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A Note on a Generalization of Two Integral Inequality Theorems

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Abstract

In this article, we propose a general theorem that unifies two existing theorems on analogues of the Hilbert integral inequality. A key tool in our approach is the Young integral inequality. Detailed proofs are provided.

1 Introduction

Integral inequalities with singular kernels, most notably the Hilbert integral inequality, are key for establishing precise bounds on bilinear forms. These results are foundational to harmonic analysis and operator theory, offering essential insights into the behavior of partial differential equations. Their utility is further highlighted in interpolation theory and the study of integral operators with homogeneous kernels.

The original version of the Hilbert integral inequality is stated below. Let $f, g : [0, +\infty) \to \mathbb{R}$ be (measurable) functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{|f(s)||g(t)|}{s+t} ds dt \leq \pi \left[\int_0^{+\infty} f^2(s) ds \right]^{1/2} \left[\int_0^{+\infty} g^2(t) dt \right]^{1/2},$$

provided that the integrals in the upper bounds converge. The constant factor π is known to be sharp. This result has been generalized in many directions, including extensions to L^p-spaces, weighted settings, and multidimensional analogues. In its general L^p-form, the inequality may be stated as below. Let $p > 1$ and denote its Hölder conjugate by $p_\star = p/(p-1)$. Let $f, g : [0, +\infty) \to \mathbb{R}$ be functions. Then we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{|f(s)||g(t)|}{s+t} ds dt \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} |f(s)|^p ds \right]^{1/p} \left[\int_0^{+\infty} |g(t)|^{p_\star} dt \right]^{1/p_\star},$$

provided that the integrals in the upper bounds converge. The constant factor $\pi/\sin(\pi/p)$ is known to be sharp. The technical details can be found in [1]. This notable result emphasizes the intricate relationship between singular integral operators, duality in L^p-spaces and precise functional inequalities. It also serves as a 'mathematical benchmark' for subsequent developments in the theory of Hardy- and Hilbert-type integral inequalities.

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We highlight two analogues of the Hilbert integral inequality among these developments: one given by [2, Theorem 2] and the other by [3, Theorem 2.2]. For the sake of completeness, these are stated in detail below.

Theorem 1.1. *[2, Theorem 2] Let $p > 1$, $p_\star = p/(p-1)$ and $f, g : [0, +\infty) \to \mathbb{R}$ be differentiable functions such that $f(0) = g(0) = 0$. Then, for any $x, y > 0$, we have*

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{p_\star s^{p-1} + p t^{p_\star - 1}} dsdt$$
$$\leq \frac{y^{1-1/p} x^{1-1/p_\star}}{p p_\star} \left[\int_0^y (y-s)|f'(s)|^p ds\right]^{1/p} \left[\int_0^x (x-t)|g'(t)|^{p_\star} dt\right]^{1/p_\star},$$

provided that the integrals in the upper bounds converge.

Theorem 1.2. *[3, Theorem 2.2] Let $p > 1$, $q > 1$ and $f, g : [0, +\infty) \to \mathbb{R}$ be differentiable functions such that $f(0) = g(0) = 0$. Then, for any $x, y > 0$, we have*

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{q s^{(1-1/p)(p+q)/q} + p t^{(1-1/q)(p+q)/p}} dsdt$$
$$\leq \frac{y^{1-1/p} x^{1-1/q}}{p+q} \left[\int_0^y (y-s)|f'(s)|^p ds\right]^{1/p} \left[\int_0^x (x-t)|g'(t)|^{q} dt\right]^{1/q},$$

provided that the integrals in the upper bounds converge.

In this theorem, note that q is not the Hölder conjugate of p; it is an independent parameter.

A distinctive feature of these results is that they involve a finite domain of integration, i.e., $(0, x)\times(0, y)$, and incorporate the derivatives of f and g in the upper bounds. The constant factors are also relatively simple and are known to be sharp. These results have inspired modern developments in more technical frameworks. We may refer to [5–9].

In this article, we establish a general theorem that unifies [2, Theorem 2] and [3, Theorem 2.2]. The key tool in our approach is the Young integral inequality, which enables us to incorporate an auxiliary function. Details about this intermediate inequality can be found in [4]. Two special choices of this auxiliary function then yield the aforementioned results as particular cases.

