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Exponential Type Interpolative Contraction Mapping Theorems for the Kannan, Berinde Weak, and Ciric-Reich-Rus Operators in Metric Spaces with Application

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Abstract

In this paper, we introduce the notion of an exponential interpolative type contraction operator, and prove the Kannan, Berinde weak, and Ciric-Reich-Rus fixed point theorems for such operators in the setting of metric spaces. Finally, we apply the exponential interpolative Kannan contraction mapping theorem to the Fredholm integral equation.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X,d) be a complete metric space, and let $T : X \mapsto X$ be a mapping such that there exists $K < \frac{1}{2}$ satisfying*

$$d(Tx,Ty) \leq K[d(x,Tx) + d(y,Ty)]$$

for all $x, y \in X$. Then T has a unique fixed point $v \in X$, and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to v, and

$$d(T^{n+1}x, v) \leq K \cdot \left(\frac{K}{1-K}\right)^n d(x,Tx), \quad n = 0, 1, 2, \cdots .$$

Definition 1.2. [2] Let (X,d) be a metric space. A map $T : X \mapsto X$ is called a *weak contraction* if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx) \text{ for all } x,y \in X. \tag{1.1}$$

Theorem 1.3. *[2] Let (X,d) be a complete metric space and $T : X \mapsto X$ be a weak contraction, that is, a mapping satisfying (1.1) with $\delta \in (0,1)$ and some $L \geq 0$. Then*

(a) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$.

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(b) *For any* $x_0 \in X$, *the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$, *converges to some* $x^* \in F(T)$.

(c) *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1),\ n = 0, 1, 2, \cdots$$

and

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n),\ n = 1, 2, \cdots$$

hold, where $\delta \in (0, 1)$.

Theorem 1.4. *[2] Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *be a weak contraction for which there exist* $\theta \in (0, 1)$ *and some* $L_1 \geq 0$ *such that*

$$d(Tx, Ty) \leq \theta d(x, y) + L_1 d(x, Tx),\ for\ all\ x, y \in X.$$

Then

(a) T *has a unique fixed point, that is,* $F(T) = \{x^*\}$.

(b) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* $x_{n+1} = Tx_n$ *converges to* x^*, *for any* $x_0 \in X$.

(c) *The a priori and a posteriori estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1),\ n = 0, 1, 2, \cdots$$

and

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n),\ n = 1, 2, \cdots$$

hold.

(d) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \theta d(x_{n-1}, x^*),\ n = 1, 2, \cdots.$$

Theorem 1.5. *([3]- [7]) In the framework of a complete metric space* (X, d), *if* $T : X \mapsto X$ *forms a Ciric-Reich-Rus contraction mapping, that is,*

$$d(Tx, Ty) \leq \lambda[d(x, y) + d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, *where* $\lambda \in [0, \frac{1}{3})$, *then* T *possesses a unique fixed point.*

Definition 1.6. [8] Let (X, d) be a metric space. We say that the self-mapping $T : X \mapsto X$ is an *interpolative Kannan type contraction*, if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha}$$

for all $x, y \in X$, $x, y \notin Fix(T)$, where $Fix(T) = \{x \in X : Tx = x\}$.

Theorem 1.7. *[8] Let (X, d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

Definition 1.8. [9] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an *interpolative Berinde weak operator* if it satisfies

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Definition 1.9. [9] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an *alternate interpolative Berinde weak operator* if it satisfies

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in (0, 1)$, for all $x, y \in X \backslash Fix(T)$.

Theorem 1.10. *[9] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X, d) is complete, then the fixed point of T exists.*

Definition 1.11. [10] Let (X, d) be a branciari metric space. A self-mapping T on X is called an *interpolative Ciric-Reich-Rus type contraction*, if there exists $\lambda \in [0, 1)$ and positive reals α, β with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{\beta} d(y, Ty)^{1-\alpha-\beta}$$

for all $x, y \in X \backslash Fix(T)$.

