - rW_0)\,G_n + W_0\,(G_{n+1} + sG_{n-1})$$

i.e.,

$$2W_n = (2W_1 - rW_0)\,G_n + W_0H_n \tag{4.1}$$

where

$$G_{n+1} + sG_{n-1} = H_n$$

as given in Corollay 1.5 (a). Therefore,

$$2W_{n+m} = (2W_{m+1} - rW_m)G_n + W_mH_n$$

is obtained from (4.1) by shifting the initial terms of $\{W_n\}$ to W_m, W_{m+1} which completes the proof.

Second proof. Using Binet's formulas for W_n, G_n, H_n: If $\alpha \neq \beta$, then

$$\begin{aligned} 2W_{n+m} &= 2\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n+m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n+m}\right) \\ &= (2\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m+1} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m+1}\right) \\ &\quad -(\alpha + \beta)\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m}\right))\frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &\quad + \left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m}\right)(\alpha^n + \beta^n) \\ &= (2W_{m+1} - rW_m)G_n + W_mH_n. \end{aligned}$$

If $\alpha = \beta$, i.e., $-s = \alpha^2$, the same identity holds after simplification.

$$\begin{aligned} 2W_{n+m} &= 2((n+m)W_1 - \alpha\,((n+m) - 1)\,W_0)\alpha^{(n+m)-1} \\ &= (2 \times ((m+1)W_1 - \alpha\,((m+1) - 1)\,W_0)\alpha^{(m+1)-1} \\ &\quad -2\alpha(mW_1 - \alpha\,(m-1)\,W_0)\alpha^{m-1}) \times n\alpha^{n-1} + (mW_1 - \alpha\,(m-1)\,W_0)\alpha^{m-1} \times 2\alpha^n \\ &= (2W_{m+1} - rW_m)G_n + W_mH_n. \end{aligned}$$

Third proof. By induction on n.

(ii) Using Binet's formulas for W_n, G_n, H_n: If $\alpha \neq \beta$, then

$$\begin{aligned} 2(-s)^mW_{n-m} &= 2(\alpha\beta)^m\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n-m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n-m}\right) \\ &= -(2\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m+1} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m+1}\right) \\ &\quad -(\alpha + \beta)\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m}\right))\frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &\quad +2\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n}\right)(\alpha^m + \beta^m) \\ &\quad -\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{m}\right)(\alpha^n + \beta^n) \\ &= -(2W_{m+1} - rW_m)G_n + 2W_nH_m - W_mH_n. \end{aligned}$$

If $\alpha = \beta$, i.e., $-s = \alpha^2$, the same identity holds after simplification:

$$\begin{aligned}
2(-s)^m W_{n-m} &= 2(\alpha^2)^m ((n-m)W_1 - \alpha\,((n-m)-1)\,W_0)\alpha^{(n-m)-1} \\
&= -(2((m+1)W_1 - \alpha\,((m+1)-1)\,W_0)\alpha^{(m+1)-1} \\
&\quad -2\alpha(mW_1 - \alpha\,(m-1)\,W_0)\alpha^{m-1})n\alpha^{n-1} \\
&\quad +2(nW_1 - \alpha\,(n-1)\,W_0)\alpha^{n-1} \times 2\alpha^m - (mW_1 - \alpha\,(m-1)\,W_0)\alpha^{m-1} \times 2\alpha^n \\
&= -(2W_{m+1} - rW_m)G_n + 2W_nH_m - W_mH_n.
\end{aligned}$$

(iii) Replace m by $-m$ in (ii) and use the identities

$$W_{-n} = -(-s)^{-n}(W_n - H_nW_0), \qquad H_{-n} = (-s)^{-n}H_n, \qquad G_{-n} = -(-s)^{-n}G_n,$$

which are given in Theorem 1.7 and Corollary 1.8.

(iv) Replace m by $-m$ in (i) and use the identities given in (iii).

(b)

(i) Using Binet's formulas for W_n, G_n, H_n: If $\alpha \neq \beta$, then

$$\begin{aligned}
2W_{n+m} &= 2\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n+m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n+m}\right) \\
&= \left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n}\right)(\alpha^m + \beta^m) \\
&\quad +(W_1(\alpha^n + \beta^n) - \alpha\beta W_0(\alpha^{n-1} + \beta^{n-1}))\frac{\alpha^m - \beta^m}{\alpha - \beta} \\
&= W_nH_m + (W_1H_n + sW_0H_{n-1})G_m.
\end{aligned}$$

If $\alpha = \beta$ (i.e., $-s = \alpha^2$), the same identity holds after simplification:

$$\begin{aligned}
2W_{n+m} &= 2((n+m)W_1 - \alpha\,((n+m)-1)\,W_0)\alpha^{(n+m)-1} \\
&= (nW_1 - \alpha\,(n-1)\,W_0)\alpha^{n-1} \times 2\alpha^m \\
&\quad +(W_1 \times 2\alpha^n - \alpha^2 \times W_0 \times 2\alpha^{n-1}) \times m\alpha^{m-1} \\
&= W_nH_m + (W_1H_n + sW_0H_{n-1})G_m.
\end{aligned}$$

(ii) *First proof.* Using Binet's formulas for W_n, G_n, H_n: If $\alpha \neq \beta$, then

$$\begin{aligned}
2(-s)^m W_{n-m} &= 2(\alpha\beta)^m\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n-m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n-m}\right) \\
&= \left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n}\right)(\alpha^m + \beta^m) \\
&\quad -(W_1(\alpha^n + \beta^n) - \alpha\beta W_0(\alpha^{n-1} + \beta^{n-1}))\frac{\alpha^m - \beta^m}{\alpha - \beta} \\
&= W_nH_m - (W_1H_n + sW_0H_{n-1})G_m.
\end{aligned}$$

If $\alpha = \beta$ (i.e., $-s = \alpha^2$), the same identity holds after simplification.

$$\begin{aligned}
2(-s)^m W_{n-m} &= 2(\alpha^2)^m ((n-m)W_1 - \alpha\,((n-m)-1)\,W_0)\alpha^{n-m-1} \\
&= (nW_1 - \alpha\,(n-1)\,W_0)\alpha^{n-1} \times 2\alpha^m - (W_1 \times 2\alpha^n - \alpha^2 W_0 \times 2\alpha^{n-1}) \times m\alpha^{m-1} \\
&= W_nH_m - (W_1H_n + sW_0H_{n-1})G_m.
\end{aligned}$$

This completes the proof.

Second proof. Replace m by $-m$ in (i) and use the identities given in (a)(iii).

(iii) Substituting $H_{n-1} = \frac{1}{s}(H_{n+1} - rH_n)$ into (i) yields the desired identity.

(iv) Substituting $H_{n-1} = \frac{1}{s}(H_{n+1} - rH_n)$ into (ii) yields the desired identity.

(v) Combining (i) with (a)(i), we obtain

$$(2W_{m+1} - rW_m)G_n + W_m H_n = W_n H_m + (W_1 H_n + sW_0 H_{n-1})G_m$$

which implies

$$(2W_{m+1} - rW_m)G_n - (W_1 H_n + sW_0 H_{n-1})G_m = W_n H_m - W_m H_n.$$

(c)

(i) *First proof.* Using Binet's formulas for W_n and G_n: If $\alpha \neq \beta$, then

$$\begin{aligned} W_{n+m} &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n+m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n+m} \\ &= \left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n+1} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n+1}\right)\frac{\alpha^m - \beta^m}{\alpha - \beta} \\ &\quad -\alpha\beta\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{n} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{n}\right)\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \\ &= W_{n+1}G_m + sW_n G_{m-1}. \end{aligned}$$

If $\alpha = \beta$ (i.e., $-s = \alpha^2$), the same identity holds after simplification:

$$\begin{aligned} W_{n+m} &= ((n+m)W_1 - \alpha\left((n+m) - 1\right)W_0)\alpha^{n+m-1} \\ &= ((n+1)W_1 - \alpha\left((n+1) - 1\right)W_0)\alpha^{n+1-1} \times m\alpha^{m-1} \\ &\quad -\alpha^2(nW_1 - \alpha\left(n - 1\right)W_0)\alpha^{n-1} \times (m-1)\alpha^{m-1-1} \\ &= W_{n+1}G_m + sW_n G_{m-1}. \end{aligned}$$

Moreover, setting $G_{m-1} = \dfrac{1}{s}(G_{m+1} - rG_m)$ in

$$W_{n+m} = W_{n+1}G_m + sW_n G_{m-1}$$

yields

$$W_{n+m} = W_{n+1}G_m + W_n(G_{m+1} - rG_m).$$

This completes the proof.

Second proof. Setting $n \mapsto n+1$ and $m \mapsto m-1$ in the identity

$$W_{n+m} = W_n G_{m+1} + sW_{n-1}G_m,$$

which is given in Theorem 1.10, we obtain

$$W_{n+m} = W_{n+1}G_m + sW_n G_{m-1}.$$

(ii) *First proof.* Using Binet's formulas for W_n and G_n: If $\alpha \neq \beta$, then

$$\begin{aligned} W_{n+1}G_m + (-s)^m W_{n-m} &= \left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{n+1} - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{n+1}\right)\frac{\alpha^m-\beta^m}{\alpha-\beta} \\ &\quad +\alpha^m\beta^m\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{n-m} - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{n-m}\right) \\ &= \left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{n} - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{n}\right)\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} = W_nG_{m+1} \end{aligned}$$

i.e.,

$$W_nG_{m+1} = W_{n+1}G_m + (-s)^m W_{n-m}$$

If $\alpha = \beta$ (i.e., $-s = \alpha^2$), the same identity holds after simplification:

$$\begin{aligned} W_{n+1}G_m + (-s)^m W_{n-m} &= ((n+1)W_1 - \alpha\left((n+1)-1\right)W_0)\alpha^{n+1-1} \times m\alpha^{m-1} \\ &\quad +\alpha^{2m}((n-m)W_1 - \alpha\left((n-m)-1\right)W_0)\alpha^{n-m-1} \\ &= (nW_1 - \alpha\left(n-1\right)W_0)\alpha^{n-1} \times (m+1)\alpha^{m+1-1} = W_nG_{m+1}. \end{aligned}$$

This completes the proof.

Second proof. Replace m by $-m$ in (i) and use the identities given in (a)(iii). Indeed, Indeed,

$$W_{n-m} = W_{n+1}G_{-m} + sW_nG_{-m-1},$$

and so

$$W_{n-m} = W_{n+1}\left(-(-s)^{-m}G_m\right) + sW_n\left(-(-s)^{-(m+1)}G_{m+1}\right)$$

which implies

$$(-s)^m W_{n-m} = -W_{n+1}G_m + W_nG_{m+1}.$$

(d) Note that the proofs of (iii) and (iv) have already been established by mathematical induction in Theorem 2.1. Here, we provide alternative proofs.

(i) Using (b)(i), together with (1.13), (1.14) and (1.15) we obtain the required identity. In particular, substituting $W_{n-1} = \frac{1}{s}(W_{n+1} - rW_n)$ into

$$(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} = ((-W_1 + rW_0)W_{m+1} + sW_0W_m)W_n + s(W_0W_{m+1} - W_1W_m)W_{n-1}$$

yields (d) (i).

(ii) Using (b)(ii), together with (1.13), (1.14) and (1.15) we obtain required identity. Substituting $W_{n-1} = \frac{1}{s}(W_{n+1} - rW_n)$ into

$$(-W_1^2+sW_0^2+rW_0W_1)(-s)^m W_{n-m} = (-W_1W_{m+1}+W_m(rW_1+sW_0))W_n - s(W_0W_{m+1}-W_1W_m)W_{n-1}$$

yields (d) (ii).

(iii) *First proof.* Substitute $W_{n-1} = \frac{1}{s}(W_{n+1} - rW_n)$ into (i).

Second proof. Using (a)(i), together with (1.13), (1.14) and (1.15) we obtain the required identity.

Third proof. Using (1.15) and (c) (i), we obtain the required identity. In particular, substituting $W_{m-1} = \frac{1}{s}(W_{m+1} - rW_m)$ into

$$(-W_1^2 + sW_0^2 + rW_0W_1)W_{n+m} = (W_0W_{m+1} - W_1W_m)W_{n+1} + s(W_0W_m - W_1W_{m-1})W_n$$

yields (d) (iii).

(iv) *First proof.* Substitute $W_{n-1}=\frac{1}{s}(W_{n+1}-rW_n)$ into (ii).

Second proof. Using (a) (ii) together with (1.13), (1.14) and (1.15) we obtain the required identity.

Third proof. Using (1.15) and (c) (ii), we obtain the required identity. In particular, substituting $W_{m+2}=rW_{m+1}+sW_m$ into

$$(-W_1^2+sW_0^2+rW_0W_1)(-s)^mW_{n-m}=-(W_0W_{m+1}-W_1W_m)W_{n+1}+(W_0W_{m+2}-W_1W_{m+1})W_n$$

yields (d) (iv).

(e)

(i) *First proof.* Apply Binet's formulas for W_n and H_n. *Second proof.* Add side by side (b)(i) and (b)(ii), or (b)(iii) and (b)(iv), or (a)(iii) and (a)(vi). *Third proof.* Use induction on n.

(ii) *First proof.* Apply Binet's formulas for W_n, G_n and H_n. If $\alpha\neq\beta$ then

$$\begin{aligned}
& W_{n+m}-(-s)^mW_{n-m} \\
= & \left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{n+m}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{n+m}\right)-(\alpha\beta)^m\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{n-m}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{n-m}\right) \\
= & \left(W_1\left(\alpha^n+\beta^n\right)-\left(\alpha^n\beta+\alpha\beta^n\right)W_0\right)\frac{\alpha^m-\beta^m}{\alpha-\beta} \\
= & \left(\left(\alpha^n+\beta^n\right)W_1-\alpha\beta\left(\alpha^{n-1}+\beta^{n-1}\right)W_0\right)\frac{\alpha^m-\beta^m}{\alpha-\beta} \\
= & (W_1H_n+sW_0H_{n-1})G_m
\end{aligned}$$

If $\alpha=\beta, i.e.,$. In this case $-s=\alpha^2$. Then, we get

$$\begin{aligned}
W_{n+m}-(-s)^mW_{n-m} &= ((n+m)W_1-\alpha\left((n+m)-1\right)W_0)\alpha^{n+m-1} \\
&\quad -\alpha^{2m}((n-m)W_1-\alpha\left((n-m)-1\right)W_0)\alpha^{n-m-1} \\
&= (W_1\times 2\alpha^n-\alpha^2W_0\times 2\alpha^{n-1})m\alpha^{m-1} \\
&= (W_1H_n+sW_0H_{n-1})G_m.
\end{aligned}$$

This completes the proof.