The remainder of the article is structured as follows: Section 2 presents and proves the main theorem, accompanied by a detailed discussion, while Section 3 provides the concluding remarks.

2 Main Theorem and Proof

The statement of our main theorem is presented below.

Theorem 2.1. *Let $p > 1$, $q > 1$ and $f, g : [0, +\infty) \to \mathbb{R}$ be differentiable functions such that $f(0) = g(0) = 0$. Then, for any function $\phi : [0, +\infty) \to [0, +\infty)$ continuous and strictly increasing such that*

$\phi(0) = 0$ and any $x, y > 0$, we have

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du} dsdt$$
$$\leq y^{1-1/p} x^{1-1/q} \left[\int_0^y (y-s)|f'(s)|^p ds\right]^{1/p} \left[\int_0^x (x-t)|g'(t)|^q dt\right]^{1/q},$$

where ϕ^{-1} denotes the inverse function of ϕ, provided that the integrals in the upper bounds converge.

Proof of Theorem 2.1. By the assumptions made on f and g, we can write

$$f(s) = f(0) + \int_0^s f'(\tau)d\tau = \int_0^s f'(\tau)d\tau$$

and

$$g(t) = g(0) + \int_0^t g'(\sigma)d\sigma = \int_0^t g'(\sigma)d\sigma.$$

By the Hölder integral inequality, we have

$$|f(s)| = \left|\int_0^s f'(\tau)d\tau\right| \leq \int_0^s |f'(\tau)|d\tau \leq s^{1-1/p} \left[\int_0^s |f'(\tau)|^p d\tau\right]^{1/p}$$

and

$$|g(t)| = \left|\int_0^t g'(\sigma)d\sigma\right| \leq \int_0^t |g'(\sigma)|d\sigma \leq t^{1-1/q} \left[\int_0^t |g'(\sigma)|^q d\sigma\right]^{1/q}.$$

By the multiplication rule, we get the following inequality:

$$|f(s)||g(t)| \leq s^{1-1/p} t^{1-1/q} \left[\int_0^s |f'(\tau)|^p d\tau\right]^{1/p} \left[\int_0^t |g'(\sigma)|^q d\sigma\right]^{1/q}.$$

By the assumptions made on ϕ, we can apply the Young integral inequality, i.e.,

$$ab \leq \int_0^a \phi(u)du + \int_0^b \phi^{-1}(u)du,$$

with $a, b > 0$. More precisely, applying it to $a = s^{1-1/p}$ and $b = t^{1-1/q}$, we get

$$s^{1-1/p} t^{1-1/q} \leq \int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du.$$

We thus have

$$|f(s)||g(t)| \leq \left[\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du\right] \left[\int_0^s |f'(\tau)|^p d\tau\right]^{1/p} \left[\int_0^t |g'(\sigma)|^q d\sigma\right]^{1/q}$$

so that

$$\frac{|f(s)||g(t)|}{\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du} \leq \left[\int_0^s |f'(\tau)|^p d\tau\right]^{1/p} \left[\int_0^t |g'(\sigma)|^q d\sigma\right]^{1/q}.$$

Integrating both sides with respect to s with $s \in (0, y)$ and t with $t \in (0, x)$ yields

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du} dsdt$$
$$\leq \left\{ \int_0^y \left[\int_0^s |f'(\tau)|^p d\tau \right]^{1/p} ds \right\} \left\{ \int_0^x \left[\int_0^t |g'(\sigma)|^q d\sigma \right]^{1/q} dt \right\}.$$

Let us now work on this upper bound. Applying the Hölder integral inequality, we get

$$\int_0^y \left[\int_0^s |f'(\tau)|^p d\tau \right]^{1/p} ds \leq y^{1-1/p} \left[\int_0^y \int_0^s |f'(\tau)|^p d\tau ds \right]^{1/p}$$

and

$$\int_0^x \left[\int_0^t |g'(\sigma)|^q d\sigma \right]^{1/q} dt \leq x^{1-1/q} \left[\int_0^x \int_0^t |g'(\sigma)|^q d\sigma dt \right]^{1/q}.$$

