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# Chatterjea Polynomial Contraction Mapping Theorem in Metric Spaces with Application

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**Abstract**

In this paper, we introduce the notion of polynomial Chatterjea contraction mapping in metric spaces, and obtain a fixed point theorem. Some consequences of the main result and a conjecture are stated. The conjecture is illustrated with an example, and the conjecture is used to show existence and uniqueness of solutions for a certain class of fractional differential equations.

## 1 Introduction and Preliminaries

**Theorem 1.1.** *[1] Let $(X,d)$ be a metric space and suppose $T : X \mapsto X$ is a mapping satisfying the following contractive condition*

$$d(Tx,Ty) \leq kd(x,y)$$

*for all $x,y \in X$ and $k \in [0,1)$. If $(X,d)$ is complete, then $T$ has a unique fixed point.*

**Definition 1.2.** [2] A mapping $T : X \mapsto X$ on a metric space $(X,d)$ is termed a *Kannan contraction* if there exists a constant $\lambda$ with $0 \leq \lambda < \frac{1}{2}$ such that for every pair $(x,y)$ the inequality below is valid

$$d(Tx,Ty) \leq \lambda(d(x,Tx) + d(y,Ty)).$$

**Theorem 1.3.** *[3] A metric space $(X,d)$ is complete precisely when every self-mapping $T$ that satisfies the Kannan-type contraction condition possesses fixed point.*

**Definition 1.4.** [4] Let $T$ be a self-mapping on $X$ with starting point $x_0 \in X$. The sequence $\{x_n\}$ constructed by the recursive formula

$$x_{n+1} = Tx_n, \ n \in \mathbb{N}$$

is called the *Picard iteration sequence* generated by $T$ and $x_0$.

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**Definition 1.5.** [5] A self-mapping $T$ on a metric space $(X, d)$ is termed a *polynomial contraction* if there are a collection of functions $a_1, \cdots, a_k : X \times X \mapsto \mathbb{R}_+$ and $\lambda \in [0, 1)$ such that for some natural number $k$, the inequality

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)d^i(x, y)$$

holds true.

**Theorem 1.6.** *[5] Consider a complete metric space $(X, d)$ endowed with $T : X \mapsto X$, a polynomial contraction with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto \mathbb{R}_+$. Assuming further that the conditions below are in effect*

(i) *The mapping $T$ is continuous.*

(ii) *There is an index $j \in \{1, 2, \cdots, k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.*

*Then $T$ possesses exactly one fixed point $x^*$. Moreover, for any starting value $x_0$ in $X$ the Picard sequence $\{x_n\}$ converges to $x^*$.*

**Definition 1.7.** [4] A mapping $T : X \mapsto X$ on a metric space $(X, d)$ is termed a *polynomial Kannan contraction* if there exists $\lambda$ with $0 \leq \lambda < \frac{1}{2}$, a natural number $k$, and a collection $a_i : X \times X \mapsto \mathbb{R}_+$ for $i = 1, 2, \cdots, k$ satisfying the following

$$\sum_{i=1}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda\left(\sum_{i=1}^{k} a_i(x, Tx)d^i(x, Tx) + \sum_{i=0}^{k} a_i(y, Ty)d^i(y, Ty)\right)$$

for all $x, y \in X$.

**Theorem 1.8.** *[4] Consider a complete metric space $(X, d)$ endowed with a mapping $T : X \mapsto X$ that satisfies a polynomial Kannan contractive condition with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto \mathbb{R}_+$. Assume further that the following condtions are in effect*

(i) *The coefficient function $a_i$ is symmetric in its arguments and continuous in its second argument for each $i = 1, 2, \cdots, k$.*

(ii) *There is an index $j \in \{1, 2, \cdots, k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies the inequality $a_j(x, y) \geq A_j$, for all $x, y \in X$.*

*Then $T$ posesses exactly one fixed point $x^*$. Moreover, for any starting value $x_0$ in $X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

**Theorem 1.9.** *[6] Let $(X, d)$ be a metric space. Suppose $T : X \mapsto X$ is a mapping satisfying*

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$$

*for all $x, y \in X$ and $\alpha \in [0, \frac{1}{2})$. If $(X, d)$ is complete, then $T$ has a unique fixed point.*

## 2 Main Result

**Definition 2.1.** A mapping $T : X \mapsto X$ on a metric space $(X, d)$ will be called a *polynomial Chatterjea contraction* if there exists $\lambda$ with $0 \leq \lambda < \frac{1}{2}$, a natural number $k$, and a collection $a_i : X \times X \mapsto \mathbb{R}_+$ for $i = 1, \cdots, k$ satisfying the following

$$\sum_{i=1}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda\Big(\sum_{i=1}^{k} a_i(x, Ty)d^i(x, Ty) + \sum_{i=1}^{k} a_i(y, Tx)d^i(y, Tx)\Big)$$

for all $x, y \in X$.

