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Spectral Rigidity and Geometric Quantization of Coadjoint Orbits

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Abstract

Coadjoint orbits provide a fundamental link between symplectic geometry and the representation theory of Lie groups, as formalized by the orbit method of Kirillov. In this paper, we investigate the spectral properties of the Casimir operator in relation to the geometry of coadjoint orbits and their quantization. We establish a spectral rigidity phenomenon: for compact semisimple Lie groups, the eigenvalue of the Casimir operator determines the corresponding coadjoint orbit and the associated irreducible representation. This rigidity is shown to be independent of the choice of G-invariant metric on the orbit, highlighting the intrinsic algebraic nature of the Casimir operator.

We provide explicit computations in the case of $SU(2)$, where coadjoint orbits are 2-spheres, and analyze the relationship between the Casimir operator and the Laplace–Beltrami operator under metric variations. We further extend the discussion to real semisimple Lie groups, where rigidity persists in a weaker form through the infinitesimal character and the Harish-Chandra isomorphism.

Our results clarify the role of the Casimir operator as a bridge between geometry, spectral theory, and geometric quantization.

1 Introduction

The geometric approach to representation theory establishes a deep connection between irreducible representations of Lie groups and geometric objects known as coadjoint orbits. This perspective originates from the orbit method developed by Kirillov [3, 4], and has since become a central theme in modern representation theory and symplectic geometry.

Coadjoint orbits are naturally endowed with a symplectic structure, the Kirillov–Kostant–Souriau form, which turns them into classical phase spaces [5, 7]. Through geometric quantization, these symplectic manifolds give rise to Hilbert spaces and unitary representations of the underlying Lie group [8].

A fundamental role in this framework is played by the Casimir operator, an element of the center of the universal enveloping algebra $Z(U(\mathfrak{g}))$. By Schur's lemma, it acts as a scalar on each irreducible representation [10, 11]. In the case of compact semisimple Lie groups, irreducible representations are

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classified by highest weights, and the eigenvalue of the Casimir operator is explicitly given by

$$d\pi_\lambda(C) = \langle \lambda, \lambda + 2\rho \rangle,$$

where ρ is the half-sum of positive roots.

This eigenvalue reflects the geometry of the corresponding coadjoint orbit, suggesting a deep link between spectral data and geometric structure.

The main objective of this paper is to investigate this relationship through the notion of *spectral rigidity.* We show that, for compact semisimple Lie groups, the Casimir eigenvalue determines the coadjoint orbit and the associated irreducible representation. This rigidity is remarkable in that it is independent of the choice of G-invariant metric on the orbit, in contrast with geometric operators such as the Laplace–Beltrami operator, whose spectrum depends on the metric.

We illustrate these phenomena through the explicit example of $SU(2)$, where coadjoint orbits are 2-spheres and the Casimir eigenvalue $j(j+1)$ determines both the representation and the geometry. We also analyze how the Laplacian varies under metric deformations, highlighting the robustness of the Casimir spectrum.

Finally, we extend the discussion to real semisimple Lie groups, where the rigidity phenomenon persists in a weaker form. In this setting, the joint spectrum of the center determines the infinitesimal character of a representation via the Harish-Chandra isomorphism [10,12], and thus corresponds to a class of coadjoint orbits rather than a single one.

This work contributes to a better understanding of the interplay between representation theory, symplectic geometry, and spectral analysis, and clarifies the role of the Casimir operator as a fundamental invariant in geometric quantization. To our knowledge, this rigidity phenomenon has not been explicitly formulated in this geometric-quantization framework.

2 Basic Definitions

We give the brief description of the basic notions [1,2].

2.1 Lie groups and Lie algebras

Definition 2.1. A *Lie group* is a smooth manifold G equipped with a group structure such that the multiplication

$$G \times G \to G$$

and inversion

$$G \to G$$

are smooth maps.

Definition 2.2. The *Lie algebra* of a Lie group G is the tangent space at the identity

$$\mathfrak{g} = T_e G$$

equipped with the Lie bracket defined by the commutator of left-invariant vector fields.

2.2 Adjoint and coadjoint actions

Definition 2.3. The *adjoint action* of a Lie group G on its Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{Ad}_g(X) = gXg^{-1}.$$

Definition 2.4. The *coadjoint action* of G on the dual space $\mathfrak{g}^*$ is defined by

$$\mathrm{Ad}^*_g(\xi)(X) = \xi(\mathrm{Ad}_{g^{-1}} X)$$

for $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$.

2.3 Coadjoint orbits

Definition 2.5. Let $\xi \in \mathfrak{g}^*$. The set

$$\mathcal{O}_\xi = \{\mathrm{Ad}^*_g(\xi) \mid g \in G\}$$

is called the *coadjoint orbit* through ξ.

Proposition 2.6. *Every coadjoint orbit is a smooth manifold.*

Proof. Let

$$G_\xi = \{g \in G \mid \mathrm{Ad}^*_g(\xi) = \xi\}$$

be the stabilizer subgroup of ξ. Then the coadjoint orbit can be identified with the homogeneous space

$$\mathcal{O}_\xi \simeq G/G_\xi.$$

Step 1: G_ξ is a Lie subgroup. Since G_ξ is the isotropy subgroup of a smooth action of the Lie group G, it is a closed Lie subgroup. Hence, G_ξ is a smooth submanifold of G.

Step 2: Quotient by a Lie subgroup is smooth. The coset space G/G_ξ is a smooth manifold of dimension $\dim G - \dim G_\xi$, and the projection

$$\pi : G \to G/G_\xi, \quad g \mapsto gG_\xi$$

is a smooth submersion.

Step 3: Diffeomorphism with the orbit. Define

$$\Phi : G/G_\xi \to \mathcal{O}_\xi, \qquad gG_\xi \mapsto \mathrm{Ad}^*_g(\xi).$$

This map is well defined: if $gG_\xi = hG_\xi$, then $h^{-1}g \in G_\xi$, so

$$\mathrm{Ad}^*_{h^{-1}g}(\xi) = \xi.$$

Hence

$$\mathrm{Ad}^*_g(\xi) = \mathrm{Ad}^*_h\big(\mathrm{Ad}^*_{h^{-1}g}(\xi)\big) = \mathrm{Ad}^*_h(\xi).$$

The map is smooth because it is induced by the smooth action

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \qquad (g, \eta) \mapsto \mathrm{Ad}^*_g(\eta),$$

together with the quotient projection $G \to G/G_\xi$.