Second proof. Subtract side by side (b)(i) and (b)(ii), or (b)(iii) and (b)(iv), or (a)(iii) and (a)(vi). *Third proof.* Use induction on n.

(iii) Add side by side (c)(i) and (c)(ii).

(iv) Subtract side by side (c)(i) and (c)(ii).

(f)

(i) Add side by side (e) (i) and (e) (iv). Note also that if we exchange n and m in (a) (i) we obtain

$$\begin{aligned}
2W_{n+m} &= (2W_{n+1}-rW_n)G_m+W_nH_m \\
&= 2G_mW_{n+1}+(H_m-rG_m)W_n.
\end{aligned}$$

(ii) Subtract side by side (e) (i) and (e) (iv). □

From Theorem 4.1, we get the following Corollary. To prove (a) (ii) 3. in the following Corollary we use the identity

$$G_n^2H_m^2 - H_n^2G_m^2 = 4((-s)^mG_n^2 - (-s)^nG_m^2)$$

which is given in Lemma 1.13 (f) (ii).

Corollary 4.2. *Let $n, m, k \in \mathbb{Z}$. Then*

(a)

(i) $2G_{n+m} = (2G_{m+1} - rG_m)G_n + G_mH_n$.

(ii)

1. $G_{n+m} + (-s)^mG_{n-m} = G_nH_m$.
2. $G_{n+m} - (-s)^mG_{n-m} = H_nG_m$
3. $4(-s)^mG_{n+m}G_{n-m} = G_n^2H_m^2 - H_n^2G_m^2 = (G_nH_m + G_mH_n)(G_nH_m - G_mH_n) = 4((-s)^mG_n^2 - (-s)^nG_m^2)$.
   i.e. $G_{n+m}G_{n-m} = G_n^2 - (-s)^{n-m}G_m^2$
4. $G_{n+m}^2 - s^{2m}G_{n-m}^2 = G_nH_nG_mH_m = G_{2n}G_{2m}$.
5. $G_{n+m}^2 + s^{2m}G_{n-m}^2 = \frac{1}{2}(G_n^2H_m^2 + H_n^2G_m^2)$.
6.

$$\begin{aligned} 2G_{n+m} &= G_nH_m + H_nG_m, \\ 2(-s)^mG_{n-m} &= G_nH_m - G_mH_n. \end{aligned}$$

7. $4(G_{n+m}^3 - (-s)^{3m}G_{n-m}^3) = (3G_n^2H_m^2 + G_m^2H_n^2)G_mH_n$.
8. $4(G_{n+m}^3 + (-s)^{3m}G_{n-m}^3) = (3G_m^2H_n^2 + G_n^2H_m^2)G_nH_m$.

(iii) $(-s)^mG_{n-m} = G_nG_{m+1} - G_{n+1}G_m$.

(iv) $G_n(H_m - G_{m+1}) = sG_{n-1}G_m + (-s)^mG_{n-m}$.

(v) $G_{nm+k} = G_{n(m-1)}H_{n+k} - (-s)^{n+k}G_{n(m-2)-k}$.

(b)

(i) $2H_{n+m} = (2H_{m+1} - rH_m)G_n + H_mH_n = (r^2 + 4s)G_mG_n + H_mH_n$.

(ii)

1. $H_{n+m} + (-s)^mH_{n-m} = H_nH_m$.
2. $H_{n+m} - (-s)^mH_{n-m} = (rH_n + 2sH_{n-1})G_m = (r^2 + 4s)G_nG_m$.
3. $4(-s)^mH_{n+m}H_{n-m} = H_n^2H_m^2 - (r^2 + 4s)^2G_n^2G_m^2$.
4. $H_{n+m}^2 - s^{2m}H_{n-m}^2 = (r^2 + 4s)G_nH_nG_mH_m = (r^2 + 4s)G_{2n}G_{2m}$.
5. $H_{n+m}^2 + s^{2m}H_{n-m}^2 = \frac{1}{2}(H_n^2H_m^2 + (r^2 + 4s)^2G_n^2G_m^2)$.
6.

$$\begin{aligned} 2H_{n+m} &= H_nH_m + (rH_n + 2sH_{n-1})G_m, \\ 2(-s)^mH_{n-m} &= H_nH_m - (rH_n + 2sH_{n-1})G_m. \end{aligned}$$

7. $4(H_{n+m}^3 - (-s)^{3m}H_{n-m}^3) = (r^2 + 4s)(3H_n^2H_m^2 + (r^2 + 4s)^2G_n^2G_m^2)G_nG_m$.

8. $4(H_{n+m}^3 + (-s)^{3m} H_{n-m}^3) = (H_n^2 H_m^2 + 3(r^2+4s)^2 G_n^2 G_m^2) H_n H_m$.

(iii) $(-s)^m H_{n-m} = H_n G_{m+1} - H_{n+1} G_m$.

(iv) $H_n(H_m - G_{m+1}) = sH_{n-1}G_m + (-s)^m H_{n-m}$.

(v) $H_{nm+k} = H_{n(m-1)} H_{n+k} - (-s)^{n+k} H_{n(m-2)-k}$.

(c)

(i) $2G_{m+n} = H_m G_n + G_m H_n$.

(ii) $2H_{m+n} = (r^2+4s) G_m G_n + H_m H_n$.

(iii) $(r^2+4s)(G_n H_m - (-s)^m G_{n-m}) = H_{n+1} H_m + sH_n H_{m-1}$.

(iv) $(r^2+4s) G_{n+m} = H_{n+m+1} + (-s)^m H_{n-m+1} + sH_n H_{m-1}$.

(v)

1. $(r^2+4s) G_{n+m+1} = H_{n+1} H_{m+1} + sH_n H_m$.
2. $(r^2+4s) G_n = H_{n-m+1} H_m + sH_{n-m} H_{m-1}$.
3. $(r^2+4s) G_{n+m-1} = H_n H_m + sH_{n-1} H_{m-1}$.
4. $(r^2+4s) G_n = H_m H_{n-m+1} + sH_{m-1} H_{n-m}$.

(vi)

1. $(r^2+4s) G_{2n} = H_{n+1} H_n + sH_n H_{n-1}$.
2. $(r^2+4s) G_{2n+1} = rH_{2n+1} + 2sH_{2n} = H_{n+1}^2 + sH_n^2 = H_{n+2} H_n + sH_{n+1} H_{n-1}$.
3. $(r^2+4s) G_{3n} = rH_{3n} + 2sH_{3n-1} = H_{2n+1} H_n + sH_{2n} H_{n-1} = H_{n+1} H_{2n} + sH_n H_{2n-1}$.
4. $(r^2+4s) G_{3n+1} = rH_{3n+1} + 2sH_{3n} = H_{2n+2} H_n + sH_{2n+1} H_{n-1} = H_{2n+1} H_{n+1} + sH_{2n} H_n$.
5. $(r^2+4s) G_{3n+2} = rH_{3n+2} + 2sH_{3n+1} = H_{2n+2} H_{n+1} + sH_{2n+1} H_n = H_{2n+3} H_n + sH_{2n+2} H_{n-1}$.
6. $(r^2+4s) G_{4n} = rH_{4n} + 2sH_{4n-1} = H_{3n} H_{n+1} + sH_{3n-1} H_n = H_{2n+1} H_{2n} + sH_{2n} H_{2n-1}$.
7. $(r^2+4s) G_{5n} = rH_{5n} + 2sH_{5n-1} = H_{4n} H_{n+1} + sH_{4n-1} H_n = H_{3n+1} H_{2n} + sH_{3n} H_{2n-1}$.

(d)

(i)

1. $G_{2n} = G_n H_n$.
2. $G_{2n} = G_{n-1} H_{n+1} + r(-s)^{n-1}$.
3. $G_{2n} = G_{n+1} H_{n-1} - r(-s)^{n-1}$.
4. $G_{2n} = G_{n-2} H_{n+2} + r\left(r^2+2s\right)(-s)^{n-2}$.

(ii)

1. $H_{2n} = H_n^2 - 2(-s)^n$.
2. $2H_{2n} = (r^2+4s) G_n^2 + H_n^2$.
3. $8(-s)^n H_{2n} = H_n^4 - (r^2+4s)^2 G_n^4 = (H_n^2 - (r^2+4s) G_n^2)(H_n^2 + (r^2+4s) G_n^2)$.
4. $H_n^2 = (r^2+4s) G_n^2 + 4(-s)^n$.
5. $H_{2n} = (r^2+4s) G_{n+1} G_{n-1} + (-s)^{n-1}(r^2+2s)$.
6. $H_{2n} = (r^2+4s) G_n^2 + 2(-s)^n$.
7. $H_{2n} = H_{n+1} H_{n-1} - (-s)^{n-1}(r^2+2s)$.

(iii)

1. $4G_{3n} = (r^2+4s)G_n^3 + 3H_n^2 G_n$ *and* $2G_{3n} = G_n(H_n^2 + H_{2n})$ *and* $4G_{3n} = G_n(3H_n^2 + (r^2+4s)G_n^2)$.
2. $G_{3n} = G_{2n}H_n - (-s)^n G_n = G_n H_n^2 - (-s)^n G_n = (H_n^2 - (-s)^n)G_n$.
3. $G_{3n} = G_n(H_{2n} + (-s)^n)$.
4. $G_{3n} = r\left(r^2+3s\right)G_{3n-3} + s^3 G_{3n-6}$.
5. $G_{3n} = (r^2+4s)G_n^3 + 3(-s)^n G_n$.
6. $G_{3n} = H_{2n}G_n + (-s)^n G_n$.
7. $G_{3n} = H_{2n+1}G_{n-1} + (-s)^{n-1}G_{n+2}$.
8. $G_{3n} = G_{2n+1}H_{n-1} - (-s)^{n-1}G_{n+2}$.

(iv)

1. $4H_{3n} = 3(r^2+4s)G_n^2 H_n + H_n^3$ *and* $2H_{3n} = H_n((r^2+4s)G_n^2 + H_{2n})$.
2. $H_{3n} = (r^2+4s)G_{2n}G_n + (-s)^n H_n = ((r^2+4s)G_n^2 + (-s)^n)H_n$.
3. $H_{3n} = H_{2n}H_n - (-s)^n H_n = (H_n^2 - 3(-s)^n)H_n$.
4. $H_{3n} = (r^2+4s)G_{2n+1}G_{n-1} + (-s)^{n-1}H_{n+2}$.
5. $H_{3n} = H_{2n+1}H_{n-1} - (-s)^{n-1}H_{n+2}$.

(v)

1. $2G_{4n} = H_n G_n(H_n^2 + (r^2+4s)G_n^2) = G_{2n}(H_n^2 + (r^2+4s)G_n^2)$.
2. $G_{4n} = H_{3n}G_n + (-s)^n G_{2n} = (H_n^2 - 2(-s)^n)G_{2n}$.
3. $G_{4n} = G_{3n}H_n - (-s)^n G_{2n} = (H_n^2 - 2(-s)^n)G_{2n}$.
4. $G_{4n} = G_{2n}H_{2n} = (H_n^2 - 2(-s)^n)G_{2n}$.

(vi)

1. $8H_{4n} = 6(r^2+4s)G_n^2 H_n^2 + (r^2+4s)^2 G_n^4 + H_n^4$.
2. $H_{4n} = (r^2+4s)G_{2n}^2 + 2s^{2n}$.
3. $H_{4n} = (r^2+4s)G_{3n}G_n + (-s)^n H_{2n} = (H_n^2 - 2\left(-s\right)^n)^2 - 2\left(-s\right)^{2n}$.
4. $H_{4n} = H_{3n}H_n - (-s)^n H_{2n} = (H_n^2 - 2\left(-s\right)^n)^2 - 2\left(-s\right)^{2n}$.

(vii)

1. $16G_{5n} = 10(r^2+4s)G_n^3 H_n^2 + (r^2+4s)^2 G_n^5 + 5H_n^4 G_n$.
2. $G_{5n} = H_{3n}G_{2n} + (-s)^{2n}G_n = ((H_n^2 - 3(-s)^n)H_n^2 + (-s)^{2n})G_n$.
3. $G_{5n} = G_{3n}H_{2n} - (-s)^{2n}G_n = ((H_n^2 - (-s)^n)^2 - (-s)^n H_n^2)G_n$.

(viii)

1. $16H_{5n} = 10(r^2+4s)G_n^2 H_n^3 + 5(r^2+4s)^2 G_n^4 H_n + H_n^5$.
2. $H_{5n} = (r^2+4s)G_{3n}G_{2n} + (-s)^{2n}H_n = ((H_n^2 - (-s)^n)(H_n^2 - 4(-s)^n) + (-s)^{2n})H_n$.
3. $H_{5n} = H_{3n}H_{2n} - (-s)^{2n}H_n = ((H_n^2 - 3(-s)^n)(H_n^2 - 2(-s)^n) - (-s)^{2n})H_n$.

(e)

(i)

1. $2G_{2n+1} = G_{n+1}H_n + G_n H_{n+1}$.
2. $G_{2n+1} = H_{n+1}G_n + (-s)^n$.

3. $G_{2n+1} = G_{n+1}H_n - (-s)^n$.
4. $G_{2n+1} = \frac{1}{2}(H_n^2 + rG_nH_n - 2(-s)^n) = \frac{1}{2}(H_{2n} + rG_{2n})$.

(ii)

1. $2H_{2n+1} = (r^2 + 4s)G_nG_{n+1} + H_nH_{n+1}$.
2. $H_{2n+1} = (r^2 + 4s)G_{n+1}G_n + r(-s)^n$.
3. $H_{2n+1} = H_{n+1}H_n - r(-s)^n$.
4. $H_{2n+1} = \frac{1}{2}((r^2 + 4s)G_{2n} + rH_{2n}) = \frac{1}{2}((r^2 + 4s)G_nH_n + r(H_n^2 - 2(-s)^n))$.

(iii)

1. $4G_{3n+1} = G_{n+1}H_n^2 + G_nH_nH_{n+1} + (r^2 + 4s)G_n^2G_{n+1} + G_nH_nH_{n+1}$.
2. $G_{3n+1} = H_{2n+1}G_n + (-s)^nG_{n+1} = ((r^2 + 4s)G_n^2 + (-s)^n)G_{n+1} + r(-s)^nG_n$.
3. $G_{3n+1} = G_{2n+1}H_n - (-s)^nG_{n+1} = (H_n^2 - (-s)^n)G_{n+1} - (-s)^nH_n$.
4. $G_{3n+1} = H_{3n} + (-s)G_{3n-1} = \frac{1}{2}(H_{3n} + rG_{3n})$.