*Remark* 2.2. The generalized Chatterjea type condition introduced above encompasses various new classes of contractive mappings as particular instances:

(*i*) Setting $k = 1$ and $a_1 \equiv 1$ yields the classical Chatterjea contraction originally introduced by Chatterjea [6], which satisfies

$$d(Tx, Ty) \leq \lambda(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$.

(*ii*) Choosing $k = 2$, $a_1 \equiv 0$ and $a_2 \equiv 1$ leads to the pure quadratic Chatterjea contraction, that is, a self-mapping $T$ on a metric $(X, d)$ for which there exist a constant $\lambda \in [0, \frac{1}{2})$ such that

$$d^2(Tx, Ty) \leq \lambda[d^2(x, Ty) + d^2(y, Tx)]$$

for all $x, y \in X$.

(*iii*) Similarly, setting $k = 3$, $a_1 \equiv a_2 \equiv 0$ and $a_3 \equiv 1$ gives rise to the pure cubic Chatterjea contraction, where a self-mapping $T$ on $(X, d)$ satisfies

$$d^3(Tx, Ty) \leq \lambda[d^3(x, Ty) + d^3(y, Tx)]$$

for all $x, y \in X$ and for a certain $\lambda \in [0, \frac{1}{2})$.

(*iv*) More generally, setting $k = m$, with $a_j \equiv 0$, for $1 \leq j \leq m - 1$ and $a_m \equiv 1$ yields a pure Chatterjea contraction of $m$-power, defined by the condition

$$d^m(Tx, Ty) \leq \lambda[d^m(x, Ty) + d^m(y, Tx)]$$

for all $x, y \in X$ and a certain $\lambda \in [0, \frac{1}{2})$.

**Theorem 2.3.** *Consider a complete metric space $(X, d)$ endowed with a mapping $T : X \mapsto X$ that satisfies a polynomial Chatterjea contractive condition with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto \mathbb{R}_+$. Assume further the conditions below are in effect* (*i*) *The coefficient function $a_i$ is symmetric in its arguments and continuous in its second argument for each $i = 1, 2, \cdots, k$.*

(*ii*) *There is an index $j \in \{1, 2, \cdots, k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies the inequality $a_j(x, y) \geq A_j$ for all $x, y \in X$.*

*Then $T$ possesses exactly one fixed point $x^*$. Moreover, for any starting value $x_0$ in $X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

*Proof.* Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}_{n=0}^{\infty}$ recursively by $x_{n+1} = Tx_n$ for all $n$ and also define

$$P_n = \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}).$$

Letting $x = x_n$ and $y = x_{n+1}$ in Definition 2.1 yields

$$\begin{aligned}\sum_{i=1}^{k} a_i(x_{n+1}, x_{n+2})d^i(x_{n+1}, x_{n+2}) &\leq \lambda\Bigg(\sum_{i=1}^{k} a_i(x_n, x_{n+2})d^i(x_n, x_{n+2}) + \sum_{i=1}^{k} a_i(x_{n+1}, x_{n+1})d^i(x_{n+1}, x_{n+1})\Bigg) \\ &\leq \lambda\Bigg(\sum_{i=1}^{k}\{a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}) + a_i(x_{n+1}, x_{n+2})d^i(x_{n+1}, x_{n+2})\}\Bigg)\end{aligned}$$

that is, $P_{n+1} \leq \lambda(P_n + P_{n+1})$ which implies that

$$P_{n+1} \leq \frac{\lambda}{1-\lambda}P_n = \gamma P_n$$

where $\gamma = \frac{\lambda}{1-\lambda} < 1$, since $\lambda < \frac{1}{2}$. It follows by induction that

$$P_n \leq \gamma^n P_0$$

for all $n \geq 0$. Now from (ii), there is an index $j$ such that $a_j(x_n, x_{n+1}) \geq A_i > 0$. Then we have

$$A_j d^j(x_n, x_{n+1}) \leq \sum_{i=1}^{k} a_i(x_n, x_{n+1})d^i(x_n, x_{n+1}) = P_n \leq \gamma^n P_0$$