It is surjective by definition of the orbit:

$$\mathcal{O}_\xi = \{\mathrm{Ad}^*_g(\xi) \mid g \in G\}.$$

It is injective: if

$$\Phi(gG_\xi) = \Phi(hG_\xi),$$

then

$$\mathrm{Ad}^*_g(\xi) = \mathrm{Ad}^*_h(\xi), \quad \text{so} \quad \mathrm{Ad}^*_{h^{-1}g}(\xi) = \xi.$$

Thus $h^{-1}g \in G_\xi$, which implies $gG_\xi = hG_\xi$. Therefore Φ is a smooth bijection. Since the quotient map

$$\pi : G \to G/G_\xi$$

is a smooth submersion and $\Phi \circ \pi(g) = \mathrm{Ad}^*_g(\xi)$, the inverse map

$$\Phi^{-1} : \mathcal{O}_\xi \to G/G_\xi, \qquad \mathrm{Ad}^*_g(\xi) \mapsto gG_\xi$$

is also smooth. Hence Φ is a diffeomorphism.

We conclude that, $\mathcal{O}_\xi$ is a smooth manifold of dimension

$$\dim \mathcal{O}_\xi = \dim G - \dim G_\xi.$$

□

3 Symplectic Structure of Coadjoint Orbits

3.1 Tangent space to a coadjoint orbit

Let $\mathcal{O}_\xi$ be a coadjoint orbit. Define

$$X^{\#}(\eta) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}^*(\exp(tX))(\eta) = \mathrm{ad}^*_X(\eta).$$

Then the tangent space at $\eta \in \mathcal{O}_\xi$ is given by

$$T_\eta \mathcal{O}_\xi = \{X^{\#}(\eta) = \mathrm{ad}^*_X(\eta) : X \in \mathfrak{g}\}.$$

The coadjoint action $\mathrm{ad}^*_X : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by

$$\langle \mathrm{ad}^*_X(\eta), Y\rangle = -\langle \eta, [X,Y]\rangle, \quad \forall Y \in \mathfrak{g}.$$

Here, the pairing $\langle \cdot, \cdot \rangle$ is the *canonical duality pairing* between the Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$.

That is,

$$\langle \eta, Y\rangle := \eta(Y), \qquad \eta \in \mathfrak{g}^*,\ Y \in \mathfrak{g}.$$

So $\mathrm{ad}^*_X(\eta)$ is the unique element of $\mathfrak{g}^*$ whose value on any $Y \in \mathfrak{g}$ is $-\eta([X,Y])$.

The notation $\langle \eta, Y\rangle$ emphasizes that this is a bilinear pairing

$$\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}, \qquad (\eta, Y) \mapsto \eta(Y),$$

rather than an inner product. It is simply the natural evaluation map.

It follows that $\mathrm{ad}^*_X(\eta)$ represents the tangent vector at η generated by the infinitesimal action of X.

Definition 3.1. Let $\mathcal{O}_\xi$ be a coadjoint orbit. The *Kirillov–Kostant–Souriau symplectic form* is defined by

$$\omega_\xi(X^{\#}, Y^{\#}) = \langle \xi, [X,Y]\rangle.$$

where $X^{\#}, Y^{\#}$ are the vector fields on $\mathcal{O}_\xi$ induced by the Lie algebra action $\mathfrak{g}$.

Proposition 3.2. *The form ω_ξ is closed and nondegenerate, hence $(\mathcal{O}_\xi, \omega_\xi)$ is a symplectic manifold.*

Proof. We prove this by showing:

- the nondegeneracy argument via the stabilizer subalgebra.
- and closedness argument via the Jacobi identity.

Step 1: Nondegeneracy.

At a point $\eta \in \mathcal{O}_\xi$, tangent vectors are of the form

$$T_\eta \mathcal{O}_\xi = \{X^\#(\eta) = \mathrm{ad}^*_X(\eta) : X \in \mathfrak{g}\}.$$

Suppose $\omega_\xi(X^\#, Y^\#) = 0$ for all $Y \in \mathfrak{g}$. Then

$$\langle \xi, [X, Y] \rangle = 0 \quad \forall Y \in \mathfrak{g}.$$

By definition, $X \in \mathfrak{g}_\xi$, the stabilizer subalgebra of ξ, so $X^\# = 0$. Hence ω_ξ is nondegenerate on $T_\eta \mathcal{O}_\xi \simeq \mathfrak{g}/\mathfrak{g}_\xi$.

Step 2: Closedness.

For $X, Y, Z \in \mathfrak{g}$, the exterior derivative $d\omega_\xi$ is given by the Chevalley-Eilenberg formula:

$$d\omega_\xi(X^\#, Y^\#, Z^\#) = \sum_{\text{cyclic}} X^\# \cdot \omega_\xi(Y^\#, Z^\#) - \omega_\xi([X^\#, Y^\#], Z^\#) + \cdots$$

Using the Jacobi identity of the Lie bracket $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, one finds that

$$d\omega_\xi(X^\#, Y^\#, Z^\#) = 0 \quad \forall X, Y, Z \in \mathfrak{g}.$$

Thus ω_ξ is closed: $d\omega_\xi = 0$. Since ω_ξ is both nondegenerate and closed, we conclude that $(\mathcal{O}_\xi, \omega_\xi)$ is a symplectic manifold. □

4 The Casimir Operator

4.1 Definition

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra with basis $\{X_i\}$ and dual basis $\{X^i\}$ with respect to the Killing form. The *Casimir operator* is the element $C = \sum_i X_i X^i$ in the universal enveloping algebra $U(\mathfrak{g})$.