(iv)

1. $4H_{3n+1} = 2(r^2 + 4s)G_nG_{n+1}H_n + (r^2 + 4s)G_n^2H_{n+1} + H_n^2H_{n+1}$.
2. $H_{3n+1} = (r^2 + 4s)G_{2n+1}G_n + (-s)^nH_{n+1} = ((r^2 + 4s)G_n^2 + (-s)^n)H_{n+1} + (r^2 + 4s)(-s)^nG_n$.
3. $H_{3n+1} = H_{2n+1}H_n - (-s)^nH_{n+1} = (H_n^2 - (-s)^n)H_{n+1} - r(-s)^nH_n$.
4. $H_{3n+1} = (r^2 + 4s)G_{3n} - sH_{3n-1} = \frac{1}{2}((r^2 + 4s)G_{3n} + rH_{3n}) = \frac{1}{2}((r^2 + 4s)(H_n^2 - (-s)^n)G_n + r(H_n^2 - 3(-s)^n)H_n)$.

(v)

1. $4G_{3n+2} = 2G_{n+1}H_nH_{n+1} + G_nH_{n+1}^2 + (r^2 + 4s)G_nG_{n+1}^2$.
2. $G_{3n+2} = H_{2n+1}G_{n+1} + (-s)^{n+1}G_n = ((r^2 + 4s)G_{n+1}^2 + (-s)^{n+1})G_n + r(-s)^nG_{n+1}$.
3. $G_{3n+2} = G_{2n+1}H_{n+1} - (-s)^{n+1}G_n = (H_{n+1}^2 - (-s)^{n+1})G_n + (-s)^nH_{n+1}$.
4. $G_{3n+2} = H_{3n}G_2 + s^2G_{3n-2} = \frac{1}{2}(G_2H_{3n} + H_2G_{3n}) = \frac{1}{2}(G_2(H_n^2 - 3(-s)^n)H_n + H_2(H_n^2 - (-s)^n)G_n)$.

(vi)

1. $4H_{3n+2} = (r^2 + 4s)G_{n+1}^2H_n + 2(r^2 + 4s)G_nG_{n+1}H_{n+1} + H_nH_{n+1}^2$.
2. $H_{3n+2} = (r^2 + 4s)G_{2n+1}G_{n+1} + (-s)^{n+1}H_n = ((r^2 + 4s)G_{n+1}^2 + (-s)^{n+1})H_n - (r^2 + 4s)(-s)^nG_{n+1}$.
3. $H_{3n+2} = H_{2n+1}H_{n+1} - (-s)^{n+1}H_n = (H_{n+1}^2 - (-s)^{n+1})H_n - r(-s)^nH_{n+1}$.
4. $H_{3n+2} = \frac{1}{2}((r^2 + 4s)G_2G_{3n} + H_2H_{3n}) = \frac{1}{2}((r^2 + 4s)(H_n^2 - (-s)^n)G_2G_n + (H_n^2 - 3(-s)^n)H_2H_n)$.

(vii)

1. $8G_{4n+1} = G_{n+1}H_n^3 + 3(r^2 + 4s)G_n^2G_{n+1}H_n + (r^2 + 4s)G_n^3H_{n+1} + 3G_nH_n^2H_{n+1}$.
2. $G_{4n+1} = H_{3n+1}G_n + (-s)^nG_{2n+1} = (H_nH_{n+1} - r(-s)^n)G_nH_n + (-s)^{2n}$.
3. $G_{4n+1} = H_{3n+1}G_n + (-s)^nG_{2n+1} = \left(H_n^2 - (-s)^n\right)H_{n+1}G_n + (-s)^n(-rG_n + G_{n+1})H_n - (-s)^{2n}$.
4. $G_{4n+1} = G_{3n+1}H_n - (-s)^nG_{2n+1} = \left(H_n^2 - (-s)^n\right)G_{n+1}H_n - (-s)^nH_n^2 - (-s)^nG_nH_{n+1} - (-s)^{2n}$.
5. $G_{4n+1} = G_{3n+1}H_n - (-s)^nG_{2n+1} = (H_n^2 - 2(-s)^n)G_{n+1}H_n - (-s)^nH_n^2 + (-s)^{2n}$.
6. $G_{4n+1} = H_{4n} - sG_{4n-1} = \frac{1}{2}(H_{4n}G_1 + H_1G_{4n}) = \frac{1}{2}((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n} + r(H_n^2 - 2(-s)^n)G_{2n})$.

(viii)

1. $8H_{4n+1} = 3(r^2 + 4s)G_nG_{n+1}H_n^2 + 3(r^2 + 4s)G_n^2H_nH_{n+1} + (r^2 + 4s)^2G_n^3G_{n+1} + H_n^3H_{n+1}$.
2. $H_{4n+1} = (r^2 + 4s)G_{3n+1}G_n + (-s)^nH_{2n+1} = (r^2 + 4s)G_nG_{n+1}H_n^2 - (r^2 + 4s)(-s)^nG_nH_n + r(-s)^{2n}$.

3. $H_{4n+1} = (r^2+4s)G_{3n+1}G_n + (-s)^n H_{2n+1} = (r^2+4s)(G_nG_{n+1}H_n^2 - (-s)^n G_nG_{n+1} - (-s)^n G_nH_n) + (-s)^n H_nH_{n+1} - r(-s)^{2n}$.
4. $H_{4n+1} = (H_{3n+1}H_n - (-s)^n H_{2n+1}) = -(r^2+4s)(-s)^n G_{n+1}G_n + (H_n^2 - (-s)^n)H_nH_{n+1} - r(-s)^n(H_n^2 + (-s)^n)$.
5. $H_{4n+1} = (H_{3n+1}H_n - (-s)^n H_{2n+1}) = (H_n^2 - 2(-s)^n)H_nH_{n+1} - r(-s)^n(H_n^2 - (-s)^n)$.
6. $H_{4n+1} = (r^2+4s)G_{4n}G_1 - sH_{4n-1} = \frac{1}{2}((r^2+4s)G_{4n}G_1 + H_1H_{4n}) = \frac{1}{2}((r^2+4s)(H_n^2 - 2(-s)^n)G_{2n} + r((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n}))$.

(ix)

1. $8G_{4n+2} = 2G_{n+1}H_n^2H_{n+1} + 2G_nH_nH_{n+1}^2 + 2(r^2+4s)G_n^2G_{n+1}H_{n+1} + (2(r^2+4s)G_nG_{n+1}^2H_n$.
2. $G_{4n+2} = H_{3n+1}G_{n+1} + (-s)^{n+1}G_{2n} = ((H_n^2 - (-s)^n)H_{n+1} - r(-s)^n H_n)G_{n+1} + (-s)^{n+1}G_nH_n$.
3. $G_{4n+2} = G_{3n+1}H_{n+1} - (-s)^{n+1}G_{2n} = ((H_n^2 - (-s)^n)G_{n+1} - (-s)^n H_n)H_{n+1} - (-s)^{n+1}G_nH_n$.
4. $G_{4n+2} = rH_{4n} + s^2G_{4n-2} = \frac{1}{2}(H_{4n}G_2 + H_2G_{4n}) = \frac{1}{2}(r((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n}) + (r^2+2s)(H_n^2 - 2(-s)^n)G_{2n})$.

(x)

1. $8H_{4n+2} = (r^2+4s)G_{n+1}^2H_n^2 + (r^2+4s)G_n^2H_{n+1}^2 + 4(r^2+4s)G_nG_{n+1}H_nH_{n+1} + (r^2+4s)^2G_n^2G_{n+1}^2 + H_n^2H_{n+1}^2$.
2. $H_{4n+2} = (r^2+4s)G_{2n+1}^2 - 2s^{2n+1}$.
3. $H_{4n+2} = (r^2+4s)G_{3n+1}G_{n+1} + (-s)^{n+1}H_{2n} = (r^2+4s)((H_n^2 - (-s)^n)G_{n+1} - (-s)^n H_n)G_{n+1} + (-s)^{n+1}(H_n^2 - 2(-s)^n)$.
4. $H_{4n+2} = H_{3n+1}H_{n+1} - (-s)^{n+1}H_{2n} = ((H_n^2 - (-s)^n)H_{n+1} - r(-s)^n H_n)H_{n+1} - (-s)^{n+1}(H_n^2 - 2(-s)^n)$.
5. $H_{4n+2} = r(r^2+4s)G_{4n} + (-s)^2H_{4n-2} = \frac{1}{2}((r^2+4s)G_2G_{4n} + H_2H_{4n}) = \frac{1}{2}(r(r^2+4s)(H_n^2 - 2(-s)^n)G_{2n} + (r^2+2s)((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n}))$.

(xi)

1. $8G_{4n+3} = G_{n+1}H_{n+2}H_n^2 + 2G_nH_nH_{n+1}H_{n+2} + (r^2+4s)G_n^2G_{n+1}H_{n+2} + 2(r^2+4s)G_nG_{n+1}G_{n+2}H_n + (r^2+4s)G_n^2G_{n+2}H_{n+1} + G_{n+2}H_n^2H_{n+1}$.
2. $G_{4n+3} = H_{3n+2}G_{n+1} + (-s)^{n+1}G_{2n+1} = (H_nH_{n+1} - r(-s)^n)G_{n+1}H_{n+1} - s(-s)^n G_nH_{n+1} + s(-s)^n H_nG_{n+1} - s(-s)^{2n}$.
3. $G_{4n+3} = H_{3n+2}G_{n+1} + (-s)^{n+1}G_{2n+1} = (H_nH_{n+1} - r(-s)^n)G_{n+1}H_{n+1} + s(-s)^{2n}$.
4. $G_{4n+3} = G_{3n+2}H_{n+1} - (-s)^{n+1}G_{2n+1} = (G_nH_{n+1} + (-s)^n)H_{n+1}^2 + 2s(-s)^n G_nH_{n+1} + s(-s)^{2n}$.
5. $G_{4n+3} = G_{3n+2}H_{n+1} - (-s)^{n+1}G_{2n+1} = (G_nH_{n+1} + (-s)^n)H_{n+1}^2 + s(-s)^n G_nH_{n+1} + s(-s)^n H_nG_{n+1} - s(-s)^{2n}$.
6. $G_{4n+3} = (r^2+s)H_{4n} + (-s)^3G_{4n-3} = \frac{1}{2}(G_3H_{4n} + H_3G_{4n}) = \frac{1}{2}((r^2+s)((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n}) + r(r^2+3s)((H_n^2 - 2(-s)^n)G_{2n}))$.

(xii)

1. $8H_{4n+3} = (r^2+4s)G_{n+1}G_{n+2}H_n^2 + 2(r^2+4s)G_nG_{n+2}H_nH_{n+1} + (r^2+4s)^2G_n^2G_{n+1}G_{n+2} + 2(r^2+4s)G_nG_{n+1}H_nH_{n+2} + (r^2+4s)G_n^2H_{n+1}H_{n+2} + H_n^2H_{n+1}H_{n+2}$.
2. $H_{4n+3} = (r^2+4s)G_{3n+2}G_{n+1} + (-s)^{n+1}H_{2n+1} = (r^2+4s)G_nG_{n+1}H_{n+1}^2 + (-s)^n(r^2+4s)G_{n+1}H_{n+1} - rs(-s)^{2n}$.
3. $H_{4n+3} = (r^2+4s)G_{3n+2}G_{n+1} + (-s)^{n+1}H_{2n+1} = (r^2+4s)H_{n+1}^2G_{n+1}G_n + s(-s)^n(r^2+4s)G_{n+1}G_n + (-s)^n(r^2+4s)G_{n+1}H_{n+1} - s(-s)^n H_nH_{n+1} + rs(-s)^{2n}$.

4. $H_{4n+3} = H_{3n+2}H_{n+1} - (-s)^{n+1}H_{2n+1} = H_{n+1}^2(H_nH_{n+1} - r(-s)^n) + s(-s)^n(r^2+4s)G_{n+1}G_n + s(-s)^nH_nH_{n+1} + rs(-s)^{2n}$.
5. $H_{4n+3} = H_{3n+2}H_{n+1} - (-s)^{n+1}H_{2n+1} = (H_nH_{n+1} - r(-s)^n)H_{n+1}^2 + 2s(-s)^nH_nH_{n+1} - rs(-s)^{2n}$.
6. $H_{4n+3} = (r^2+4s)(r^2+s)G_{4n} + (-s)^3H_{4n-3} = \frac{1}{2}((r^2+4s)G_{4n}G_3 + H_3H_{4n}) = \frac{1}{2}((r^2+4s)(r^2+s)(H_n^2 - 2(-s)^n)G_{2n} + r(r^2+3s)((H_n^2 - 2(-s)^n)^2 - 2(-s)^{2n}))$.

(f)

(i)

1. $G_{n+5} + s^2G_{n+1} = (r^2+2s)G_{n+3}$.
2. $G_{n+5} - s^2G_{n+1} = rH_{n+3}$.
3. $H_{n+5} + s^2H_{n+1} = (r^2+2s)H_{n+3}$.
4. $H_{n+5} - s^2H_{n+1} = r(rH_{n+3} + 2sH_{n+2}) = r(r^2+4s)G_{n+3}$.

(ii)

1. $G_{2n+6} + s^2G_{2n+2} = (r^2+2s)G_{2n+4}$.
2. $G_{2n+6} - s^2G_{2n+2} = rH_{2n+4}$
3. $H_{2n+6} + s^2H_{2n+2} = (r^2+2s)H_{2n+4}$.
4. $H_{2n+6} - s^2H_{2n+2} = r(rH_{2n+4} + 2sH_{2n+3}) = r(r^2+4s)G_{2n+4}$.

(g)

(i)

1. $G_{n+1} - sG_{n-1} = rG_n$.
2. $G_{n+1} + sG_{n-1} = H_n$.
3. $4sG_{n+1}G_{n-1} = H_n^2 - r^2G_n^2 = (H_n + rG_n)(H_n - rG_n) = 4(sG_n^2 + (-s)^n)$
   i.e. $G_{n+1}G_{n-1} = G_n^2 - (-s)^{n-1}$.
4. $G_{n+1}^2 - s^2G_{n-1}^2 = rG_nH_n = rG_{2n}$.
5. $G_{n+1}^2 + s^2G_{n-1}^2 = \frac{1}{2}(r^2G_n^2 + H_n^2)$.
6. $4(G_{n+1}^3 - (-s)^3G_{n-1}^3) = (3r^2G_n^2 + H_n^2)H_n$.
7. $4(G_{n+1}^3 + (-s)^3G_{n-1}^3) = r(r^2G_n^2 + 3H_n^2)G_n$.