Theorem 1.12. *[10] Let $T : X \mapsto X$ be an interpolative Ciric-Reich-Rus type contraction on a complete Branciari distance space (X, d). Then T has a fixed point in X.*

Definition 1.13. [11] Let (X, d) be a metric space. A map $T : X \mapsto X$ is called an *alternate interpolative Ciric-Reich-Rus operator* if there exists $\lambda \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{3}} d(x, Tx)^{\frac{1}{3}} d(y, Ty)^{\frac{1}{3}}$$

for all $x, y \in X \backslash Fix(T)$.

Theorem 1.14. *[11] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an alternate interpolative Ciric-Reich-Rus operator. If (X, d) is complete, then the fixed point exist.*

2 Main Result

Theorem 2.1. *(Exponential Interpolative Kannan Contraction Mapping Theorem) Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an exponential interpolative Kannan contraction operator, that is, T satisfies*

$$e^{d(Tx,Ty)} \leq k e^{\alpha d(x,Tx)+(1-\alpha)d(y,Ty)},$$

where $k \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X \backslash Fix(T)$. If (X, d) is complete, then the fixed point of T is unique in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n, and also define

$$P_n = e^{d(x_n, x_{n+1})}.$$

Letting $x = x_{n-1}$ and $y = x_n$ in the contractive definition of the theorem, we have

$$e^{d(x_n, x_{n+1})} \leq k e^{\alpha d(x_n, x_{n-1}) + (1-\alpha) d(x_n, x_{n+1})}.$$

From which it follows that

$$e^{d(x_n, x_{n+1})} \leq k^{\frac{1}{\alpha}} e^{d(x_n, x_{n-1})}$$

that is, $P_n \leq k^{\frac{1}{\alpha}} P_{n-1}$. By induction, we have

$$d(x_n, x_{n+1}) \leq P_n \leq k^{\frac{n}{\alpha}} P_0.$$

Thus, making use of the above inequality and the triangle inequality, we obtain for all $n \geq 0$ and $m \geq 1$, that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq P_0 (k^{\frac{n}{\alpha}} + k^{\frac{n+1}{\alpha}} + \cdots + k^{\frac{n+m-1}{\alpha}}) \\ &= P_0 k^{\frac{n}{\alpha}} \frac{1 - k^{\frac{m}{\alpha}}}{1-k} \\ &\leq P_0 \frac{k^{\frac{n}{\alpha}}}{1-k} \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), ensuring its convergence to a point $x^* \in X$. To confirm that x^* is a fixed point, we substitute $x = x^*$ and $y = x_n$ in the contractive definition of the theorem, then we have

$$e^{d(Tx^*, x_{n+1})} \leq k e^{\alpha d(Tx^*, x^*) + (1-\alpha) d(x_{n+1}, x_n)}.$$

Passing to the limit as $n \to \infty$ in the above inequality, we deduce that

$$e^{d(Tx^*, x^*)} \leq k e^{\alpha d(Tx^*, x^*)}$$

which implies $e^{(1-\alpha) d(Tx^*, x^*)} \leq 1$ (since $k < 1$). It follows that $d(Tx^*, x^*) = 0$, that is, $Tx^* = x^*$, and so x^* is a fixed point of T. For uniqueness, assume that $Tx = x$ and $Ty = y$, with $x \neq y$. Then, from the contractive definition of the theorem, we have

$$e^{d(x,y)} = e^{d(Tx, Ty)} \leq k e^{\alpha d(x,x) + (1-\alpha) d(y,y)}$$

which implies $e^{d(x,y)} \leq 1$, since $k < 1$. Thus, $d(x, y) = 0$, that is, $x = y$, and the fixed point is unique. This completes the proof. □

Theorem 2.2. *(Exponential (Alternate) Interpolative Berinde Weak Contraction Mapping Theorem) Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an exponential interpolative Berinde weak contraction operator, that is, T satisfies*

$$e^{d(Tx,Ty)} \leq ke^{\frac{1}{2}[d(x,y)+d(x,Tx)]}$$

where $k \in [0, 1)$, for all $x, y \in X \backslash Fix(T)$. If (X, d) is complete, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n, and also define

$$P_n = e^{d(x_n,x_{n+1})}.$$

Letting $x = x_{n-1}$ and $y = x_n$ in the contractive definition of the theorem, we have

$$\begin{aligned} e^{d(x_n,x_{n+1})} &\leq ke^{\frac{1}{2}[d(x_{n-1},x_n)+d(x_{n-1},x_n)]} \\ &= ke^{d(x_{n-1},x_n)}. \end{aligned}$$