Proposition 4.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. The Casimir element $C = \sum_{i=1}^n X_i X^i$ lies in the center of $U(\mathfrak{g})$, i.e, $C \in Z(U(\mathfrak{g}))$.*

Proof. Let $X \in \mathfrak{g}$. We compute the commutator

$$[X, C] = \sum_i [X, X_i X^i].$$

Using the identity

$$[X, AB] = [X, A]B + A[X, B],$$

we obtain

$$[X, C] = \sum_i ([X, X_i]X^i + X_i[X, X^i]).$$

Write

$$[X, X_i] = \sum_j a_{ij} X_j.$$

Then

$$\sum_i [X, X_i]X^i = \sum_{i,j} a_{ij} X_j X^i.$$

Similarly,

$$[X, X^i] = -\sum_j a_{ji} X^j.$$

Hence

$$\sum_i X_i[X, X^i] = -\sum_{i,j} a_{ji} X_i X^j.$$

Using the invariance of the Killing form,

$$B([X, Y], Z) = B(Y, [X, Z]),$$

one finds that the two sums cancel, giving

$$[X, C] = 0.$$

Therefore the Casimir element commutes with every element of $\mathfrak{g}$, and hence

$$C \in Z(U(\mathfrak{g})).$$

□

4.2 Consequence: Schur's lemma

Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be an irreducible representation. Since C lies in the center of $U(\mathfrak{g})$, the operator $\pi(C)$ commutes with all operators $\pi(X)$ for $X \in \mathfrak{g}$. By Schur's lemma, it follows that

$$\pi(C) = c\,\mathrm{Id}_V$$

for some scalar c [10–12]. Thus the Casimir operator acts as a scalar on any irreducible representation.

5 Geometric Quantization

Definition 5.1. A coadjoint orbit $(\mathcal{O}, \omega)$ is said to satisfy the *integrality condition* if

$$[\omega] \in H^2(\mathcal{O}, \mathbb{Z}).$$

This condition ensures the existence of a prequantum line bundle and associates a Hilbert space to the orbit [5, 7].

Definition 5.2. The process of associating a Hilbert space and operators to a symplectic manifold is called *geometric quantization* [9, 11].

6 The Notion of Casimir Rigidity

We first introduce the notion of *spectral rigidity.* Let D_g be a family of G-equivariant differential operators depending on a G-invariant geometric structure g (typically a Riemannian metric) on a homogeneous space or coadjoint orbit. We say that the spectrum is *rigid* if the distinguished eigenvalues attached to irreducible G-representations remain unchanged under any deformation of the G-invariant geometry. In this sense, the spectral data are determined by representation theory rather than by the particular metric realization.

The Casimir operator $C \in Z(U(\mathfrak{g}))$ is the prototypical example of such rigidity. Since C lies in the center of the universal enveloping algebra, it acts by a scalar on every irreducible representation by Schur's lemma [10–12]. Thus, for any irreducible representation π_λ, one has

$$d\pi_\lambda(C) = c_\lambda \, \mathrm{Id}.$$

- The scalar c_λ depends only on the representation π_λ, hence only on the highest weight (or infinitesimal character), and not on the choice of basis.
- Once the Casimir element is fixed by an Ad-invariant form on $\mathfrak{g}$ (for example the Killing form in the semisimple case), the eigenvalue c_λ is independent of any further choice of G-invariant metric on the coadjoint orbit $\mathcal{O}_\lambda$.
- This scalar is therefore *spectrally rigid*: it remains unchanged under deformations of the G-invariant geometry of $\mathcal{O}_\lambda$.

Geometrically, on the coadjoint orbit $\mathcal{O}_\lambda \simeq G/G_\lambda$, the quadratic Casimir corresponds to the Laplace–Beltrami operator of the canonical G-invariant metric induced by the same invariant bilinear form used to define C. With the convention

$$C = \sum_i X_i^2$$

for an orthonormal basis $\{X_i\}$ of $\mathfrak{g}$, one has

$$d\pi_\lambda(C) = -\Delta_g$$

(up to the standard sign convention for the Laplacian; some authors write $d\pi_\lambda(C) = \Delta_g$).

If the metric g is varied within the class of G-invariant metrics, the operator Δ_g changes in general, whereas the central element C and its scalar action c_λ remain unchanged. Hence the Casimir encodes intrinsic algebraic information about the representation, independent of the particular geometric deformation.

7 Casimir Rigidity and Geometric Quantization

7.1 The Casimir as a quantum observable

Let G be a compact semisimple Lie group with Lie algebra $\mathfrak{g}$. Each coadjoint orbit $\mathcal{O}_\xi \subset \mathfrak{g}^*$ carries a natural symplectic structure ω_ξ, making it a classical phase space.

In this classical setting, smooth functions $f : \mathcal{O}_\xi \to \mathbb{R}$ represent observables. Among these, invariant functions under the coadjoint action of G play a distinguished role. In particular, the quadratic invariant induced by the Killing form,

$$C(\xi) = \langle \xi, \xi \rangle,$$

is constant along each orbit.

Geometric quantization associates to $(\mathcal{O}_\xi, \omega_\xi)$ a Hilbert space $\mathcal{H}_\xi$, together with a correspondence between classical observables and quantum operators. Under this correspondence, invariant functions give rise to central operators.

In particular, the Casimir operator acts on the quantum Hilbert space obtained from geometric quantization of a coadjoint orbit [4,5,7,9,11]. This illustrates the spectral rigidity: the eigenvalue depends solely on the orbit and the corresponding representation, independent of the choice of G-invariant metric.

7.2 Link with representation theory

According to the orbit method [4], coadjoint orbits $\mathcal{O}_\lambda$ correspond to irreducible representations π_λ of G. The eigenvalue of the Casimir operator is given by the highest weight λ [10–12], which encodes the representation completely.

Thus, the spectrum of the Casimir operator is entirely determined by the geometric data of the orbit. In particular, it provides an invariant that encodes the representation at the spectral level.

7.3 Spectral rigidity and geometric interpretation

The Casimir operator exhibits a remarkable rigidity property: its eigenvalue is independent of any choice of G-invariant metric on the coadjoint orbit [4,7,10,11]. While the Laplace-Beltrami operator Δ_g depends explicitly on the metric, the Casimir spectrum captures intrinsic algebraic information.

7.4 A rigidity theorem in geometric quantization

We now formulate a result that makes precise the relation between Casimir rigidity and geometric quantization.

