

Hadamard-Hermite Integral Inequality: An Integration by Parts Point of View

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Abstract

This article focuses on creating new integral inequalities based on, and derived from, the classical Hermite-Hadamard integral inequality for convex functions. A key aspect of these developments is the use of integration by parts. One of the results obtained provides an alternative perspective under a specific smoothness assumption.

1 Introduction

Convex functions are one of the most important classes of functions in mathematical analysis, optimization theory and economics. The formal definition of a convex function is provided below. Let $I \subseteq \mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is said to be convex on I if, for any $x, y \in I$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (1)$$

Geometrically, this means that the line segment joining any two points on the graph of f lies above or on the graph itself.

Convex functions exhibit a rich structure that gives rise to a variety of important inequalities. One such inequality is the well-known Hermite-Hadamard integral inequality, which establishes a relationship between the value of a convex function at the midpoint of an interval, its integral mean and its endpoint values. A formal statement of this inequality is given in the theorem below.

Theorem 1.1 (Hermite-Hadamard integral inequality). *Let $a, b \in \mathbb{R}$ with $b > a$ and $f : [a, b] \rightarrow \mathbb{R}$ be a convex function. Then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}. \quad (2)$$

The Hermite-Hadamard integral inequality is a useful estimate for the integral mean of a convex function. It has a wide range of applications in pure and applied analysis. Many generalizations and refinements of this classical result have been established over the years, involving various classes of convexity, such as s -convexity, quasi-convexity and φ -convexity, as well as extensions using fractional integrals and other methods. See, e.g., [1–19].

In this article, we present a new derivation of the Hermite-Hadamard integral inequality using an integration-by-parts approach that emphasizes the derivative of f . This also yields new bounds under additional smoothness assumptions.

The remainder of the article is as follows: Section 2 presents our main theorem and its detailed proof, followed by two propositions. Section 3 provides a conclusion.

2 Results

2.1 A general theorem

Beyond the concept of a convex function, the theorem below proposes an integral result involving the integral mean and endpoint values of a function.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, $f : [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable function and*

$$m = \frac{a + b}{2}.$$

Then we have

$$\begin{aligned} & \frac{1}{2}(f(a) + f(b)) - \frac{1}{b - a} \int_a^b f(x) dx \\ &= \frac{1}{b - a} \int_a^b (x - m)(f'(x) - f'(m)) dx. \end{aligned}$$

Proof. An integration by parts with the weight function $x - m$ gives

$$\begin{aligned} \int_a^b f(x) dx &= [(x - m)f(x)]_{x=a}^{x=b} - \int_a^b (x - m)f'(x) dx \\ &= \left(b - \frac{a + b}{2}\right) f(b) - \left(a - \frac{a + b}{2}\right) f(a) - \int_a^b (x - m)f'(x) dx \\ &= \frac{b - a}{2}(f(a) + f(b)) - \int_a^b (x - m)f'(x) dx. \end{aligned} \tag{3}$$

The definition of m gives

$$\int_a^b x dx = \left[\frac{x^2}{2}\right]_{x=a}^{x=b} = \frac{b^2 - a^2}{2} = \frac{b + a}{2} \int_a^b dx = m \int_a^b dx.$$

Therefore, we have

$$\int_a^b (x - m)f'(m)dx = f'(m) \int_a^b (x - m)dx = f'(m) \left(\int_a^b xdx - m \int_a^b dx \right) = 0. \tag{4}$$

It follows from Equations (3) and (4) that

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b f(x)dx + \int_a^b (x - m)f'(m)dx \\ &= \frac{b - a}{2}(f(a) + f(b)) - \int_a^b (x - m)f'(x)dx + \int_a^b (x - m)f'(m)dx \\ &= \frac{b - a}{2}(f(a) + f(b)) - \int_a^b (x - m)(f'(x) - f'(m))dx. \end{aligned}$$

Dividing by $b - a$ and rearranging, we derive

$$\begin{aligned} \frac{1}{2}(f(a) + f(b)) - \frac{1}{b - a} \int_a^b f(x)dx \\ = \frac{1}{b - a} \int_a^b (x - m)(f'(x) - f'(m))dx. \end{aligned}$$

This concludes the proof. □

The main advantage of this integral representation lies in the presence of the difference terms $x - m$ and $f'(x) - f'(m)$, which can be effectively manipulated within the integral to derive refined inequalities. Two propositions illustrating this idea are presented in the subsection below.

2.2 Two propositions

The proposition below reestablishes one side of the Hermite-Hadamard integral inequality by means of Theorem 2.1.