(ii)

1. $H_{n+1} - sH_{n-1} = rH_n$.
2. $H_{n+1} + sH_{n-1} = (rH_n + 2sH_{n-1}) = (r^2+4s)G_n$.
3. $4sH_{n+1}H_{n-1} = (r^2+4s)^2G_n^2 - r^2H_n^2$.
4. $H_{n+1}^2 - s^2H_{n-1}^2 = r(r^2+4s)G_nH_n = r(r^2+4s)G_{2n}$.
5. $H_{n+1}^2 + s^2H_{n-1}^2 = \frac{1}{2}(r^2H_n^2 + (r^2+4s)^2G_n^2)$.
6. $4(H_{n+1}^3 + s^3H_{n-1}^3) = (r^2+4s)(3r^2H_n^2 + (r^2+4s)^2G_n^2)G_n$.
7. $4(H_{n+1}^3 - s^3H_{n-1}^3) = r(r^2H_n^2 + 3(r^2+4s)^2G_n^2)H_n$.

Proof.

(a) Take $W_n = G_n$ in Theorem 4.1.

(b) Take $W_n = H_n$ in Theorem 4.1.

(c)

(i) This follows from (a)(i) by using the identity

$$H_m = (2G_{m+1} - rG_m).$$

(ii) This follows from (b)(i) by using the identity

$$(r^2 + 4s)G_m = (2H_{m+1} - rH_m).$$

(iii) Using (a)(ii)-1 and the identity $G_{n+m} = G_n H_m - (-s)^m G_{n-m}$, we obtain

$$(r^2 + 4s)(G_n H_m - (-s)^m G_{n-m}) = H_{n+1}H_m + sH_n H_{m-1}.$$

(iv) Using (a)(ii)-1 and the identity $H_{n+m} = H_n H_m - (-s)^m H_{n-m}$, i.e.,

$$H_{n+m+1} = H_{n+1}H_m - (-s)^m H_{n-m+1},$$

which implies $H_{n+1}H_m = H_{n+m+1} + (-s)^m H_{n-m+1}$, we obtain

$$(r^2 + 4s)G_{n+m} = H_{n+m+1} + (-s)^m H_{n-m+1} + sH_n H_{m-1}.$$

(v) Since $(r^2 + 4s)G_n = 2H_{n+1} - rH_n$, from (a)(ii)-1 we derive additional formulas for the sequence G_n: (i) take $n \mapsto n$, $m \mapsto m-1$; (ii) $n \mapsto n-m$, $m \mapsto m$; (iii) $n \mapsto n-1$, $m \mapsto m$; (iv) $n \mapsto m-1$, $m \mapsto n-m+1$.

(vi) Further substitutions in (a)(ii)-1 yield additional identities: (i) $n \mapsto n$, $m \mapsto n+1$; (ii) $n \mapsto 2n$, $m \mapsto 1$ (and $n \mapsto n$, $m \mapsto n+1$; $n \mapsto n+1$, $m \mapsto n$); (iii) $n \mapsto 3n-1$, $m \mapsto 1$ (and $n \mapsto 2n$, $m \mapsto n$; $n \mapsto n$, $m \mapsto 2n$); (iv) $n \mapsto 3n$, $m \mapsto 1$ (and $n \mapsto 2n+1$, $m \mapsto n$; $n \mapsto 2n$, $m \mapsto n+1$); (v) $n \mapsto 3n+1$, $m \mapsto 1$ (and $n \mapsto 2n+1$, $m \mapsto n+1$; $n \mapsto 2n+2$, $m \mapsto n$); (vi) $n \mapsto 4n-1$, $m \mapsto 1$ (and $n \mapsto 3n-1$, $m \mapsto n+1$; $n \mapsto 2n$, $m \mapsto 2n$); (vii) $n \mapsto 5n-1$, $m \mapsto 1$ (and $n \mapsto 4n-1$, $m \mapsto n+1$; $n \mapsto 3n$, $m \mapsto 2n$).

(d)

(i) (1) Take $n \mapsto n$, $m \mapsto n$ in (c)(i). (2) Take $n \mapsto n+1$, $m \mapsto n-1$ in (a)(ii)-2. (3) Take $n \mapsto n+1$, $m \mapsto n-1$ in (a)(ii)-1. (4) Take $n \mapsto n+2$, $m \mapsto n-2$ in (a)(ii)-2.

(ii) (1) Take $n \mapsto n$, $m \mapsto n$ in (b)(ii)-1. (2) Take $n \mapsto n$, $m \mapsto n$ in (c)(ii). (3) Take $n \mapsto n$, $m \mapsto n$ in (b)(ii)-3. (4) Use (1) and (2). (5) Take $n \mapsto n+1$, $m \mapsto n-1$ in (b)(ii)-1. (6) Take $n \mapsto n$, $m \mapsto n$ in (b)(ii)-1. (7) Take $n \mapsto n+1$, $m \mapsto n-1$ in (b)(ii)-2.

(iii) (1) Take $n \mapsto 2n$, $m \mapsto n$ in (c)(i), then use (i) and (ii). (2) Take $n \mapsto 2n$, $m \mapsto n$ in (a)(ii)-1. (3) Take $n \mapsto 2n$, $m \mapsto n$ in (a)(ii)-2. (4) Take $n \mapsto 3n-3$, $m \mapsto 3$ in (a)(ii)-1. (5) Using the identities

$$4G_{3n} = G_n(3H_n^2 + (r^2 + 4s)G_n^2), \qquad G_{3n} = G_n H_n^2 - (-s)^n G_n,$$

i.e.,

$$3G_{3n} = 3(G_n H_n^2 - (-s)^n G_n),$$

we obtain

$$\begin{aligned} G_{3n} &= 4G_{3n} - 3G_{3n} = G_n(3H_n^2 + (r^2 + 4s)G_n^2) - 3(G_n H_n^2 - (-s)^n G_n) \\ &= (r^2 + 4s)G_n^3 + 3(-s)^n G_n. \end{aligned}$$

(6) Take $n \mapsto 2n$, $m \mapsto n$ in (a)(ii)-1. (7) Take $n \mapsto 2n+1$, $m \mapsto n-1$ in (a)(ii)-1. (8) Take $n \mapsto 2n+1$, $m \mapsto n-1$ in (a)(ii)-2.

(iv) (1) Take $n \mapsto 2n$, $m \mapsto n$ in (c)(ii), then use (ii) and (iii). (2) Take $n \mapsto 2n$, $m \mapsto n$ in (b)(ii)-1 and use (i)-1. (3) Take $n \mapsto 2n$, $m \mapsto n$ in (b)(ii)-2 and use (ii)-1. (4) Take $n \mapsto 2n+1$, $m \mapsto n-1$ in (b)(ii)-1. (5) Take $n \mapsto 2n+1$, $m \mapsto n-1$ in (b)(ii)-2.

(v) (1) Take $n \mapsto 3n$, $m \mapsto n$ in (c)(i), then use (iii) and (iv). (2) Take $n \mapsto 3n$, $m \mapsto n$ in (a)(ii)-1 and use (iv)-3. (3) Take $n \mapsto 3n$, $m \mapsto n$ in (a)(ii)-2 and use (iii)-2. (4) Use (i)-1 and (ii)-1.

(vi) (1) Take $n \mapsto 3n$, $m \mapsto n$ in (c)(ii), then use (iii) and (iv). (2) Take $n \mapsto 2n$, $m \mapsto 2n$ in (a)(ii)-2. (3) Take $n \mapsto 3n$, $m \mapsto n$ in (b)(ii)-1 and use (ii)-1 and (ii)-4. (4) Take $n \mapsto 3n$, $m \mapsto n$ in (b)(ii)-2 and use (ii)-1 and (ii)-4.

(vii) (1) Take $n \mapsto 4n$, $m \mapsto n$ in (c)(i), then use (v) and (vi). (2) Take $n \mapsto 3n$, $m \mapsto 2n$ in (a)(ii)-1 and use (i)-1 and (iv)-3. (3) Take $n \mapsto 3n$, $m \mapsto 2n$ in (a)(ii)-2 and use (ii)-1 and (iii)-2.

(viii) (1) Take $n \mapsto 4n$, $m \mapsto n$ in (c)(ii), then use (v) and (vi). (2) Take $n \mapsto 3n$, $m \mapsto 2n$ in (b)(ii)-1 and use (i)-1 and (iii)-2. (3) Take $n \mapsto 3n$, $m \mapsto 2n$ in (b)(ii)-2 and use (ii)-1 and (iv)-3.

$$\begin{aligned}
G_{n+m} &= H_nG_m + (-s)^m G_{n-m},\\
G_{n+m} &= G_nH_m - (-s)^m G_{n-m},\\
H_{n+m} &= (r^2+4s)G_nG_m + (-s)^m H_{n-m},\\
H_{n+m} &= H_nH_m - (-s)^m H_{n-m}.
\end{aligned}$$

(e) **(i)** Take $n \mapsto n+1$, $m \mapsto n$ in (c)(i). (2) Take $n \mapsto n+1$, $m \mapsto n$ in (a)(ii)-1. (3) Take $n \mapsto n+1$, $m \mapsto n$ in (a)(ii)-2. (4) Take $n \mapsto 2n$, $m \mapsto 1$ in (a)(ii)-1 and use the identities

$$sG_{2n-1} = G_{2n+1} - rG_{2n}, \qquad H_{2n} = H_n^2 - 2(-s)^n.$$

(ii) (1) Take $n \mapsto n+1$, $m \mapsto n$ in (c)(ii). (2) Take $n \mapsto n+1$, $m \mapsto n$ in (b)(ii)-1. (3) Take $n \mapsto n+1$, $m \mapsto n$ in (b)(ii)-2. (4) Take $n \mapsto 2n$, $m \mapsto 1$ in (b)(ii)-1 and use (i)-1 and (ii)-1.

(iii) (1) Take $n \mapsto 2n+1$, $m \mapsto n$ in (c)(i), then use (i) and (ii). (2) Take $n \mapsto 2n+1$, $m \mapsto n$ in (a)(ii)-1 and use (ii)-2. (3) Take $n \mapsto 2n+1$, $m \mapsto n$ in (a)(ii)-2 and use (ii)-3. (4) Take $n \mapsto 3n$, $m \mapsto 1$ in (a)(ii)-1 and use the identity $sG_{3n-1} = G_{3n+1} - rG_{3n}$.

(iv) (1) Take $n \mapsto 2n+1$, $m \mapsto n$ in (c)(ii), then use (i) and (ii). (2) Take $n \mapsto 2n+1$, $m \mapsto n$ in (b)(ii)-1 and use (ii)-2. (3) Take $n \mapsto 2n+1$, $m \mapsto n$ in (b)(ii)-2 and use (ii)-3. (4) Take $n \mapsto 3n$, $m \mapsto 1$ in (b)(ii)-1 and use the identity $sH_{3n-1} = H_{3n+1} - rH_{3n}$ together with (d)(iii)-2 and (d)(iv)-3.

(v) (1) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (c)(i), then use (i) and (ii). (2) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (a)(ii)-1 and use (ii)-2. (3) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (a)(ii)-2 and use (i)-2. (4) Take $n \mapsto 3n$, $m \mapsto 2$ in (a)(ii)-1 and use the identity

$$G_{3n-2} = \frac{1}{s^2}(-G_{3n+2} + \left(r^2+2s\right)G_{3n})$$

together with (d)(iii)-2 and (d)(iv)-3.

(vi) (1) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (c)(ii), then use (i) and (ii). (2) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (b)(ii)-1 and use (i)-3. (3) Take $n \mapsto 2n+1$, $m \mapsto n+1$ in (b)(ii)-2 and use (ii)-3. (4) Take $n \mapsto 3n$, $m \mapsto 2$ in (b)(ii)-1 and use (d)(iii)-2 and (d)(iv)-3.

(vii) (1) Take $n \mapsto 3n+1$, $m \mapsto n$ in (c)(i), then use (iii) and (iv). (2) Take $n \mapsto 3n+1$, $m \mapsto n$ in (a)(ii)-1 and use (i)-2 and (iv)-3. (3) Take $n \mapsto 3n+1$, $m \mapsto n$ in (a)(ii)-1 and use (i)-3 and (iv)-3. (4) Take $n \mapsto 3n+1$, $m \mapsto n$ in (a)(ii)-2 and use (iii)-3 and (i)-2. (5) Take $n \mapsto 3n+1$, $m \mapsto n$ in (a)(ii)-2 and use (iii)-3 and (i)-3. (6) Take $n \mapsto 4n$, $m \mapsto 1$ in (a)(ii)-1 and use the identity $sG_{4n-1} = G_{4n+1} - rG_{4n}$ together with (d)(v)-3 and (d)(vi)-3.

(viii) (1) Take $n \mapsto 3n+1$, $m \mapsto n$ in (c)(ii), then use (iii) and (iv). (2) Take $n \mapsto 3n+1$, $m \mapsto n$ in (b)(ii)-1 and use (iii)-3 and (i)-2. (3) Take $n \mapsto 3n+1$, $m \mapsto n$ in (b)(ii)-2 and use (iii)-3 and (i)-3. (4) Take $n \mapsto 3n+1$, $m \mapsto n$ in (b)(ii)-1 and use (iv)-3 and (ii)-2. (5) Take $n \mapsto 3n+1$, $m \mapsto n$ in (b)(ii)-2 and use (iv)-3 and (ii)-3. (6) Take $n \mapsto 4n$, $m \mapsto 1$ in (b)(ii)-1 and use the identity $sH_{4n-1} = H_{4n+1} - rH_{4n}$ together with (d)(v)-3 and (d)(vi)-3.

(ix) (1) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (c)(i), then use (iii) and (iv). (2) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (a)(ii)-1 and use (d)(i)-1 and (iv)-3. (3) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (a)(ii)-2 and use (d)(i)-1 and (iii)-3. (4) Take $n \mapsto 4n$, $m \mapsto 2$ in (a)(ii)-1 and use (d)(v)-3 and (d)(vi)-3.

(x) (1) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (c)(ii), then use (iii) and (iv). (2) Take $n \mapsto 2n+1$, $m \mapsto 2n+1$ in (a)(ii)-2. (3) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (b)(ii)-1 and use (iii)-3 and (d)(ii)-1. (4) Take $n \mapsto 3n+1$, $m \mapsto n+1$ in (b)(ii)-2 and use (iv)-3 and (d)(ii)-1. (5) Take $n \mapsto 4n$, $m \mapsto 2$ in (b)(ii)-1 and use (d)(v)-3 and (d)(vi)-3.