Theorem 7.1. *Let G be a compact semisimple Lie group and $\mathcal{O}_\lambda$ a coadjoint orbit. Let $(\mathcal{H}_\lambda, \pi_\lambda)$ be the representation obtained by geometric quantization of $\mathcal{O}_\lambda$ [4, 7, 10, 11].*

Then:

1. *The Casimir operator $C \in Z(U(\mathfrak{g}))$ acts on $\mathcal{H}_\lambda$ by a scalar c_λ depending only on the orbit $\mathcal{O}_\lambda$.*
2. *This eigenvalue is independent of any G-invariant metric chosen on $\mathcal{O}_\lambda$.*
3. *If g is the metric induced by the Killing form, the Laplace–Beltrami operator Δ_g reproduces the Casimir spectrum.*
4. *For any other G-invariant metric g', the Laplacian $\Delta_{g'}$ does not reproduce the Casimir spectrum exactly.*

Proof. The first statement follows from the fact that C belongs to the center $Z(U(\mathfrak{g}))$, hence acts by scalars in any irreducible representation.

The second statement follows from the purely algebraic nature of C, which does not depend on any geometric structure.

For the third statement, when g is induced by the Killing form, the Laplacian coincides (up to normalization) with the action of the Casimir operator on functions over the orbit.

Finally, for a general G-invariant metric g', the Laplacian depends on the metric coefficients, leading to a deformation of the spectrum, which no longer matches the Casimir eigenvalue. ☐

7.5 Conceptual interpretation

The above theorem highlights a fundamental dichotomy [4,7,9,11]:

$$\text{geometry (metric-dependent)} \quad \text{vs.} \quad \text{algebra (rigid)}.$$

Geometric quantization provides a bridge between these two perspectives:

$$(\mathcal{O}_\lambda, \omega_\lambda) \longrightarrow \mathcal{H}_\lambda \longrightarrow \pi_\lambda.$$

Within this framework, the Casimir operator emerges as a quantization of a classical invariant whose spectrum is rigid and encodes the representation, independently of the geometric realization of the orbit.

8 A Rigidity Theorem via Geometric Quantization

We now formulate a rigidity result linking the Casimir spectrum, coadjoint orbits, and geometric quantization [4–7, 9].

Theorem 8.1. *Let G be a compact semisimple Lie group and let $\mathcal{O}_\lambda$ be a coadjoint orbit equipped with its canonical symplectic form ω.*

Assume that the Casimir operator takes a fixed value

$$d\pi_\lambda(C) = c.$$

Then:

1. *The coadjoint orbit $\mathcal{O}_\lambda$ is uniquely determined (up to isomorphism) by c.*
2. *The geometric quantization of $(\mathcal{O}_\lambda, \omega)$ produces a unique irreducible representation V_λ.*
3. *Any small deformation of ω preserving the Casimir value c does not change the resulting representation.*

Proof. The Casimir operator corresponds to the invariant quadratic function

$$Q(\xi) = \langle \xi, \xi \rangle.$$

Thus, the coadjoint orbit $\mathcal{O}_\lambda$ is contained in the level set

$$Q(\xi) = c.$$

In the compact semisimple case, coadjoint orbits are classified by such invariants, so c determines $\mathcal{O}_\lambda$.

The symplectic form ω depends on λ, and its cohomology class determines the prequantum line bundle. Since this class is fixed by c, the quantization space is uniquely determined.

Finally, small deformations preserving c do not change the cohomology class, hence the resulting representation remains unchanged. □

9 Example: Case of SU(2): Representations Spin-j

9.1 The Lie algebra of SU(2)

The Lie algebra $\mathfrak{su}(2)$ admits the following matrix realization [11,13,14]:

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the commutation relations

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

An alternative realization is given by differential operators:

$$J_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad J_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad J_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

which satisfy

$$[J_x, J_y] = J_z$$

or equivalently $[J_i, J_j] = i\epsilon_{ijk}J_k$. This algebra is 3-dimensional and semisimple.

9.2 The quadratic Casimir operator

The quadratic Casimir operator is defined by

$$C = J_x^2 + J_y^2 + J_z^2.$$

One verifies directly that

$$[J_x, C] = [J_y, C] = [J_z, C] = 0,$$

which shows that C belongs to the center of $U(\mathfrak{su}(2))$. In another words, it commutes with all generators and therefore acts as a scalar on any irreducible representation [11,13].

9.3 Representations: Spin-j

Let V_j be an irreducible representation. For spin $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, the irreducible representation V_j has dimension $2j+1$. Define

$$J_\pm = J_x \pm iJ_y.$$

Then

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z.$$

The eigenvalues of J_z are

$$m = -j, -j+1, \ldots, j.$$

We choose a highest weight vector [13, 14] v_j such that:

$$J_z v_j = j v_j, \qquad J_+ v_j = 0.$$

Define:

$$v_m = (J_-)^{j-m} v_j, \quad m = j, j-1, \ldots, -j.$$

Then $\{v_m\}$ forms a basis of V_j, and $\dim V_j = 2j+1$. The action is given by:

$$J_z v_m = m v_m,$$

$$J_+ v_m = \sqrt{(j-m)(j+m+1)}\, v_{m+1},$$

$$J_- v_m = \sqrt{(j+m)(j-m+1)}\, v_{m-1}.$$

9.3.1 Casimir in terms of raising and lowering operators

The Casimir operator can be rewritten as [11, 13, 14].

$$C = J_z^2 + \frac{1}{2}(J_+ J_- + J_- J_+).$$

9.3.2 Casimir eigenvalue

Let $\pi_j : SU(2) \to GL(V_j)$ be the irreducible representation of spin j, where $\dim V_j = 2j+1$.

Let

$$d\pi_j : \mathfrak{su}(2) \to \mathrm{End}(V_j)$$

be the induced Lie algebra representation.

The quadratic Casimir operator is

$$C = J_x^2 + J_y^2 + J_z^2.$$

A fundamental result in representation theory states that the Casimir operator acts as a scalar on irreducible representations.

Proposition 9.1. *In the spin-j representation of $SU(2)$, the Casimir operator acts as [13–15]*

$$d\pi_j(C) = j(j+1)\,\mathrm{Id}_{V_j}$$

Proof. Let $\{v_m\}_{m=-j}^{j}$ be a basis of V_j consisting of eigenvectors of $d\pi_j(J_z)$:

$$d\pi_j(J_z)v_m = mv_m.$$

Using the standard relations of the Lie algebra $\mathfrak{su}(2)$ and the raising and lowering operators, one obtains

$$d\pi_j(C)v_m = j(j+1)v_m$$

for all $m = -j, \ldots, j$.

Hence the Casimir operator acts as the scalar $j(j+1)$ on V_j.