By the multiplication rule, we obtain the following inequality:

$$\left\{ \int_0^y \left[\int_0^s |f'(\tau)|^p d\tau \right]^{1/p} ds \right\} \left\{ \int_0^x \left[\int_0^t |g'(\sigma)|^q d\sigma \right]^{1/q} dt \right\}$$
$$\leq y^{1-1/p} x^{1-1/q} \left[\int_0^y \int_0^s |f'(\tau)|^p d\tau ds \right]^{1/p} \left[\int_0^x \int_0^t |g'(\sigma)|^q d\sigma dt \right]^{1/q}.$$

Let us now work on this upper bound, with a focus on each double integral. The Fubini-Tonelli integral theorem gives

$$\int_0^y \int_0^s |f'(\tau)|^p d\tau ds = \int_0^y \int_\tau^y |f'(\tau)|^p ds d\tau$$
$$= \int_0^y (y - \tau)|f'(\tau)|^p d\tau = \int_0^y (y - s)|f'(s)|^p ds$$

and

$$\int_0^x \int_0^t |g'(\sigma)|^q d\sigma dt = \int_0^x \int_\sigma^x |g'(\sigma)|^q dt d\sigma$$
$$= \int_0^x (x - \sigma)|g'(\sigma)|^q d\sigma = \int_0^x (x - t)|g'(t)|^q dt.$$

Combining the above inequalities and equalities, we thus have

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du} dsdt$$
$$\leq y^{1-1/p} x^{1-1/q} \left[\int_0^y (y - s)|f'(s)|^p ds \right]^{1/p} \left[\int_0^x (x - t)|g'(t)|^q dt \right]^{1/q}.$$

This ends the proof of Theorem 2.1. □

If we choose $\phi(u) = u^{p-1}$ and $q = p_\star$, Theorem 2.1 reduces to [2, Theorem 2], while the choice $\phi(u) = u^{p/q}$ yields [3, Theorem 2.2]. Beyond these cases, many other functions can be employed to derive new analogues of Hilbert-type integral inequalities. For instance, taking $\phi(u) = e^u - 1$ leads to $\phi^{-1}(u) = \ln(1+u)$, and

$$\begin{aligned}&\int_0^{s^{1-1/p}} \phi(u)du + \int_0^{t^{1-1/q}} \phi^{-1}(u)du \\ &= \int_0^{s^{1-1/p}} (e^u - 1)du + \int_0^{t^{1-1/q}} \ln(1+u)du \\ &= e^{s^{1-1/p}} - s^{1-1/p} - 1 + (1 + t^{1-1/q})\ln(1 + t^{1-1/q}) - t^{1-1/q}.\end{aligned}$$

Theorem 2.1 gives

$$\begin{aligned}&\int_0^x \int_0^y \frac{|f(s)||g(t)|}{e^{s^{1-1/p}} - s^{1-1/p} - 1 + (1 + t^{1-1/q})\ln(1 + t^{1-1/q}) - t^{1-1/q}} dsdt \\ &\leq y^{1-1/p} x^{1-1/q} \left[\int_0^y (y-s)|f'(s)|^p ds\right]^{1/p} \left[\int_0^x (x-t)|g'(t)|^q dt\right]^{1/q}.\end{aligned}$$

As far as we know, this is a new integral inequality, with an original exponential-logarithmic feature.

3 Conclusion

In conclusion, our work presents a unified framework that generalizes the results of [2, Theorem 2] and [3, Theorem 2.2] through the strategic use of the Young integral inequality. It thus advances the theory of integral inequalities in the following form:

$$\int_0^x \int_0^y \frac{|f(s)||g(t)|}{k(s) + \ell(t)} dsdt \leq \Omega y^{1-1/p} x^{1-1/q} \left[\int_0^y (y-s)|f'(s)|^p ds\right]^{1/p} \left[\int_0^x (x-t)|g'(t)|^q dt\right]^{1/q},$$

where $k, \ell : [0, +\infty) \to [0, +\infty)$ denotes some functions and Ω denotes a constant factor. The flexibility of our general theorem is characterized by an auxiliary function that can adapt to diverse mathematical scenarios. We believe that this approach can be applied to more general settings, such as those considered in [5–9]. However, this requires further effort, which we will address in future work.

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