From the above inequality, we have $P_n \leq kP_{n-1}$, and by induction we have

$$d(x_n, x_{n+1}) \leq P_n \leq k^n P_0.$$

Thus, making use of the above inequality, and the triangle inequality, we obtain for all $n \geq 0$ and $m \geq 1$ that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq P_0(k^n + k^{n+1} + \cdots + k^{n+m-1}) \\ &= P_0 k^n \frac{1 - k^m}{1 - k} \\ &\leq P_0 \frac{k^n}{1 - k} \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), ensuring its convergence to a point $x^* \in X$. To confirm that x^* is a fixed point, we substitute $x = x^*$ and $y = x_n$ in the contractive definition of the theorem, then we have

$$e^{d(Tx^*,x_{n+1})} \leq ke^{\frac{1}{2}[d(x^*,x_n)+d(x^*,Tx^*)]}.$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain that

$$e^{d(Tx^*,x^*)} \leq ke^{\frac{1}{2}d(x^*,Tx^*)}$$

which implies that $e^{\frac{1}{2}d(x^*,Tx^*)} \leq 1$, since $k < 1$. Thus, $d(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$, and so x^* is a fixed point of T. For uniqueness, assume that $Tx = x$ and $Ty = y$, with $x \neq y$. Then from the contractive definition of the theorem, we have

$$e^{d(x,y)} = e^{d(Tx,Ty)} \leq ke^{\frac{1}{2}[d(x,y)+d(x,x)]} = ke^{\frac{1}{2}d(x,y)}$$

which implies $e^{\frac{1}{2}d(x,y)} \leq 1$, since $k < 1$. Thus, $d(x, y) = 0$, that is, $x = y$, and the fixed point is unique. This completes the proof. □

Theorem 2.3. *(Exponential (Alternate) Interpolative Ciric-Reich-Rus Contraction Mapping Theorem) Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an exponential interpolative Ciric-Reich-Rus contraction operator, that is, T satisfies*

$$e^{d(Tx,Ty)} \leq ke^{\frac{1}{3}[d(x,y)+d(x,Tx)+d(y,Ty)]},$$

where $k \in [0, 1)$, for all $x, y \in X \backslash Fix(T)$. If (X, d) is complete, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all n, and also define

$$P_n = e^{d(x_n,x_{n+1})}.$$

Letting $x = x_{n-1}$ and $y = x_n$ in the contractive definition of the theorem, we obtain

$$\begin{aligned} e^{d(x_n,x_{n+1})} &\leq ke^{\frac{1}{3}[d(x_{n-1},x_n)+d(x_{n-1},x_n)+d(x_n,x_{n+1})]} \\ &\leq ke^{\frac{1}{3}[3d(x_{n-1},x_n)]} \\ &= ke^{d(x_{n-1},x_n)}. \end{aligned}$$

From the above inequality, we have $P_n \leq kP_{n-1}$. By induction, we have

$$d(x_n, x_{n+1}) < P_n \leq k^n P_0.$$

Thus, making use of the above inequality and the triangle inequality, we obtain for $n \geq 0$ and $m \geq 1$ that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq P_0(k^n + k^{n+1} + \cdots + k^{n+m-1}) \\ &= P_0 k^n \frac{1-k^m}{1-k} \\ &\leq P_0 \frac{k^n}{1-k} \to 0 \ as \ n, m \to \infty. \end{aligned}$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), ensuring its convergence to a point $x^* \in X$. To confirm that x^* is a fixed point, substitute $x = x^*$ and $y = x_n$ in the contractive definition of the theorem. Then, we have

$$e^{d(Tx^*,x_{n+1})} \leq ke^{\frac{1}{3}[d(x^*,x_n)+d(x^*,Tx^*)+d(x_n,x_{n+1})]}.$$