□

The fact that the Casimir operator acts as the scalar $j(j+1)$ on V_j encodes a deep relationship between representation theory, spectral theory, and geometry. Indeed, this scalar completely characterizes the irreducible representation, determines the spectrum of the Casimir operator, and coincides with the value of the invariant quadratic function on the corresponding coadjoint orbit. This establishes a direct link between algebraic data and geometric structure, and explains the spectral rigidity of the Casimir operator.

9.4 Coadjoint orbits of SU(2)

Elements of $\mathfrak{su}(2)^*$ can be identified with $\mathbb{R}^3$. The coadjoint action of $SU(2)$ is equivalent to rotations in $\mathbb{R}^3$.

The coadjoint orbit of SU(2) associated with spin j is the 2-sphere

$$\mathcal{O}_j = \{\xi \in \mathfrak{su}(2)^* : \|\xi\| = j\},$$

equipped with the Kirillov-Kostant-Souriau symplectic form [4, 5, 11, 13].

9.5 Reconstruction of coadjoint orbits from the Casimir spectrum

Using the identification $\mathfrak{su}(2)^* \cong \mathbb{R}^3$, the quadratic Casimir function is given by

$$Q(\xi) = x^2 + y^2 + z^2.$$

This function is invariant under the coadjoint action, and its level sets define the coadjoint orbits:

$$\mathcal{O}_c = \{\xi \in \mathbb{R}^3 \mid Q(\xi) = c\}.$$

For the irreducible representation of spin j, the Casimir eigenvalue is

$$c = j(j+1).$$

Therefore, the corresponding coadjoint orbit is

$$\mathcal{O}_j = \{\xi \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = j(j+1)\}.$$

Hence,

$$\mathcal{O}_j \simeq S^2_{\sqrt{j(j+1)}}.$$

This shows that the spectrum of the Casimir operator completely determines the geometry of the coadjoint orbit [4, 5, 11, 13].

9.6 Geometric quantization of coadjoint orbits

We now explain how irreducible representations arise from the geometric quantization of coadjoint orbits.

Symplectic structure

Let $\mathcal{O}_\xi$ be a coadjoint orbit. It carries a natural symplectic form defined by

$$\omega_\xi(X^\#, Y^\#) = \langle \xi, [X, Y] \rangle.$$

In the case of $\mathfrak{su}(2)$, the orbit is a sphere

$$\mathcal{O}_j \simeq S^2_r,$$

and the symplectic form is proportional to the area form:

$$\omega = r\,\omega_{\text{area}}.$$

Quantization condition

The integrality condition requires:

$$\frac{1}{2\pi}\int_{S^2} \omega \in \mathbb{Z}.$$

Since $\int_{S^2} \omega = 4\pi r$, we obtain: $2r \in \mathbb{Z}$. Thus $r = \frac{n}{2}, \quad n \in \mathbb{N}$.

Quantum Hilbert space

One constructs a line bundle $L \to S^2$ with curvature ω, and defines the quantum space

$$\mathcal{H} = H^0(S^2, L).$$

A standard result gives:

$$\dim \mathcal{H} = 2j + 1.$$

Therefore,

$$\mathcal{H} \cong V_j.$$

This shows that irreducible representations of $SU(2)$ are obtained by quantizing coadjoint orbits. Combined with the fact that the Casimir operator determines the orbit, this establishes a deep correspondence between spectral data and geometric quantization [4–6, 13]..

9.7 Discussion: Relation with the Laplacian

For a compact Lie group G and a coadjoint orbit $\mathcal{O}_\xi$, the Casimir operator $C \in Z(U(\mathfrak{g}))$ acts by scalars in each irreducible representation, while the coadjoint orbit carries a natural G-invariant metric [10]. In another words, the case of $G = SU(2)$ illustrates here the construction of a natural G-invariant metric on a coadjoint orbit.

The Lie algebra $\mathfrak{su}(2)$ has a standard basis

$$\{J_x, J_y, J_z\}, \quad [J_x, J_y] = J_z, \quad [J_y, J_z] = J_x, \quad [J_z, J_x] = J_y.$$

The coadjoint orbit associated with spin j is a 2-sphere:

$$\mathcal{O}_j \simeq S^2_r, \quad r = \sqrt{j(j+1)}.$$

Its stabilizer is

$$G_\xi \simeq U(1), \quad \text{so that } S^2_r \simeq SU(2)/U(1).$$

The Killing form κ is given by

$$\kappa(X, Y) = -2\,\mathrm{Tr}(XY), \quad X, Y \in \mathfrak{su}(2),$$

using 2 × 2 matrix representation.

For $X \in \mathfrak{su}(2)$ and $\eta \in S^2_r \subset \mathbb{R}^3$, the fundamental vector field is

$$X^{\#}(\eta) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*(\exp(tX))(\eta) = X \times \eta,$$

where × is the usual vector product in $\mathbb{R}^3$. This defines a tangent vector to the sphere S_r^2.

Using the Killing form, the natural SU(2)-invariant metric on S_r^2 is

$$g_\eta(X^\#(\eta), Y^\#(\eta)) = \kappa(X, Y) \quad \text{which implies that} \quad g_\eta(u, v) = u \cdot v,$$

where u, v are tangent vectors.

- This is exactly the standard metric on the 2-sphere [5, 7], scaled by r^2
- It is invariant under SU(2) action:

$$g_{g_\eta}\big((\mathrm{Ad}_g X)^\#, (\mathrm{Ad}_g Y)^\#\big) = g_\eta(X^\#, Y^\#), \quad \forall g \in SU(2)$$

We can compute the expression in spherical coordinates. For

$$\eta = r(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta),$$

the tangent vectors are ∂_θ and ∂_ϕ, and the metric reads

$$g = r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

This metric is naturally induced by the Killing form via the fundamental vector fields, and hence is SU(2)-invariant.

The Laplace-Beltrami operator $\Delta_{\mathcal{O}_\xi}$ associated with this metric is closely related to the Casimir:

$$\Delta_{\mathcal{O}_\xi} \sim d\pi_\xi(C),$$

up to a normalization factor [16, 17].

Physically, the Casimir corresponds to the total kinetic energy (e.g., total spin or angular momentum), whereas the Laplacian is the geometric operator on functions over the orbit. For homogeneous orbits, the spectra are proportional:

$$\text{eigenvalues of the Laplacian} \propto \text{Casimir eigenvalues.}$$

If the metric on the orbit is varied, the Laplacian Δ_g changes, but the Casimir C remains central and invariant. This highlights the spectral rigidity: the Casimir spectrum is more robust than the Laplacian spectrum and encodes intrinsic geometric information.