(xi) (1) Take $n \mapsto 3n+1$, $m \mapsto n+2$ in (c)(i), then use (iii) and (iv). (2) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (a)(ii)-1 and use (vi)-3 and (i)-2. (3) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (a)(ii)-1 and use (vi)-3 and (i)-3. (4) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (a)(ii)-2 and use (v)-3 and (i)-2. (5) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (a)(ii)-2 and use (v)-3 and (i)-3. (6) Take $n \mapsto 4n$, $m \mapsto 3$ in (a)(ii)-1 and use (d)(v)-3 and (d)(vi)-3.

(xii) (1) Take $n \mapsto 3n+1$, $m \mapsto n+2$ in (c)(ii), then use (iii) and (iv). (2) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (b)(ii)-1 and use (v)-3 and (ii)-2. (3) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (b)(ii)-1 and use (v)-3 and (ii)-3. (4) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (b)(ii)-2 and use (vi)-3 and (ii)-2. (5) Take $n \mapsto 3n+2$, $m \mapsto n+1$ in (b)(ii)-2 and use (vi)-3 and (ii)-3. (6) Take $n \mapsto 4n$, $m \mapsto 3$ in (b)(ii)-1 and use (d)(v)-3 and (d)(vi)-3.

(f)

(i)

1. Take $n \to n+3$ and $m = 2$ in (a) (ii)-1.
2. Take $n \to n+3$ and $m = 2$ in (a) (ii)-2.
3. Take $n \to n+3$ and $m = 2$ in (a) (iii)-1.
4. Take $n \to n+3$ and $m = 2$ in (a) (iii)-2.

(ii)

1. Take $n \to 2n+4$ and $m = 2$ in (a) (ii)-1.
2. Take $n \to 2n+4$ and $m = 2$ in (a) (ii)-2.
3. Take $n \to 2n+4$ and $m = 2$ in (a) (iii)-1.
4. Take $n \to 2n+4$ and $m = 2$ in (a) (iii)-2.

(g)

(i) Take $m = 1$ in (a).

(ii) Take $m = 1$ in (b). □

From Theorem 4.1 the following results follow.

Corollary 4.3. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i) $W_n H_n = W_{2n} + (-s)^n W_0$.

(ii) $W_n G_{n+1} = W_{n+1} G_n + (-s)^n W_0$.

(iii) $W_{n+1} H_n = W_{2n+1} + (-s)^n W_1$.

(iv) $W_{n+1} G_{n+1} = W_{n+2} G_n + (-s)^n W_1$.

(b)

(i) $G_{2n} = G_n H_n$.

(ii) $G_{2n+1} = G_{n+1} H_n - (-s)^n$.

(iii) $G_{n+1}^2 = G_{n+2} G_n + (-s)^n$.

(c)

(i) $H_{2n} = H_n^2 - 2(-s)^n = (r^2 + 4s)G_n^2 + 2(-s)^n$.

(ii) $H_n G_{n+1} = H_{n+1} G_n + 2(-s)^n$.

(iii) $H_{2n+1} = H_{n+1} H_n - r(-s)^n$.

(iv) $H_{n+1} G_{n+1} = H_{n+2} G_n + r(-s)^n$.

(d) $H_n^2 = (r^2 + 4s)G_n^2 + 4(-s)^n$,
i.e., $H_n^2 - (r^2 + 4s)G_n^2 = 4(-s)^n$.

Proof.

(a)

(i) Take $n \to n$ and $m = n$ in Theorem 4.1 (b) (i) which gives $H_{2n} = H_n^2 - 2(-s)^n$. Subtracting this from $2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2$ yields $H_{2n} = (r^2 + 4s)G_n^2 + 2(-s)^n$.

(ii) Take $n \to n$ and $m = n$ in Theorem 4.1 (c).

(iii) Take $n \to n + 1$ and $m = n$ in Theorem 4.1 (b).

(iv) Take $n \to n + 1$ and $m = n$ in Theorem 4.1 (c).

(b) Take $W_n = G_n$ in (a). Note that identity (ii) can also be obtained using Binet's formulas as follows:
For $\alpha \neq \beta$,

$$(-s)^n = \alpha^n \beta^n = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}(\alpha^n + \beta^n) - \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} = G_{n+1} H_n - G_{2n+1}.$$

For $\alpha = \beta$,

$$(-s)^n = \alpha^n \beta^n = \alpha^{2n} = (n+1)\alpha^{(n+1)-1} \times 2\alpha^n - (2n+1)\alpha^{(2n+1)-1} = G_{n+1} H_n - G_{2n+1}.$$

(c) Take $W_n = H_n$ in (b).

(d) Subtract (c) (i) twice from the identity $2H_{2n} = (r^2 + 4s)G_n^2 + H_n^2$ which was given in a previous corollary. Alternatively, the proof can be given as follows:

$$4(-s)^n = 4\alpha^n \beta^n = 2\alpha^n.2\beta^n = (H_n + \sqrt{r^2 + 4s}G_n)(H_n - \sqrt{r^2 + 4s}G_n) = H_n^2 - (r^2 + 4s)G_n^2$$

by Lemma 1.13. □

Note that all identities in the previous corollary can also be proved by induction.

We now recollect several identities from Theorem 4.1 for the sake of completeness.

Lemma 4.4. *For all integers m, n, we have the following identities*

(a)

(i)

1. $2W_{n+m} = 2G_nW_{m+1} + (H_n - rG_n)W_m$
2. $2W_{n+m} = W_nH_m + (W_1H_n + sW_0H_{n-1})G_m$
3. $2W_{n+m} = W_nH_m + (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m$
4. $W_{n+m} = W_{n+1}G_m + W_n(G_{m+1} - rG_m)$
5. $2W_{n+m} = 2G_mW_{n+1} + (H_m - rG_m)W_n$

(ii)

1. $2(-s)^mW_{n-m} = -(2W_{m+1} - rW_m)G_n + 2W_nH_m - W_mH_n$
2. $2(-s)^mW_{n-m} = W_nH_m - (W_1H_n + sW_0H_{n-1})G_m$
3. $2(-s)^mW_{n-m} = W_nH_m - (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m$
4. $(-s)^mW_{n-m} = -W_{n+1}G_m + W_nG_{m+1}$
5. $2(-s)^mW_{n-m} = -2G_mW_{n+1} + (H_m + rG_m)W_n$

(b)

(i)

1. $2G_{n+m} = 2G_nG_{m+1} + (H_n - rG_n)G_m$.
2. $2G_{n+m} = G_nH_m + H_nG_m$.
3. $G_{n+m} = G_{n+1}G_m + G_n(G_{m+1} - rG_m)$.
4. $2G_{n+m} = 2G_mG_{n+1} + (H_m - rG_m)G_n$.

(ii)

1. $2(-s)^mG_{n-m} = -(2G_{m+1} - rG_m)G_n + 2G_nH_m - G_mH_n$.
2. $2(-s)^mG_{n-m} = G_nH_m - H_nG_m$.
3. $(-s)^mG_{n-m} = -G_{n+1}G_m + G_nG_{m+1}$.
4. $2(-s)^mG_{n-m} = -2G_mG_{n+1} + (H_m + rG_m)G_n$.

(c)

(i)

1. $2H_{n+m} = 2G_nH_{m+1} + (H_n - rG_n)H_m$.
2. $2H_{n+m} = H_nH_m + (rH_n + 2sH_{n-1})G_m$.
3. $2H_{n+m} = H_nH_m + (2H_{n+1} - rH_n)G_m$.
4. $H_{n+m} = H_{n+1}G_m + H_n(G_{m+1} - rG_m)$.
5. $2H_{n+m} = 2G_mH_{n+1} + (H_m - rG_m)H_n$.

(ii)

1. $2(-s)^m H_{n-m} = -(2H_{m+1} - rH_m)G_n + 2H_nH_m - H_mH_n.$
2. $2(-s)^m H_{n-m} = H_nH_m - (rH_n + 2sH_{n-1})G_m.$
3. $2(-s)^m H_{n-m} = H_nH_m - (2H_{n+1} - rH_n)G_m.$
4. $(-s)^m H_{n-m} = -H_{n+1}G_m + H_nG_{m+1}.$
5. $2(-s)^m H_{n-m} = -2G_mH_{n+1} + (H_m + rG_m)H_n.$

(a) The proofs was given in Theorem 4.1.

(i) Identities correspond to Theorem 4.1 parts (a)(i), (b)(i), (b)(iii), (c)(i), (f)(i).

(ii) Identities correspond to Theorem 4.1 parts (a)(ii), (b)(ii), (b)(iv), (c)(ii), (f)(ii).

(b) Setting $W_n = G_n$ with $G_0 = 0$ and $G_1 = 1$ in (a).

(c) Setting $W_n = H_n$ with $H_0 = 2$ and $H_1 = r$ in (a). □

As an application of Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii), namely

$$W_{n+m} + (-s)^m W_{n-m} = W_n^2 H_m^2,$$

and

$$W_{n+m} - (-s)^m W_{n-m} = (W_1H_n + sW_0H_{n-1})G_m = (W_0H_{n+1} + (W_1 - rW_0)H_n)G_m,$$

the following corollary presents several identities for the generalized Fibonacci (Horadam) polynomials W_n, together with their two special cases G_n and H_n.

Corollary 4.5. *For all integers m, n we have the following identities*

(a) $4(-s)^m W_{n+m}W_{n-m} = W_n^2H_m^2 - (W_1H_n + sW_0H_{n-1})^2G_m^2.$

(b) $W_{n+m}^2 - (-s)^{2m}W_{n-m}^2 = (W_1H_n + sW_0H_{n-1})G_mH_mW_n = (W_1H_n + sW_0H_{n-1})W_nG_{2m}.$

(c) $W_{n+m}^2 + (-s)^{2m}W_{n-m}^2 = \frac{1}{2}(W_n^2H_m^2 + (W_1H_n + sW_0H_{n-1})^2G_m^2).$

(d) $4(W_{n+m}^3 - (-s)^{3m}W_{n-m}^3) = G_m(W_1H_n + sW_0H_{n-1})(3W_n^2H_m^2 + (W_1H_n + sW_0H_{n-1})^2G_m^2).$

(e) $4(W_{n+m}^3 + (-s)^{3m}W_{n-m}^3) = (W_n^2H_m^2 + 3(W_1H_n + sW_0H_{n-1})^2G_m^2)W_nH_m.$

(f) $(-W_1^2 + sW_0^2 + rW_0W_1)(W_{n+m} + (-s)^mW_{n-m}) = W_n((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m).$

(g) $(-W_1^2 + sW_0^2 + rW_1W_0)(W_{n+m} - (-s)^mW_{n-m}) = (rW_n + 2sW_{n-1})(W_0W_{m+1} - W_1W_m).$

(h) $4(-s)^m(-W_1^2 + sW_0^2 + rW_1W_0)^2W_{n+m}W_{n-m} = ((-2W_1 + rW_0)W_{m+1} + (rW_1 + 2sW_0)W_m)^2W_n^2 - (rW_n + 2sW_{n-1})^2(W_0W_{m+1} - W_1W_m)^2.$

(i) $(-W_1^2+sW_0^2+rW_0W_1)^2(W_{n+m}^2-(-s)^{2m}W_{n-m}^2) = (rW_n+2sW_{n-1})(W_0W_{m+1}-W_1W_m)((-2W_1+rW_0)W_{m+1}+(rW_1 + 2sW_0)W_m)W_n.$

(j) $(-W_1^2+sW_0^2+rW_0W_1)^2(W_{n+m}^2+(-s)^{2m}W_{n-m}^2) = \frac{1}{2}(W_n^2((-2W_1+rW_0)W_{m+1}+(rW_1+2sW_0)W_m)^2+(rW_n + 2sW_{n-1})^2(W_0W_{m+1} - W_1W_m)^2).$

(k) $4(-W_1^2+sW_0^2+rW_0W_1)^3(W_{n+m}^3-(-s)^{3m}W_{n-m}^3) = (W_0W_{m+1}-W_1W_m)(rW_n+2sW_{n-1})(3W_n^2((-2W_1+rW_0)W_{m+1}+(rW_1+2sW_0)W_m)^2+(rW_n+2sW_{n-1})^2(W_0W_{m+1}-W_1W_m)^2)$.

(l) $4(-W_1^2+sW_0^2+rW_0W_1)^3(W_{n+m}^3+(-s)^{3m}W_{n-m}^3) = (W_n^2((-2W_1+rW_0)W_{m+1}+(rW_1+2sW_0)W_m)^2+3(rW_n+2sW_{n-1})^2(W_0W_{m+1}-W_1W_m)^2)W_n((-2W_1+rW_0)W_{m+1}+(rW_1+2sW_0)W_m)$.

Proof.

(a) Substituting $a=W_{n+m}$ and $b=(-s)^mW_{n-m}$ in the identity

$$4ab=(a+b)^2-(a-b)^2$$

and using Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii), we get the required identity

$$\begin{aligned} 4(-s)^mW_{n+m}W_{n-m} &= (W_{n+m}+(-s)^mW_{n-m})^2-(W_{n+m}-(-s)^mW_{n-m})^2 \\ &= W_n^2H_m^2-(W_1H_n+sW_0H_{n-1})^2G_m^2. \end{aligned}$$

(b) By applying Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii)) and using the identity

$$a^2-b^2=(a+b)(a-b)$$

we get

$$\begin{aligned} W_{n+m}^2-(-s)^{2m}W_{n-m}^2 &= =(W_{n+m}+(-s)^mW_{n-m})(W_{n+m}-(-s)^mW_{n-m}) \\ &= W_nH_m(W_1H_n+sW_0H_{n-1})G_m \\ &= (W_1H_n+sW_0H_{n-1})G_mH_mW_n. \end{aligned}$$

(c) Substituting $a=W_{n+m}$ and $b=(-s)^mW_{n-m}$ in the identity

$$a^2+b^2=(a+b)^2-2ab$$

or and using Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii), we get the required identity

$$\begin{aligned} W_{n+m}^2+s^{2m}W_{n-m}^2 &= W_{n+m}^2+(-s)^{2m}W_{n-m}^2 \\ &= W_n^2H_m^2-\frac{1}{2}(W_n^2H_m^2-(W_1H_n+sW_0H_{n-1})^2G_m^2) \\ &= \frac{1}{2}(W_n^2H_m^2+(W_1H_n+sW_0H_{n-1})^2G_m^2). \end{aligned}$$

(d) Substituting $a=W_{n+m}$ and $b=(-s)^mW_{n-m}$ in the identity

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)((a+b)^2-ab)$$

and using (a) with Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii), we obtain the required identity.