Proposition 2.2. *Let $a, b \in \mathbb{R}$ with $b > a$ and $f : [a, b] \rightarrow \mathbb{R}$ be a twice differentiable convex function. Then Theorem 2.1 implies the following side of the Hermite-Hadamard integral inequality:*

$$\frac{1}{2}(f(a) + f(b)) \geq \frac{1}{b - a} \int_a^b f(x)dx.$$

Proof. Let $m = (a + b)/2$. Since f is a twice differentiable convex function, f' is non-decreasing. This implies that, for any $x \in [m, b]$, we have $f'(x) \geq f'(m)$, and, for any $x \in [a, m]$, we have $f'(x) \leq f'(m)$. Therefore, for any $x \in [a, b]$, we have

$$(x - m)(f'(x) - f'(m)) \geq 0.$$

Theorem 2.1 implies that

$$\begin{aligned} & \frac{1}{2}(f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{b-a} \int_a^b (x-m)(f'(x) - f'(m)) dx \geq 0, \end{aligned}$$

so that

$$\frac{1}{2}(f(a) + f(b)) \geq \frac{1}{b-a} \int_a^b f(x) dx.$$

This completes the proof. \square

The proposition below provides a simple upper bound on the absolute value of the difference between the integral mean and the endpoint values of a function, provided that an appropriate smoothness assumption is imposed on f' .

Proposition 2.3. *Let $a, b \in \mathbb{R}$ with $b > a$ and $f : [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable function such that f' is Hölder-continuous, i.e., there exist $\alpha, \beta > 0$ such that, for any $x, y \in [a, b]$,*

$$|f'(x) - f'(y)| \leq \alpha|x - y|^\beta.$$

Then we have

$$\left| \frac{1}{2}(f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{\alpha}{\beta+2} \left(\frac{b-a}{2} \right)^{\beta+1}.$$

Proof. Let $m = (a + b)/2$. It follows from Theorem 2.1 that

$$\begin{aligned} & \frac{1}{2}(f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{b-a} \int_a^b (x-m)(f'(x) - f'(m)) dx. \end{aligned}$$

Therefore, by the triangle inequality and the Hölder-continuity of f' , we have

$$\begin{aligned} & \left| \frac{1}{2}(f(a) + f(b)) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ &= \left| \frac{1}{b-a} \int_a^b (x-m)(f'(x) - f'(m)) dx \right| \\ &\leq \frac{1}{b-a} \int_a^b |x-m| |f'(x) - f'(m)| dx \\ &\leq \frac{1}{b-a} \int_a^b |x-m| \alpha |x-m|^\beta dx \\ &= \frac{\alpha}{b-a} \int_a^b |x-m|^{\beta+1} dx. \end{aligned} \tag{5}$$

An integral calculus involving power functions and the definition of m give

$$\begin{aligned}
 \int_a^b |x - m|^{\beta+1} dx &= \left(\int_a^m (m - x)^{\beta+1} dx + \int_m^b (x - m)^{\beta+1} dx \right) \\
 &= \left(\left[-\frac{(m - x)^{\beta+2}}{\beta + 2} \right]_{x=a}^{x=m} + \left[\frac{(x - m)^{\beta+2}}{\beta + 2} \right]_{x=m}^{x=b} \right) \\
 &= \frac{1}{\beta + 2} \left((m - a)^{\beta+2} + (b - m)^{\beta+2} \right) \\
 &= \frac{2}{\beta + 2} \left(\frac{b - a}{2} \right)^{\beta+2}. \tag{6}
 \end{aligned}$$

Combining Equations (5) and (6), we get

$$\begin{aligned}
 &\left| \frac{1}{2}(f(a) + f(b)) - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
 &\leq \frac{\alpha}{b - a} \times \frac{2}{\beta + 2} \left(\frac{b - a}{2} \right)^{\beta+2} = \frac{\alpha}{\beta + 2} \left(\frac{b - a}{2} \right)^{\beta+1}.
 \end{aligned}$$

This completes the proof. \square

In particular, applying Proposition 2.3 to $\beta = 1$, corresponding to the Lipschitz-continuity on f' , we get

$$\left| \frac{1}{2}(f(a) + f(b)) - \frac{1}{b - a} \int_a^b f(x) dx \right| \leq \frac{\alpha(b - a)^2}{12}.$$

3 Conclusion

In this article, we present new integral inequalities inspired by the classical Hermite-Hadamard integral inequality for convex functions. Our approach, which is based on integration by parts, provides refined bounds that emphasize the importance of the difference terms $x - m$ and $f'(x) - f'(m)$. These results offer a unified framework for deriving sharper estimates under various smoothness assumptions. Future research could involve extending these inequalities to fractional integrals, generalized convexity classes and multidimensional settings.

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