For the orbit $\mathcal{O}_j \simeq S_r^2$ where $r = \sqrt{j(j+1)}$, the Laplacian spectrum on spherical harmonics is

$$\Delta_{S^2} Y_{l,m} = l(l+1) Y_{l,m}, \quad l = 0, 1, \ldots$$

while the Casimir acts by

$$d\pi_j(C) = j(j+1).$$

Hence, the Laplacian reproduces the Casimir spectrum on the orbit functions, illustrating the deep link between algebraic and geometric analysis.

This illustrates the general principle of the orbit method [4], where representations correspond to coadjoint orbits and the Casimir encodes their geometry.

Remark 9.2. If G is a compact semisimple Lie group, $\mathcal{O}_\xi \simeq G/G_\xi$ a coadjoint orbit, and $C \in Z(U(\mathfrak{g}))$ the Casimir operator. The unique natural choice of G-invariant metrics on $\mathcal{O}_\xi$ which reproduce the Casimir spectrum exactly via the Laplacian, is the metric induced by the Killing form. This metric ensures that the Laplace-Beltrami operator Δ_g coincides with the action of the Casimir on functions over the orbit. The overall scale can be adjusted so that the eigenvalues match exactly the Casimir eigenvalues.

9.8 Casimir rigidity versus Laplacian under metric variation

We illustrate explicitly how the Casimir eigenvalue remains invariant even if the G-invariant metric on the coadjoint orbit is varied.

Orbite: SU(2) **and** S^2**.** Consider the coadjoint orbit $\mathcal{O}_j \simeq S^2$ corresponding to the spin-j representation. The standard SU(2)-invariant metric is

$$g = r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $r = \sqrt{j(j+1)}$ [4, 7].

The Laplace-Beltrami operator for this metric reads

$$\Delta_g f = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2},$$

which is the standard Laplacian on the sphere [16].

Its eigenfunctions are the spherical harmonics $Y_{l,m}(\theta,\phi)$ with eigenvalues [11, 13]

$$\Delta_g Y_{l,m} = \frac{l(l+1)}{r^2} Y_{l,m}, \quad l = 0, 1, 2, \dots$$

Variation of G**-invariant metric.** Now consider a different SU(2)-invariant metric of the form

$$g' = r^2(a\, d\theta^2 + b\, \sin^2\theta\, d\phi^2), \quad a, b > 0.$$

The Laplacian becomes

$$\Delta_{g'} f = \frac{1}{r^2 a\, \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 b\, \sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

Its spectrum now depends explicitly on the constants a and b, demonstrating that the Laplace-Beltrami operator is sensitive to the choice of metric [16, 17].

Casimir invariance. Despite this variation, the Casimir operator

$$C = J_x^2 + J_y^2 + J_z^2 \in Z(U(\mathfrak{su}(2)))$$

acts in the spin-j representation as

$$d\pi_j(C) = j(j+1)\,\mathrm{Id},$$

independent of the choice of metric [10, 11]. Thus, while the Laplacian spectrum changes with the metric, the Casimir spectrum remains rigid.

We conclude that this explicit calculation illustrates that:

- Only the Killing-induced metric reproduces exactly the Casimir spectrum and the weights of the representation.
- Any other G-invariant metric deforms the Laplacian and breaks this exact correspondence.

10 Casimir Rigidity for Real Semisimple Groups

The rigidity phenomenon described for compact groups extends in a weaker form to real semisimple Lie groups. The central idea is that the joint spectrum of the center of the universal enveloping algebra determines the infinitesimal character of an irreducible representation, which in turn corresponds to a class of coadjoint orbits [4, 10, 11, 13, 17, 18].

Theorem 10.1. *Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$, and let π be an irreducible representation of G. Then:*

1. *The center $Z(U(\mathfrak{g}))$ acts on π by scalars:*

$$d\pi(z) = \chi_\pi(z)\,\mathrm{Id}, \quad \forall z \in Z(U(\mathfrak{g})),$$

where χ_π is the infinitesimal character of π.

2. *Via the Harish-Chandra isomorphism, each $z \in Z(U(\mathfrak{g}))$ corresponds to an invariant polynomial on $\mathfrak{g}^*$, and the joint spectrum $\{\chi_\pi(z)\}_{z\in Z(U(\mathfrak{g}))}$ fixes the values of these polynomials.*

3. *Consequently, π is associated with a class of coadjoint orbits lying in the common level set defined by these invariants.*

4. *In particular, for a compact group, the quadratic Casimir alone suffices to determine the orbit and the irreducible representation uniquely, recovering the stronger rigidity.*

Proof. By Schur's lemma, the center $Z(U(\mathfrak{g}))$ acts by scalars on any irreducible representation [18]. The map $\chi_\pi : Z(U(\mathfrak{g})) \to \mathbb{C}$ encodes these scalars as the infinitesimal character.

The Harish-Chandra isomorphism identifies $Z(U(\mathfrak{g}))$ with the algebra of Weyl-invariant polynomials on a Cartan subalgebra, which can be interpreted as invariant polynomials on $\mathfrak{g}^*$ [10,17]. Hence, the joint spectrum of $Z(U(\mathfrak{g}))$ determines the level sets of these invariants.

From the orbit method perspective, irreducible representations correspond (in a generalized sense) to coadjoint orbits, and these invariant polynomials are constant along orbits [4]. Therefore, the infinitesimal character determines a family of coadjoint orbits contained in the corresponding level set.

For compact semisimple groups, invariant polynomials (in particular the quadratic Casimir) separate coadjoint orbits, so the orbit is uniquely determined. For non-compact groups, distinct orbits may share the same invariant values, so only a class of orbits is determined. This explains the weaker rigidity phenomenon [10,17]. □

Remark 10.2. This result highlights a fundamental distinction between compact and non-compact semisimple groups:

- **Compact case:** The Casimir operator (together with invariant polynomials) determines the coadjoint orbit and the corresponding irreducible representation [11,13].
- **Non-compact case:** The joint spectrum of the center determines only the infinitesimal character, which corresponds to a class of coadjoint orbits [4,10,17].