(e) Substituting $a=W_{n+m}$ and $b=(-s)^mW_{n-m}$ in the identity

$$a^3+b^3=(a+b)(a^2-ab+b^2)=(a+b)((a+b)^2-3ab)$$

and using (a) with Theorem 4.1 (e)(i) and Theorem 4.1 (e)(ii), we obtain

$$\begin{aligned} W_{n+m}^3+(-s)^{3m}W_{n-m}^3 &= W_nH_m(W_n^2H_m^2-\frac{3}{4}(W_n^2H_m^2-(W_1H_n+sW_0H_{n-1})^2G_m^2)) \\ &= \frac{1}{4}(W_n^2H_m^2+3(W_1H_n+sW_0H_{n-1})^2G_m^2)W_nH_m. \end{aligned}$$

(f) By using Theorem 4.1 (e)(i) and (1.13) we get required identity.

(g) By using Theorem 4.1 (e)(ii), (1.14) and (1.15) we get required identity.

(h) By using (a), (1.13), (1.14) and (1.15) we get required identity.

(i) By using (b), (1.13), (1.14) and (1.15) we get required identity.

(j) By using (c), (1.13), (1.14) and (1.15) we get required identity.

(k) By using (d), (1.13), (1.14) and (1.15) we get required identity.

(l) By using (e), (1.13), (1.14) and (1.15) we get required identity. □

5 Identities: Group IV

The following theorem establishes several identities for W_{kn+m}, together with their two special cases G_{kn+m} and H_{kn+m}.

Theorem 5.1. *Let $n, m, k \in \mathbb{Z}$. Then, the following formulas hold.*

(a)

(i) $W_{kn+m} = \frac{1}{2}((W_1 H_{kn} + sW_0 H_{kn-1})G_m + H_m W_{kn})$.

(ii) $G_{kn+m} = \frac{1}{2}(G_m H_{kn} + H_m G_{kn})$.

(iii) $H_{kn+m} = \frac{1}{2}((r^2 + 4s)G_m G_{kn} + H_m H_{kn})$.

(b) **(i)** $W_n = \frac{1}{2}(W_0 H_n + (2W_1 - rW_0)G_n)$.

(ii) $W_{kn} = \frac{1}{2}(W_0 H_{kn} + (2W_1 - rW_0)G_{kn})$.

(iii) $W_{kn+m} = \frac{1}{2}(W_0 H_{kn+m} + (2W_1 - rW_0)G_{kn+m})$.

(c)

(i) $W_{kn+m} = \frac{1}{4}((2W_1 G_m + W_0 H_m) H_{kn} - (rW_0 - W_1 H_0) H_m G_{kn} + 2sW_0 G_m H_{kn-1})$.

(ii) $H_{kn+m} = \frac{1}{2}((H_m + rG_m)H_{kn} + 2sG_m H_{kn-1})$.

Proof.

(a)

(i) We use Binet's formulas of W_n, G_n and H_n. If $\alpha \neq \beta$ (in this case $s = -\alpha\beta$), then we get

$$\begin{aligned}
W_{kn+m} &= \left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{kn+m} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{kn+m}\right) \\
&= \frac{1}{2}((W_1(\alpha^{kn} + \beta^{kn}) + (-\alpha\beta)W_0(\alpha^{kn-1} + \beta^{kn-1}))\left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) \\
&\quad +(\alpha^m + \beta^m)\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^{kn} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^{kn}\right)) \\
&= \frac{1}{2}((W_1 H_{kn} + sW_0 H_{kn-1})G_m + H_m W_{kn}),
\end{aligned}$$

If $\alpha = \beta$ (in this case $s = -\alpha^2$) then we obtain

$$\begin{aligned} W_{kn+m} &= ((kn+m)W_1 - \alpha\,((kn+m)-1)\,W_0)\alpha^{kn+m-1} \\ &= \frac{1}{2}((W_1 \times 2\alpha^{kn} + (-\alpha^2)W_0 \times 2\alpha^{kn-1}) \times m\alpha^{m-1} \\ &\quad +2\alpha^m(knW_1 - \alpha\,(kn-1)\,W_0)\alpha^{kn-1}) \\ &= \frac{1}{2}((W_1 H_{kn} + sW_0 H_{kn-1})G_m + H_m W_{kn}) \end{aligned}$$

This completes the proof.

(ii) Take $W_n = G_n$ in (i).

(iii) Take $W_n = H_n$ in (i). Here, we use Corollary 4.2 (g)(ii) 2., namely $(rH_n + 2sH_{n-1}) = (r^2+4s)G_n$ i.e., $(rH_{kn} + 2sH_{kn-1}) = (r^2+4s)G_{kn}$.

(b) We know from Theorem 4.1 (b) (i) that

$$2W_{n+m} = W_n H_m + (W_1 H_n + sW_0 H_{n-1})G_m.$$

Setting $n = 0, m = n$ in the last identity we get

$$2W_n = W_0 H_n + (2W_1 + sW_0 H_{-1})G_n.$$

Since $H_{-1} = -\frac{r}{s}$, we see that

$$W_n = \frac{1}{2}(W_0 H_n + (2W_1 - rW_0)G_n).$$

Now it follows that

$$W_{kn} = \frac{1}{2}(W_0 H_{kn} + (2W_1 - rW_0)G_{kn})$$

and

$$W_{kn+m} = \frac{1}{2}(W_0 H_{kn+m} + (2W_1 - rW_0)G_{kn+m}).$$

(c)

(i) Using (a) (i) and (b)(ii), we obtain

$$\begin{aligned} W_{kn+m} &= \frac{1}{2}((W_1 H_{kn} + sW_0 H_{kn-1})G_m + H_m W_{kn}) \\ &= \frac{1}{2}((W_1 H_{kn} + sW_0 H_{kn-1})G_m + H_m \frac{1}{2}(W_0 H_{kn} + (W_1 H_0 - rW_0)G_{kn})) \\ &= \frac{1}{4}((2W_1 G_m + W_0 H_m)\,H_{kn} - (rW_0 - W_1 H_0)\,H_m G_{kn} + 2sW_0 G_m H_{kn-1}). \end{aligned}$$

(ii) Set $W_n = H_n$ in (i). □

In the next theorem, we derive explicit identities for W_{kn+m}, expressed in terms of G_{kn+m} and H_{kn+m}, which follow directly from previously established relations.

Lemma 5.2. *Let $n, m, k \in \mathbb{Z}$. Then*

(a) $W_{kn+m} + (-s)^m W_{kn-m} = \frac{1}{2}((2W_1 - rW_0)(2G_{m+1} - rG_m)G_{kn} + W_0 H_m H_{kn})$

(b) $(r^2+4s)(W_{kn+m} - (-s)^m W_{kn-m}) = \frac{1}{2}((2W_1 - rW_0)(r^2+4s)G_m(2G_{kn+1} - rG_{kn}) + W_0(2H_{m+1} - rH_m)(2H_{kn+1} - rH_{kn}))$

Proof. Apply the identities

$$G_{n+m} = G_m G_{n+1} + (G_{m+1} - rG_m)G_n,$$

and

$$(r^2 + 4s)H_{n+m} = (2H_{m+1} - rH_m)H_{n+1} + (-rH_{m+1} + (r^2 + 2s)H_m)H_n,$$

given in Corollary 2.2 (a)(i) and Theorem 3.1 (b)(i), respectively, together with Theorem 5.1 (b)(iii). □

6 Identities on W_n and W_{-n}: Group V

Next, we derive several identities for the generalized Fibonacci (Horadam) polynomials.

Lemma 6.1. *Let $n, m \in \mathbb{Z}$. Then*

(a) $(r^2+4s)W_nW_m = (-rW_1+(r^2+2s)W_0)(W_{n+m}+(-s)^mW_{n-m})+(2W_1-rW_0)(W_{n+m+1}+(-s)^{m+1}W_{n-(m+1)})$.

(b)

- **(i)** $(2W_1-rW_0)W_{2n} = (r^2+4s)W_nW_{n-1}-(r^2W_0+2sW_0-rW_1)W_{2n-1}-(-s)^{n-1}(r^2W_0+2sW_0-rW_1)W_1-(-s)^n(2W_1-rW_0)W_0$.
- **(ii)** $(2W_1-rW_0)W_{3n} = (r^2+4s)W_{2n}W_{n-1}-(r^2W_0+2sW_0-rW_1)W_{3n-1}-(-s)^{n-1}(r^2W_0+2sW_0-rW_1)W_{n+1}-(-s)^n(2W_1-rW_0)W_n$.

(c)

- **(i)** $(r^2+4s)W_{n-m+1}W_{n-m} = (-rW_1+(r^2+2s)W_0)(W_{2n-2m+1}+(-s)^{n-m}W_1)+(2W_1-rW_0)(W_{2n-2m+2}+(-s)^{n-m+1}W_0)$.
- **(ii)** $(r^2+4s)W_nW_{n-1} = (-rW_1+(r^2+2s)W_0)(W_{2n-1}+(-s)^{n-1}W_1)+(2W_1-rW_0)(W_{2n}+(-s)^nW_0)$.
- **(iii)** $(r^2+4s)W_{n-m+2}W_{n-m+1} = (-rW_1+(r^2+2s)W_0)(W_{2n-2m+3}+(-s)^{n-m+1}W_1)+(2W_1-rW_0)(W_{2n-2m+4}+(-s)^{n-m+2}W_0)$.
- **(iv)** $(r^2+4s)W_{n+1}W_n = (-rW_1+(r^2+2s)W_0)(W_{2n+1}+(-s)^nW_1)+(2W_1-rW_0)(W_{2n+2}+(-s)^{n+1}W_0)$.

Proof. We use Binet's formula for W_n, which is given as

$$W_n = \begin{cases} \dfrac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \dfrac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1-\alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

(a) Note that (a) can be written as

$$(r^2+4s)W_nW_m = \Delta_1W_{n+m}+(-s)^m\Delta_1W_{n-m}+\Delta_2W_{n+m+1}+(-s)^{m+1}\Delta_2W_{n-(m+1)}$$

where

$$\Delta_1 = -rW_1+(r^2+2s)W_0, \quad \Delta_2 = 2W_1-rW_0.$$

Suppose that $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$. From Theorem 4.1 (b) we know that

$$W_nH_m = W_{n+m}+(-s)^mW_{n-m}. \tag{6.1}$$

Changing m to $m+1$ in (6.1), we obtain

$$W_nH_{m+1} = W_{n+m+1} + (-s)^{m+1}W_{n-m-1}. \tag{6.2}$$

We have also shown earlier that

$$(4s+r^2)W_n = (2W_1 - rW_0)H_{n+1} + (-rW_1 + (2s+r^2)W_0)H_n.$$

By multiplying (6.1) by $\dfrac{\Delta_1}{r^2+4s}$ and (6.2) by $\dfrac{\Delta_2}{r^2+4s}$ and adding these results we obtain the required result,

$$(r^2+4s)W_nW_m = \Delta_1W_{n+m} + (-s)^m\Delta_1W_{n-m} + \Delta_2W_{n+m+1} + (-s)^{m+1}\Delta_2W_{n-(m+1)}.$$

Suppose now that $\alpha=\beta$, i.e., $r^2+4s=0$ (and in this case $r=2\alpha$, $s=-\alpha^2$). Then

$(-rW_1 + (r^2+2s)W_0)(W_{n+m} + (-s)^mW_{n-m}) + (2W_1 - rW_0)(W_{n+m+1} + (-s)^{m+1}W_{n-(m+1)})$
$= (-2\alpha W_1 + ((2\alpha)^2 + 2(-\alpha^2))W_0)(((n+m)W_1 - \alpha\,((n+m)-1)\,W_0)\alpha^{(n+m)-1} + (\alpha^2)^m((n-m)W_1 - \alpha\,((n-m)-1)\,W_0)\alpha^{(n-m)-1}) + (2W_1 - 2\alpha W_0)(((n+m+1)W_1 - \alpha\,((n+m+1)-1)\,W_0)\alpha^{(n+m+1)-1} + (\alpha^2)^{m+1}((n-(m+1))W_1 - \alpha\,((n-(m+1))-1)\,W_0)\alpha^{(n-(m+1))-1})$
$=0$
$=0\times W_nW_m$
$=(r^2+4s)W_nW_m.$

(b) Take $n\to n$, $m=n-1$; $n\to 2n$, $m=n-1$ in (a), respectively.

(c) **(i)** Set $n\to n-m+1$ and $m\to n-m$ in (a)

(ii) Set $m=1$ in (i) or rearrange (b) (i).

(iii) Take $n\to n+1$ in (i).

(iv) Take $m=1$ in (iii). □

In particular, if we set $W_n=G_n$ with $G_0=0, G_1=1$ in Lemma 6.1, the following corollary follows.

Corollary 6.2. *Let $n,m\in\mathbb{Z}$. Then*

(a)

$$(r^2+4s)G_nG_m = -rG_{n+m} - r(-s)^mG_{n-m} + 2G_{n+m+1} + 2(-s)^{m+1}G_{n-(m+1)}.$$

(b)

$$\begin{aligned} 2G_{2n} &= (r^2+4s)G_nG_{n-1} + rG_{2n-1} + r(-s)^{n-1}, \\ 2G_{3n} &= (r^2+4s)G_{2n}G_{n-1} + rG_{3n-1} + r(-s)^{n-1}G_{n+1} - 2(-s)^nG_n. \end{aligned}$$

Restricting to the specialization $W_n=H_n$ with $H_0=2, H_1=r$ in Lemma 6.1, we infer the following corollary.

Corollary 6.3. *Let $n,m\in\mathbb{Z}$. Then*

(a)

$$H_nH_m = H_{n+m} + (-s)^mH_{n-m}.$$

(b)

$$\begin{array}{rcl} 0 &=& H_nH_{n-1} - H_{2n-1} - r(-s)^{n-1}, \\ 0 &=& H_{2n}H_{n-1} - H_{3n-1} - (-s)^{n-1}H_{n+1}, \end{array}$$

The following lemma provides results concerning W_{-n}.

Lemma 6.4. *Let $n, m \in \mathbb{Z}$. Then*

(a) $W_nW_{-n} = W_0^2 + (-W_1^2 + sW_0^2 + rW_0W_1)(-s)^{-n}G_n^2$.