Passing to the limit as $n \to \infty$ in the above inequality, we deduce that

$$e^{d(Tx^*,x^*)} \leq ke^{\frac{1}{3}d(x^*,Tx^*)}$$

which implies $e^{\frac{2}{3}d(x^*,Tx^*)} \leq 1$, since $k < 1$. Thus, $d(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$, and so x^* is a fixed point of T. For uniqueness, assume that $Tx = x$ and $Ty = y$, with $x \neq y$. Then from the contractive definition of the theorem, we have

$$\begin{aligned} e^{d(x,y)} &= e^{d(Tx,Ty)} \\ &\leq ke^{\frac{1}{3}[d(x,y)+d(x,x)+d(y,y)]} \\ &= ke^{\frac{1}{3}d(x,y)}. \end{aligned}$$

From the above inequality, we have, $e^{\frac{2}{3}d(x,y)} \leq 1$, since $k < 1$. It follows that $d(x,y) = 0$, and hence, $x = y$. So the fixed point is unique. This completes the proof. □

3 Application

We apply our result to establish an existence theorem for non-linear Fredholm integral equation. Let $Y = C[0,1]$ be a set of all real continuous functions on $[0,1]$ equipped with the metric $p(u,v) = |u - v| = \max_{t\in[0,1]} |u(t) - v(t)|$, for all $u, v \in C[0,1]$. Then (Y,p) is a complete metric space. Now we consider the non-linear Fredholm integral equation

$$u(t) = v(t) + \int_0^1 K(t,s,u(s))ds, \tag{3.1}$$

where $t, s \in [0,1]$. Assume that $K : [0,1] \times [0,1] \times Y \mapsto \mathbb{R}$ and $v : [0,1] \mapsto \mathbb{R}$ are continuous, where $v(t)$ is a given function in Y.

Theorem 3.1. *Let (Y,p) be a metric space equipped with the metric $p(u,v) = |u - v| = \max_{t\in[0,1]} |u(t) - v(t)|$ for all $u, v \in Y$, and $F : Y \mapsto Y$ be an operator on Y defined by*

$$Fu(t) = v(t) + \int_0^1 K(t,s,u(s))ds. \tag{3.2}$$

If there exists $\mu \in [0,1)$ such that for all $u, v \in Y$, $s, t \in [0,1]$, satisfying the following inequality

$$e^{|K(t,s,u(s))-K(t,s,v(s))|} \leq \mu M(u(s), v(s)),$$

where

$$M(u(s), v(s)) = e^{\alpha|u(s)-Fu(s)|+(1-\alpha)|v(s)-Fv(s)|}$$

with $\alpha \in (0,1)$. Then the integral equation (3.2) has a unique solution in Y.

Proof. From (3.1) and (3.2), we obtain

$$\begin{aligned} e^{|Fu(t)-Fv(t)|} &= e^{\left|\int_0^1 K(t,s,u(s))ds - \int_0^1 K(t,s,v(s))ds\right|} \\ &\leq e^{\int_0^1 |K(t,s,u(s))-K(t,s,v(s))|ds} \\ &\leq \int_0^1 e^{|K(t,s,u(s))-K(t,s,v(s))|} ds \\ &\leq \mu \int_0^1 e^{\alpha|u(s)-Fu(s)|+(1-\alpha)|v(s)-Fv(s)|} ds. \end{aligned}$$

Taking the maximum on both sides for all $t \in [0,1]$, we obtain

$$\begin{aligned} e^{p(Fu,Fv)} &= e^{\max_{t\in[0,1]} |Fu(t)-Fv(t)|} \\ &\leq \mu \max_{t\in[0,1]} \int_0^1 e^{\alpha|u(s)-Fu(s)|+(1-\alpha)|v(s)-Fv(s)|} ds \\ &= \mu e^{\alpha p(u,Fu)+(1-\alpha)p(v,Fv)}. \end{aligned}$$

Since $Y = C[0,1]$ is complete metric space, all the conditions of Theorem 2.1 are satisfied. Hence, the integral equation (3.2) has a unique solution in Y. □

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