In both cases, the spectrum of central elements remains rigid and independent of any choice of invariant metric, preserving the intrinsic algebraic information that characterizes the representation.

10.1 The Example of $\mathrm{SL}(2,\mathbb{R})$

In order to better understand the limitations of spectral rigidity in the non-compact setting, we analyze explicitly the case of $G = \mathrm{SL}(2,\mathbb{R})$.

The Lie Algebra and the Casimir operator

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ admits a basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

satisfying the commutation relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

The Casimir element of $\mathfrak{sl}(2, \mathbb{R})$ is defined by

$$C = \frac{1}{2}H^2 + EF + FE,$$

which commutes with all elements of the universal enveloping algebra. It belongs to the center $Z(U(\mathfrak{sl}(2, \mathbb{R})))$.

We now classify the coadjoint orbits of $G = \mathrm{SL}(2, \mathbb{R})$ using the quadratic Casimir invariant. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ with basis $\{H, E, F\}$ and dual basis $\{H^*, E^*, F^*\}$.

We make the identification $\mathfrak{sl}(2, \mathbb{R})^* \cong \mathbb{R}^3$: we choose the dual basis $\{H^*, E^*, F^*\}$ corresponding to the basis $\{H, E, F\}$ of $\mathfrak{sl}(2, \mathbb{R})$. By definition,

$$\begin{aligned} H^*(H) &= 1, \quad H^*(E) = H^*(F) = 0, \\ E^*(E) &= 1, \quad E^*(H) = E^*(F) = 0, \\ F^*(F) &= 1, \quad F^*(H) = F^*(E) = 0. \end{aligned}$$

Every linear functional $\xi \in \mathfrak{sl}(2, \mathbb{R})^*$ can then be written uniquely as

$$\xi = xH^* + yE^* + zF^*, \qquad x, y, z \in \mathbb{R}.$$

This defines a linear isomorphism

$$\mathfrak{sl}(2, \mathbb{R})^* \longrightarrow \mathbb{R}^3, \qquad \xi = xH^* + yE^* + zF^* \longmapsto (x, y, z).$$

Conversely, every triple $(x, y, z) \in \mathbb{R}^3$ determines a unique functional

$$(x, y, z) \longmapsto xH^* + yE^* + zF^*.$$

Hence $\mathfrak{sl}(2, \mathbb{R})^*$ is identified with $\mathbb{R}^3$ once the basis $\{H, E, F\}$ is fixed. In these coordinates, x, y, z are simply the components of ξ in the dual basis.

Thus, identifying $\mathfrak{sl}(2, \mathbb{R})^* \cong \mathbb{R}^3$, the quadratic invariant is given by

$$Q(\xi) = \tfrac{1}{2}x^2 + yz.$$

The coadjoint orbits are classified as follows:

- Elliptic orbits: $Q(\xi) > 0$
- Hyperbolic orbits: $Q(\xi) < 0$
- Nilpotent orbit: $Q(\xi) = 0$

Weak spectral rigidity for $\mathrm{SL}(2,\mathbb{R})$

In contrast with the compact case, the spectral rigidity of the Casimir operator for non-compact groups such as $\mathrm{SL}(2,\mathbb{R})$ is only partial.

In the compact case, irreducible representations are classified by a discrete set of highest weights λ, and the Casimir eigenvalue

$$c(\lambda) = \langle \lambda, \lambda + 2\rho \rangle$$

determines λ uniquely. This leads to strong spectral rigidity.

However, in the non-compact case, several new phenomena appear:

- The spectrum becomes continuous. For example, in the principal series of $\mathrm{SL}(2,\mathbb{R})$, one has
$$c(t) = \frac{1}{4} + t^2, \quad t \in \mathbb{R}.$$
- The Casimir eigenvalue is not injective: distinct parameters (e.g. t and $-t$) yield the same value.
- The orbit method is no longer bijective: multiple representations may correspond to the same class of coadjoint orbits.

Geometrically, the Casimir corresponds to the quadratic invariant

$$Q(\xi) = B(\xi, \xi).$$

In the compact case, level sets of Q coincide with individual coadjoint orbits, whereas in the non-compact case they contain infinitely many distinct orbits.

Therefore, the Casimir eigenvalue determines only the type of coadjoint orbit (elliptic, hyperbolic, or nilpotent), but not the orbit itself nor the corresponding representation. This phenomenon is referred to as *weak spectral rigidity.*

Theorem 10.3. *The Casimir eigenvalue determines the value of the quadratic invariant $Q(\xi)$ and therefore the type of coadjoint orbit (elliptic, hyperbolic, or nilpotent), but does not uniquely determine the orbit nor the representation.*

Proof. Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ with basis H, E, F. The quadratic invariant on $\mathfrak{g}^*$ is

$$Q(\xi) = \tfrac{1}{2}x^2 + yz, \quad \xi = xH^* + yE^* + zF^*.$$

The Casimir operator in $U(\mathfrak{g})$ is

$$C = \tfrac{1}{2}H^2 + EF + FE.$$

For any irreducible representation π, $d\pi(C) = c\,\mathrm{Id}$. By the orbit method, the Casimir corresponds to $Q(\xi)$, which is constant on each coadjoint orbit. Therefore, the value c determines $Q(\xi)$, and hence the orbit type:

$$Q(\xi) > 0 \Rightarrow \text{elliptic}, \quad Q(\xi) < 0 \Rightarrow \text{hyperbolic}, \quad Q(\xi) = 0 \Rightarrow \text{nilpotent}.$$

However, the mapping from Casimir eigenvalue to coadjoint orbit is not injective:

- Continuous parameters t in the principal series give the same orbit type.
- Multiple orbits (e.g., different sheets of hyperboloids) share the same $Q(\xi)$.
- Therefore, several distinct representations share the same eigenvalue c.

Hence, the Casimir determines only the orbit type, not the unique orbit or representation. ☐

10.2 Explicit computation: representations of $\mathrm{SL}(2,\mathbb{R})$

We illustrate the weaker form of Casimir rigidity in the non-compact case by considering a concrete representation of $G = \mathrm{SL}(2,\mathbb{R})$. Using the explicit expression of the quadratic Casimir element in $U(\mathfrak{sl}(2,\mathbb{R}))$, one can compute its action in both the discrete series and the principal series representations [10, 17].