(b) $W_nW_{-n} = W_0^2 + (W_0W_{n+1} - W_1W_n)(-s)^{-n}G_n$.

(c) $(-W_1^2 + sW_0^2 + rW_1W_0)W_nW_{-n} = (-W_1^2 + sW_0^2 + rW_1W_0)W_0^2 + (W_0W_{n+1} - W_1W_n)^2(-s)^{-n}$.

Proof.

(a) We use Binet's formulas for W_n and G_n. If $\alpha \neq \beta$, then

$$\begin{array}{rcl} W_nW_{-n} &=& \left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n\right)\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{-n} - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{-n}\right) \\ &=& \frac{1}{(\alpha-\beta)^2}\left(-\alpha^nW_1 + \beta^nW_1 + \alpha\alpha^nW_0 - \beta\beta^nW_0\right)\alpha^{-n}\beta^{-n}(\alpha^nW_1 - \beta^nW_1 + \alpha\beta^nW_0 - \alpha^n\beta W_0) \\ &=& W_0^2 + (-W_1^2 + sW_0^2 + rW_0W_1)(\alpha\beta)^{-n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)^2 \\ &=& W_0^2 + (-W_1^2 + sW_0^2 + rW_0W_1)(-s)^{-n}G_n^2. \end{array}$$

Now, suppose that $\alpha = \beta$. In this case $-s = \alpha^2$ and $r = 2\alpha$. Then, we obtain

$$\begin{array}{rcl} W_nW_{-n} &=& (nW_1 - \alpha(n-1)W_0)\alpha^{n-1}(-nW_1 - \alpha(-n-1)W_0)\alpha^{-n-1} \\ &=& W_0^2 + (-W_1^2 - \alpha^2W_0^2 + 2\alpha W_0W_1)(\alpha^2)^{-n}\left(n\alpha^{n-1}\right)^2 \\ &=& W_0^2 + (-W_1^2 + sW_0^2 + rW_0W_1)(-s)^{-n}G_n^2. \end{array}$$

This completes the proof. Note that proof can also be done by using induction.

(b) Use (a) together with the identity $(-W_1^2 + sW_0^2 + rW_1W_0)G_n = W_0W_{n+1} - W_1W_n$.

(c) Use (b) together with the same identity $(-W_1^2 + sW_0^2 + rW_1W_0)G_n = W_0W_{n+1} - W_1W_n$. □

Specializing to the case $W_n = G_n$ with $G_0 = 0, G_1 = 1$ in Lemma 6.4, we deduce the following corollary.

Corollary 6.5. *Let $n, m \in \mathbb{Z}$. Then $G_{-n} = -(-s)^{-n}G_n$.*

Setting $W_n = H_n$ with $H_0 = 2, H_1 = r$ in Lemma 6.4, we arrive at the following corollary.

Corollary 6.6. *Let $n, m \in \mathbb{Z}$. Then*

(a) $H_nH_{-n} = 4 + (r^2 + 4s)(-s)^{-n}G_n^2$.

(b) $H_nH_{-n} = 4 + (2H_{n+1} - rH_n)(-s)^{-n}G_n$.

(c) $(r^2 + 4s)H_nH_{-n} = 4(r^2 + 4s) + (2H_{n+1} - rH_n)^2(-s)^{-n}$.

7 Identities: Group VI

In this section, we derive several identities for W_n, G_n, and H_n , some obtained solely from well-known binomial identities, and others established through binomial identities in combination with Cassini's or Catalan's identities.

We begin with the following theorem, which presents several identities for W_n, G_n and H_n.

Theorem 7.1. *Let $n, m \in \mathbb{Z}$. Then*

(a)

- **(i)** $W_{n-m+2}^2 - r^2 W_{n-m+1}^2 - s^2 W_{n-m}^2 = 2rsW_{n-m+1}W_{n-m}$.
- **(ii)** $W_{n+1}^2 - r^2 W_n^2 - s^2 W_{n-1}^2 = 2rsW_nW_{n-1}$.
- **(iii)** $G_{n-m+2}^2 - r^2 G_{n-m+1}^2 - s^2 G_{n-m}^2 = 2rsG_{n-m+1}G_{n-m}$.
- **(iv)** $G_{n+1}^2 - r^2 G_n^2 - s^2 G_{n-1}^2 = 2rsG_nG_{n-1}$.
- **(v)** $H_{n-m+2}^2 - r^2 H_{n-m+1}^2 - s^2 H_{n-m}^2 = 2rsH_{n-m+1}H_{n-m}$.
- **(vi)** $H_{n+1}^2 - r^2 H_n^2 - s^2 H_{n-1}^2 = 2rsH_nH_{n-1}$.

(b)

- **(i)** $(r^2+4s)(W_{n-m+2}^2 - r^2 W_{n-m+1}^2 - s^2 W_{n-m}^2) = 2rs((-rW_1 + (r^2+2s)W_0)(W_{2n-2m+1} + (-s)^{n-m}W_1) + (2W_1 - rW_0)(W_{2n-2m+2} + (-s)^{n-m+1}W_0))$.
- **(ii)** $(r^2+4s)(W_{n+1}^2 - r^2 W_n^2 - s^2 W_{n-1}^2) = 2rs((-rW_1+(r^2+2s)W_0)(W_{2n-1}+(-s)^{n-1}W_1)+(2W_1-rW_0)(W_{2n}+(-s)^nW_0))$.
- **(iii)** $(r^2+4s)(G_{n-m+2}^2 - r^2 G_{n-m+1}^2 - s^2 G_{n-m}^2) = 2rs(2G_{2n-2m+2} - rG_{2n-2m+1} - r(-s)^{n-m})$.
- **(iv)** $(r^2+4s)(G_{n+1}^2 - r^2 G_n^2 - s^2 G_{n-1}^2) = 2rs(2G_{2n} - rG_{2n-1} - r(-s)^{n-1})$.
- **(v)** $H_{n-m+2}^2 - r^2 H_{n-m+1}^2 - s^2 H_{n-m}^2 = 2rs(H_{2n-2m+1} + r(-s)^{n-m})$.
- **(vi)** $H_{n+1}^2 - r^2 H_n^2 - s^2 H_{n-1}^2 = 2rs(H_{2n-1} + r(-s)^{n-1})$.

Proof.

(a)

- **(i)** Set $a = rW_{n-m+1},\ b = sW_{n-m}$ in the identity $(a+b)^2 - a^2 - b^2 = 2ab$. Note that in this case, $a+b = rW_{n-m+1} + sW_{n-m} = W_{n-m+2},\ ab = rsW_{n-m+1}W_{n-m}$.
- **(ii)** Replace $m=1$ in (i).
- **(iii)-(iv)** Substitute $W_n = G_n$ in (i) and (ii).
- **(v)-(vi)** Substitute $W_n = H_n$ in (i) and (ii).

(b)

- **(i)** Use (a)(i) with together Lemma 6.1 (c) (i).
- **(ii)** Replace $m=1$ in (i).

(iii)-(iv) Substitute $W_n = G_n$ in (i) and (ii).

(v)-(vi) Substitute $W_n = H_n$ in (i) and (ii).

The following theorem establishes several identities for W_n, together with G_n and H_n.

Theorem 7.2. *Let $n, m \in \mathbb{Z}$. Then*

(a)

(i) $W_{n-m+2}^3 - r^3 W_{n-m+1}^3 - s^3 W_{n-m}^3 = 3rsW_{n-m+2}W_{n-m+1}W_{n-m}$.

(ii) $W_{n+1}^3 - r^3 W_n^3 - s^3 W_{n-1}^3 = 3rsW_{n+1}W_nW_{n-1}$.

(iii) $G_{n-m+2}^3 - r^3 G_{n-m+1}^3 - s^3 G_{n-m}^3 = 3rsG_{n-m+2}G_{n-m+1}G_{n-m}$.

(iv) $G_{n+1}^3 - r^3 G_n^3 - s^3 G_{n-1}^3 = 3rsG_{n+1}G_nG_{n-1}$.

(v) $H_{n-m+2}^3 - r^3 H_{n-m+1}^3 - s^3 H_{n-m}^3 = 3rsH_{n-m+2}H_{n-m+1}H_{n-m}$.

(vi) $H_{n+1}^3 - r^3 H_n^3 - s^3 H_{n-1}^3 = 3rsH_{n+1}H_nH_{n-1}$.

(b)

(i) $W_{n+1}^3 - r(r^2+3s)W_n^3 - s^3 W_{n-1}^3 = -3rs(W_1^2 - sW_0^2 - rW_0W_1)(-s)^{n-1}W_n$.

(ii) $G_{n+1}^3 - r(r^2+3s)G_n^3 - s^3 G_{n-1}^3 = -3rs(-s)^{n-1}G_n$.

(iii) $H_{n+1}^3 - r(r^2+3s)H_n^3 - s^3 H_{n-1}^3 = 3rs(r^2+4s)(-s)^{n-1}H_n$.

(c)

(i) $(r^2+4s)(W_{n-m+2}^3 - r^3 W_{n-m+1}^3 - s^3 W_{n-m}^3) = 3rs((-rW_1 + (r^2+2s)W_0)(W_{2n-2m+3} + (-s)^{n-m+1} W_1) + (2W_1 - rW_0)(W_{2n-2m+4} + (-s)^{n-m+2} W_0))W_{n-m}$.

(ii) $(r^2+4s)(W_{n+1}^3 - r^3 W_n^3 - s^3 W_{n-1}^3) = 3rs((-rW_1 + (r^2+2s)W_0)(W_{2n+1} + (-s)^n W_1) + (2W_1 - rW_0)(W_{2n+2} + (-s)^{n+1} W_0))W_{n-1}$.

(iii) $(r^2+4s)(G_{n-m+2}^3 - r^3 G_{n-m+1}^3 - s^3 G_{n-m}^3) = 3rs(2G_{2n-2m+4} - rG_{2n-2m+3} - r(-s)^{n-m+1})G_{n-m}$.

(iv) $(r^2+4s)(G_{n+1}^3 - r^3 G_n^3 - s^3 G_{n-1}^3) = 3rs(2G_{2n+2} - rG_{2n+1} - r(-s)^n)G_{n-1}$.

(v) $H_{n-m+2}^3 - r^3 H_{n-m+1}^3 - s^3 H_{n-m}^3 = 3rs(H_{2n-2m+3} + r(-s)^{n-m+1})H_{n-m}$.

(vi) $H_{n+1}^3 - r^3 H_n^3 - s^3 H_{n-1}^3 = 3rs(H_{2n+1} + r(-s)^n)H_{n-1}$.

Proof.

(a)

(i) Set $a = rW_{n-m+1},\ b = sW_{n-m}$ in the identity $(a+b)^3 - a^3 - b^3 = 3ab(a+b)$. Note that in this case, $a + b = rW_{n-m+1} + sW_{n-m} = W_{n-m+2},\ ab = rsW_{n-m+1}W_{n-m}$.

(ii) Replace $m = 1$ in (i).

(iii)-(iv) Substitute $W_n = G_n$ in (i) and (ii).

(v)-(vi) Substitute $W_n = H_n$ in (i) and (ii).

(b)

(i) Use (a)(ii) with together Cassini's identity: $W_{n+1}W_{n-1} - W_n^2 = -(-s)^{n-1}(W_1^2 - sW_0^2 - rW_0W_1)$.

(ii) Substitute $W_n = G_n$ in (i).

(iii) Substitute $W_n = H_n$ in (i).

(c)

(i) Use (a)(i) and Lemma 6.1 (c) (iii).

(ii) Replace $m = 1$ in (i).

(iii)-(iv) Substitute $W_n = G_n$ in (i) and (ii).

(v)-(vi) Substitute $W_n = H_n$ in (i) and (ii).

The next theorem displays several identities for W_n, together with G_n and H_n.

Theorem 7.3. *Let $n, m, k \in \mathbb{Z}$. Then*

(a)

(i) $W_{n+1}^4 - r^4W_n^4 - s^4W_{n-1}^4 = 2rs(2W_{n+1}^2 - rsW_nW_{n-1})W_nW_{n-1} = 2rs(2W_{n+2}W_n - rsW_nW_{n-1} + 2(-s)^n(W_1^2 - sW_0^2 - rW_0W_1))W_nW_{n-1}$.

(ii) $G_{n+1}^4 - r^4G_n^4 - s^4G_{n-1}^4 = 2rs(2G_{n+1}^2 - rsG_nG_{n-1})G_nG_{n-1} = 2rs(2G_{n+2}G_n - rsG_nG_{n-1} + 2(-s)^n)G_nG_{n-1}$.

(iii) $H_{n+1}^4 - r^4H_n^4 - s^4H_{n-1}^4 = 2rs(2H_{n+1}^2 - rsH_nH_{n-1})H_nH_{n-1} = 2rs(2H_{n+2}H_n - rsH_nH_{n-1} - 2(-s)^n(r^2 + 4s))H_nH_{n-1}$.

(b)

(i) $W_{n+1}^4 - r^4W_n^4 - s^4W_{n-1}^4 - \frac{4}{3}(W_{n+1}^3 - r^3W_n^3 - s^3W_{n-1}^3)W_{n+1} + \frac{1}{2}(W_{n+1}^2 - r^2W_n^2 - s^2W_{n-1}^2)^2 = 0$.

(ii) $G_{n+1}^4 - r^4G_n^4 - s^4G_{n-1}^4 - \frac{4}{3}(G_{n+1}^3 - r^3G_n^3 - s^3G_{n-1}^3)G_{n+1} + \frac{1}{2}(G_{n+1}^2 - r^2G_n^2 - s^2G_{n-1}^2)^2 = 0$.

(iii) $H_{n+1}^4 - r^4H_n^4 - s^4H_{n-1}^4 - \frac{4}{3}(H_{n+1}^3 - r^3H_n^3 - s^3H_{n-1}^3)H_{n+1} + \frac{1}{2}(H_{n+1}^2 - r^2H_n^2 - s^2H_{n-1}^2)^2 = 0$.

(c)

(i) $W_{n+m}W_{n-m}W_{n+k}W_{n-k} = W_n^4 - \Delta_1W_n^2 + \Delta_2$
where

$$\begin{aligned}\Delta_1 &= (W_1^2 - sW_0^2 - rW_0W_1)((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2),\\ \Delta_2 &= (W_1^2 - sW_0^2 - rW_0W_1)^2(-s)^{2n-m-k}G_m^2G_k^2.\end{aligned}$$

(ii) $W_{n+1}W_{n-1}W_{n+2}W_{n-2} = W_n^4 - (W_1^2 - sW_0^2 - rW_0W_1)(-s)^{n-2}(r^2 - s)W_n^2 + (W_1^2 - sW_0^2 - rW_0W_1)^2(-s)^{2n-3}r^2$.

(iii) $G_{n+m}G_{n-m}G_{n+k}G_{n-k} = G_n^4 - ((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2)G_n^2 + (-s)^{2n-m-k}G_m^2G_k^2$.

(iv) $G_{n+1}G_{n-1}G_{n+2}G_{n-2} = G_n^4 - (-s)^{n-2}(r^2 - s)G_n^2 + (-s)^{2n-3}r^2$.

(v) $H_{n+m}H_{n-m}H_{n+k}H_{n-k} = H_n^4 + (r^2 + 4s)((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2)H_n^2 + (r^2 + 4s)^2(-s)^{2n-m-k}G_m^2G_k^2$.

(vi) $H_{n+1}H_{n-1}H_{n+2}H_{n-2} = H_n^4 + (r^2+4s)(-s)^{n-2}(r^2-s)H_n^2 + (r^2+4s)^2(-s)^{2n-3}r^2.$

Proof.

(a)

(i) Set $a = rW_n,\ b = sW_{n-1}$ in the identity $(a+b)^4 - a^4 - b^4 = 2ab(2(a+b)^2 - ab)$. Note that in this case, $a+b = rW_n + sW_{n-1} = W_{n+1},\ ab = rsW_nW_{n-1}$. Then use Cassini's identity:

$$W_{n+1}W_{n-1} - W_n^2 = -(-s)^{n-1}(W_1^2 - sW_0^2 - rW_0W_1).$$

(ii) Substitute $W_n = G_n$ in (i).

(iii) Substitute $W_n = H_n$ in (i).

(b)

(i) Use (a)(i) with Theorem 7.1 (a)(i) and Theorem 7.2 (a)(iv).

(ii) Substitute $W_n = G_n$ in (i).

(iii) Substitute $W_n = H_n$ in (i).

(c)

(i) Use Catalan's identity: $W_{n+m}W_{n-m} = W_n^2 - (-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$ (see Theorem 1.6).

(ii) Replace $m = 1$ in (i).

(iii)-(iv) Substitute $W_n = G_n$ in (i) and (ii).

(v)-(vi) Substitute $W_n = H_n$ in (i) and (ii).

The following theorem reveals several identities for W_n, together with G_n and H_n.

Theorem 7.4. *Let $n, m, k, p \in \mathbb{Z}$. Then*

(a) $W_{n+m}W_{n-m}W_{n+k}W_{n-k}W_{n+p}W_{n-p} = W_n^6 - \Delta_1 W_n^4 + \Delta_2 W_n^2 - \Delta_3$

where

$$\begin{aligned}
\Delta_1 &= (W_1^2 - sW_0^2 - rW_0W_1)((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2),\\
\Delta_2 &= (W_1^2 - sW_0^2 - rW_0W_1)^2((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-k-p}G_k^2G_p^2),\\
\Delta_3 &= (W_1^2 - sW_0^2 - rW_0W_1)^3(-s)^{3n-m-k-p}G_m^2G_k^2G_p^2.
\end{aligned}$$

(b) $W_{n+1}W_{n-1}W_{n+2}W_{n-2}W_{n+3}W_{n-3} = W_n^6 - (W_1^2 - sW_0^2 - rW_0W_1)(r^4 + r^2s + 2s^2)(-s)^{n-3}W_n^4 + (W_1^2 - sW_0^2 - rW_0W_1)^2(-r^6 - r^4s + s^3)s^{2n-5}W_n^2 - (W_1^2 - sW_0^2 - rW_0W_1)^3(-s)^{3n-6}r^2(r^2+s)^2.$

(c) $G_{n+m}G_{n-m}G_{n+k}G_{n-k}G_{n+p}G_{n-p} = G_n^6 - ((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2)G_n^4 + ((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-k-p}G_k^2G_p^2)G_n^2 - (-s)^{3n-m-k-p}G_m^2G_k^2G_p^2.$

(d) $G_{n+1}G_{n-1}G_{n+2}G_{n-2}G_{n+3}G_{n-3} = G_n^6 - (r^4+r^2s+2s^2)(-s)^{n-3}G_n^4 + (-r^6 - r^4s + s^3)s^{2n-5}G_n^2 - (-s)^{3n-6}r^2(r^2+s)^2.$

(e) $H_{n+m}H_{n-m}H_{n+k}H_{n-k}H_{n+p}H_{n-p} = H_n^6 + (r^2+4s)((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2)H_n^4 + (r^2+4s)^2((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-k-p}G_k^2G_p^2)H_n^2 + (r^2+4s)^3(-s)^{3n-m-k-p}G_m^2G_k^2G_p^2$.

(f) $H_{n+1}H_{n-1}H_{n+2}H_{n-2}H_{n+3}H_{n-3} = H_n^6 + (r^2+4s)(r^4+r^2s+2s^2)(-s)^{n-3}H_n^4 + (r^2+4s)^2(-r^6-r^4s+s^3)s^{2n-5}H_n^2 + (-s)^{3n-6}r^2(r^2+4s)^3(r^2+s)^2$.

Proof.

(a) Use Catalan's identity: $W_{n+m}W_{n-m} = W_n^2 - (-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$ (see Theorem 1.6).

(b) Replace $m=1$ in (a).

(c)-(d) Substitute $W_n = G_n$ in (a) and (b).

(e)-(f) Substitute $W_n = H_n$ in (a) and (b).

The following theorem exhibits several identities for W_n, together with G_n and H_n.

Theorem 7.5. *Let $n, m, k, p, q \in \mathbb{Z}$. Then*

(a) $W_{n+m}W_{n-m}W_{n+k}W_{n-k}W_{n+p}W_{n-p}W_{n+q}W_{n-q} = W_n^8 - \Delta_1W_n^6 + \Delta_2W_n^4 - \Delta_3W_n^2 + \Delta_4$

where

$$\begin{aligned}
\Delta_1 &= ((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2 + (-s)^{n-q}G_q^2)(W_1^2 - sW_0^2 - rW_0W_1),\\
\Delta_2 &= ((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-m-q}G_m^2G_q^2 + (-s)^{2n-k-p}G_k^2G_p^2\\
&\quad + (-s)^{2n-k-q}G_k^2G_q^2 + (-s)^{2n-p-q}G_p^2G_q^2)(W_1^2 - sW_0^2 - rW_0W_1)^2,\\
\Delta_3 &= ((-s)^{3n-m-k-p}G_m^2G_k^2G_p^2 + (-s)^{3n-m-k-q}G_m^2G_k^2G_q^2 + (-s)^{3n-k-p-q}G_k^2G_p^2G_q^2\\
&\quad + (-s)^{3n-m-p-q}G_m^2G_p^2G_q^2)(W_1^2 - sW_0^2 - rW_0W_1)^3,\\
\Delta_4 &= (-s)^{4n-m-k-p-q}G_m^2G_k^2G_p^2G_q^2(W_1^2 - sW_0^2 - rW_0W_1)^4.
\end{aligned}$$

(b) $W_{n+1}W_{n-1}W_{n+2}W_{n-2}W_{n+3}W_{n-3}W_{n+4}W_{n-4} = W_n^8 - \Delta_5W_n^6 + \Delta_6W_n^4 - \Delta_7W_n^2 + \Delta_8$

where

$$\begin{aligned}
\Delta_5 &= (-s)^{n-4}(r^6 + 3r^4s + 3r^2s^2 - 2s^3)(W_1^2 - sW_0^2 - rW_0W_1),\\
\Delta_6 &= s^{2n-7}(-r^{10} - 5r^8s - 11r^6s^2 - 13r^4s^3 - 8r^2s^4 + s^5)(W_1^2 - sW_0^2 - rW_0W_1)^2,\\
\Delta_7 &= (-s)^{3n-9}r^2(r^{10} + 5r^8s + 8r^6s^2 + 2r^4s^3 - 6r^2s^4 - 5s^5)(W_1^2 - sW_0^2 - rW_0W_1)^3,\\
\Delta_8 &= (-s)^{4n-10}r^4(r^2+s)^2(r^2+2s)^2(W_1^2 - sW_0^2 - rW_0W_1)^4.
\end{aligned}$$

(c) $G_{n+m}G_{n-m}G_{n+k}G_{n-k}G_{n+p}G_{n-p}G_{n+q}G_{n-q} = G_n^8 - ((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2 + (-s)^{n-q}G_q^2)G_n^6 + ((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-m-q}G_m^2G_q^2 + (-s)^{2n-k-p}G_k^2G_p^2 + (-s)^{2n-k-q}G_k^2G_q^2 + (-s)^{2n-p-q}G_p^2G_q^2)G_n^4 - ((-s)^{3n-m-k-p}G_m^2G_k^2G_p^2 + (-s)^{3n-m-k-q}G_m^2G_k^2G_q^2 + (-s)^{3n-k-p-q}G_k^2G_p^2G_q^2 + (-s)^{3n-m-p-q}G_m^2G_p^2G_q^2)G_n^2 + (-s)^{4n-m-k-p-q}G_m^2G_k^2G_p^2G_q^2$.

(d) $G_{n+1}G_{n-1}G_{n+2}G_{n-2}G_{n+3}G_{n-3}G_{n+4}G_{n-4} = G_n^8 - (-s)^{n-4}(r^6+3r^4s+3r^2s^2-2s^3)G_n^6 + s^{2n-7}(-r^{10}-5r^8s-11r^6s^2-13r^4s^3-8r^2s^4+s^5)G_n^4 - (-s)^{3n-9}r^2(r^{10}+5r^8s+8r^6s^2+2r^4s^3-6r^2s^4-5s^5)G_n^2 + (-s)^{4n-10}r^4(r^2+s)^2(r^2+2s)^2$.

(e) $H_{n+m}H_{n-m}H_{n+k}H_{n-k}H_{n+p}H_{n-p}H_{n+q}H_{n-q} = H_n^8 + ((-s)^{n-m}G_m^2 + (-s)^{n-k}G_k^2 + (-s)^{n-p}G_p^2 + (-s)^{n-q}G_q^2)(r^2+4s)H_n^6 + ((-s)^{2n-m-k}G_m^2G_k^2 + (-s)^{2n-m-p}G_m^2G_p^2 + (-s)^{2n-m-q}G_m^2G_q^2 + (-s)^{2n-k-p}G_k^2G_p^2 + (-s)^{2n-k-q}G_k^2G_q^2 + (-s)^{2n-p-q}G_p^2G_q^2)(r^2+4s)^2H_n^4 + ((-s)^{3n-m-k-p}G_m^2G_k^2G_p^2 + (-s)^{3n-m-k-q}G_m^2G_k^2G_q^2 + (-s)^{3n-k-p-q}G_k^2G_p^2G_q^2 + (-s)^{3n-m-p-q}G_m^2G_p^2G_q^2)(r^2+4s)^3H_n^2 + (-s)^{4n-m-k-p-q}G_m^2G_k^2G_p^2G_q^2(r^2+4s)^4.$

(f) $H_{n+1}H_{n-1}H_{n+2}H_{n-2}H_{n+3}H_{n-3}H_{n+4}H_{n-4} = H_n^8 + (-s)^{n-4}(r^6+3r^4s+3r^2s^2-2s^3)(r^2+4s)H_n^6 + s^{2n-7}(-r^{10} - 5r^8s - 11r^6s^2 - 13r^4s^3 - 8r^2s^4 + s^5)(r^2+4s)^2H_n^4 + (-s)^{3n-9}r^2(r^{10}+5r^8s+8r^6s^2+2r^4s^3-6r^2s^4-5s^5)(r^2+4s)^3H_n^2 + (-s)^{4n-10}r^4(r^2+s)^2(r^2+2s)^2(r^2+4s)^4.$

Proof.

(a) Use Catalan's identity: $W_{n+m}W_{n-m} = W_n^2 - (-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$ (see Theorem 1.6).

(b) Replace $m = 1$ in (a).

(c)-(d) Substitute $W_n = G_n$ in (a) and (b).

(e)-(f) Substitute $W_n = H_n$ in (a) and (b).

References

[1] Andrica, D., & Bagdasar, O. (2020). *Recurrent sequences: Key results, applications, and problems*. Springer. https://doi.org/10.1007/978-3-030-51502-7

[2] Djordjević, G. B., & Milovanović, G. V. (2014). *Special classes of polynomials*. University of Niš, Faculty of Technology, Leskovac. http://www.mi.sanu.ac.rs/~gvm/Teze/Special%20Classes%20of%20Polynomials.pdf

[3] Frei, G. (1980). Binary Lucas and Fibonacci polynomials I. *Mathematische Nachrichten*, *96*, 83–112. https://doi.org/10.1002/mana.19800960109

[4] Flórez, R., McAnally, N., & Mukherjee, A. (2018). Identities for the generalized Fibonacci polynomial. *Integers*, *18B*.

[5] He, T. X., & Shiue, P. J. S. (2009). On sequences of numbers and polynomials defined by linear recurrence relations of order 2. *International Journal of Mathematics and Mathematical Sciences*, *2009*, Article ID 709386, 21 pages. https://doi.org/10.1155/2009/709386

[6] Koshy, T. (2014). *Pell and Pell-Lucas numbers with applications*. Springer. https://doi.org/10.1007/978-1-4614-8489-9

[7] Koshy, T. (2018). *Fibonacci and Lucas numbers with applications* (Vol. 1, 2nd ed.). John Wiley & Sons. https://doi.org/10.1002/9781118742297

[8] Koshy, T. (2019). *Fibonacci and Lucas numbers with applications* (Vol. 2). John Wiley & Sons. https://doi.org/10.1002/9781118742297

[9] Soykan, Y. (2023). On generalized Fibonacci polynomials: Horadam polynomials. *Earthline Journal of Mathematical Sciences*, *11*(1), 23–114. https://doi.org/10.34198/ejms.11123.23114

[10] Vajda, S. (2008). *Fibonacci and Lucas numbers, and the golden section: Theory and applications*. Dover Publications.

[11] Wang, J. (2014). Some new results for the (p,q)-Fibonacci and Lucas polynomials. *Advances in Difference Equations*, *2014*, Article 64. https://doi.org/10.1186/1687-1847-2014-64

[12] Wang, W., & Wang, H. (2017). Generalized-Humbert polynomials via generalized Fibonacci polynomials. *Applied Mathematics and Computation*, *307*, 204–216. https://doi.org/10.1016/j.amc.2017.02.050