10.2.1 Discrete series representation

Let D_k^+ be a discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with lowest weight $k > 0$ (integer). Let v_n, $n \in \mathbb{N}$, denote the weight vectors satisfying

$$Hv_n = (k+2n)v_n, \quad Ev_n = -(n+1)(k+n)v_{n+1}, \quad Fv_n = v_{n-1}, \quad Fv_0 = 0$$

as in the standard construction of discrete series representations [10, 17].

The Casimir operator acts on D_k^+ as a scalar:

$$Cv_n = \frac{k}{2}\left(\frac{k}{2} - 1\right) v_n$$

which depends only on the infinitesimal character [17, 18].

Computation

We check explicitly on v_0:

$$\begin{aligned} Cv_0 &= \left(\frac{1}{2}H^2 + EF + FE\right) v_0 \\ &= \frac{1}{2}H^2 v_0 + EFv_0 + FEv_0 \\ &= \frac{1}{2}k^2 v_0 + E \cdot 0 + F(0) \text{ (since } Fv_0 = 0,\ Ev_0 \neq 0) \\ &= \frac{k}{2}\left(\frac{k}{2} - 1\right) v_0. \end{aligned}$$

A similar computation shows that the same scalar is obtained for all v_n, in agreement with the fact that the Casimir acts by a scalar on irreducible representations [18].

Interpretation: weaker rigidity

Unlike the compact case SU(2), the Casimir operator determines only the infinitesimal character. More precisely, for D_k^+ one has

$$\chi_{D_k^+}(C) = \frac{k}{2}\left(\frac{k}{2} - 1\right),$$

which depends only on the Harish-Chandra parameter [10, 17].

However, this does not uniquely determine a single coadjoint orbit. Invariant polynomials (such as the quadratic Casimir) are constant along coadjoint orbits, but in the non-compact setting distinct orbits may share the same values of these invariants [4].

Thus, coadjoint orbits lying in the same level set of the quadratic invariant are associated, in the sense of the orbit method, with representations having the same infinitesimal character.

Hence, for $\mathrm{SL}(2,\mathbb{R})$, the Casimir spectrum rigidly fixes the infinitesimal character, but only partially determines the geometric data, illustrating a weaker form of rigidity compared to the compact case [4, 10, 17].

10.3 Principal series representations

We now illustrate Casimir eigenvalues for the principal series of $\mathrm{SL}(2,\mathbb{R})$, which are infinite-dimensional and non-discrete [10, 17].

10.3.1 Construction of the principal series

Let $G = \mathrm{SL}(2,\mathbb{R})$ and let B be the upper-triangular Borel subgroup. Given parameters (ν, ϵ) with $\nu \in i\mathbb{R}$ (purely imaginary) and $\epsilon \in \{0,1\}$, we define a character of B by

$$\chi_{\nu,\epsilon}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{\nu}\,(\operatorname{sgn} a)^{\epsilon}.$$

The (unitary) principal series representation $\pi_{\nu,\epsilon}$ is obtained by normalized induction

$$\pi_{\nu,\epsilon} = \mathrm{Ind}_B^G(\chi_{\nu,\epsilon}).$$

Action of the Lie algebra

Let $\mathfrak{sl}(2,\mathbb{R})$ have the standard basis (H, E, F) with $[H,E] = 2E, [H,F] = -2F, [E,F] = H$ and Casimir

$$C = \frac{1}{2}H^2 + EF + FE.$$

In the principal series, a natural basis is given by v_m with $m \in \epsilon + 2\mathbb{Z}$ (weight under H). The action of H, E, F is

$$Hv_m = m\,v_m, \quad Ev_m = \frac{1}{2}(\nu + m + 1)v_{m+2}, \quad Fv_m = \frac{1}{2}(\nu - m + 1)v_{m-2}.$$

Casimir eigenvalue

A direct computation shows that the Casimir acts as a scalar:

$$Cv_m = \left(\frac{1}{2}H^2 + EF + FE\right)v_m = \frac{1}{4}(\nu^2 - 1)\,v_m$$

independently of m, in accordance with Schur's lemma.

Thus, for all v_m in the principal series, the Casimir has the constant eigenvalue

$$\chi_{\pi_{\nu,\epsilon}}(C) = \frac{1}{4}(\nu^2 - 1)$$

which depends only on the infinitesimal character of the representation.

Interpretation: weaker rigidity

As in the discrete series, the Casimir fixes the infinitesimal character $\chi_{\pi_{\nu,\epsilon}}$ but does not uniquely determine a single coadjoint orbit. In fact, for principal series, the coadjoint orbits form a continuous family, all of which share the same quadratic invariant. This demonstrates that for non-compact groups like $\mathrm{SL}(2,\mathbb{R})$, the Casimir operator exhibits a weaker form of rigidity: it constrains the representation but cannot completely capture the geometry of coadjoint orbits.

This demonstrates that for non-compact groups such as $\mathrm{SL}(2,\mathbb{R})$, the Casimir operator exhibits a weaker form of rigidity: it determines the infinitesimal character but does not completely fix the geometric structure of the corresponding coadjoint orbits [4, 10, 17].

11 Comparison between the Compact versus Non-compact Cases

The preceding examples highlight a fundamental contrast between compact and non-compact semisimple Lie groups, both at the level of representation theory and geometric quantization [4, 10, 11, 17]:

Compact case ($SU(2)$)	Non-compact case ($\mathrm{SL}(2,\mathbb{R})$)
Discrete spectrum of irreducible representations	Continuous families of representations
Orbit method essentially bijective	Orbit method non-bijective
Strong rigidity (Casimir determines orbit in rank one)	Weak rigidity (only infinitesimal character fixed)
Invariant polynomials separate orbits	Additional data required to distinguish orbits

In the compact case, irreducible representations are classified by highest weights, and the Casimir operator determines the corresponding coadjoint orbit (in rank one), yielding a strong form of rigidity [11, 13].

In contrast, for non-compact groups such as $\mathrm{SL}(2,\mathbb{R})$, representations occur in continuous families (e.g., principal series), and the orbit method provides only a partial correspondence between representations and coadjoint orbits [4, 10, 17].

Thus, while the spectrum of central elements remains invariant and encodes the infinitesimal character, it does not fully determine the geometric structure of the representation in the non-compact setting. This reflects a fundamental weakening of rigidity.

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