which implies that

$$d(x_n, x_{n+1}) \leq \left(\frac{\gamma^n P_0}{A_j}\right)^{\frac{1}{j}}.$$

Applying the triangle inequality for $m > n \geq 0$, it follows that

$$d(x_n, x_m) \leq \sum_{l=n}^{m-1} d(x_l, x_{l+1}) \leq \left(\frac{P_0}{A_j}\right)^{\frac{1}{j}} \sum_{l=n}^{\infty} \gamma^{\frac{l}{j}} = \left(\frac{P_0}{A_j}\right)^{\frac{1}{j}} \frac{\gamma^{\frac{n}{j}}}{1-\gamma^{\frac{1}{j}}}.$$

Given that $\gamma < 1$, the right-hand side approaches to zero. Thus $\{x_n\}$ is a Cauchy sequence in the complete metric space $X$, ensuring its convergence to a point $x^* \in X$. To confirm that $x^*$ is a fixed point, we substitute $x = x^*$ and $y = x_n$ in Definition 2.1, then we have

$$\sum_{i=1}^{k} a_i(Tx^*, x_{n+1})d^i(Tx^*, x_{n+1}) \leq \lambda\Bigg(\sum_{i=1}^{k} a_i(x^*, x_{n+1})d^i(x^*, x_{n+1}) + \sum_{i=1}^{k} a_i(x_n, Tx^*)d^i(x_n, Tx^*)\Bigg).$$

Passing to the limit as $n \to \infty$ and applying the continuity of $a_i$ in its second argument, we deduce that

$$\sum_{i=1}^{k} a_i(Tx^*, x^*)d^i(Tx^*, x^*) \leq \lambda\Big(\sum_{i=1}^{k} a_i(x^*, x^*)d^i(x^*, x^*) + \sum_{i=1}^{k} a_i(x^*, Tx^*)d^i(x^*, Tx^*)\Big)$$

that is,

$$P(Tx^*, x^*) \leq \lambda P(x^*, Tx^*)$$

where $P(u, v) = \sum_{i=1}^{k} a_i(u, v)d^i(u, v)$. Owing to $\lambda < 1$ and $a_i$ is symmetric in its arguments, it follows that $Tx^* = x^*$. Assume, for the purposes of proving uniqueness that $x^*$ and $y^*$ are both fixed points of $T$, then using Definition 2.1, we conclude that

$$\begin{aligned} P(x^*, y^*) &\leq \lambda(P(x^*, y^*) + P(y^*, x^*)) \\ &\leq 2\lambda P(x^*, y^*). \end{aligned}$$

Since $1 - 2\lambda \neq 0$, then $P(x^*, y^*) = 0$, which in view of (ii), leads to $x^* = y^*$, and this completes the proof. □

Now using Remark 2.2 and Theorem 2.3, we have the following

**Corollary 2.4.** *Let $(X, d)$ be a metric space. If $(X, d)$ is complete and $T : X \mapsto X$ is a Chatterjea contraction, then there is a unique element $x^*$ with $Tx^* = x^*$. For every starting value $x_0$ in $X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \ n \in \mathbb{Z}_{\geq 0}$$

*converges to $x^*$ with respect to the metric $d$.*

**Corollary 2.5.** *Let $(X, d)$ be a metric space. If $(X, d)$ is complete and $T : X \mapsto X$ is a pure quadratic Chatterjea contraction, then there is a unique element $x^*$ with $Tx^* = x^*$. For every starting value $x_0$ in $X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \ n \in \mathbb{Z}_{\geq 0}$$

*converges to $x^*$ with respect to the metric $d$.*

**Corollary 2.6.** *Let $(X, d)$ be a metric space. If $(X, d)$ is complete and $T : X \mapsto X$ is a pure cubic Chattterjea contraction, then there is a unique element $x^*$ with $Tx^* = x^*$. For every starting value $x_0$ in $X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \ n \in \mathbb{Z}_{\geq 0}$$

*converges to $x^*$ with respect to the metric $d$.*

**Corollary 2.7.** *Let $(X,d)$ be a metric space. If $(X,d)$ is complete and $T: X \mapsto X$ is a pure Chatterjea contraction of $m$ power, then there is a unique element $x^*$ with $Tx^* = x^*$. For every starting value $x_0$ in $X$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recurrence relation*

$$x_{n+1} = Tx_n, \ n \in \mathbb{Z}_{\geq 0}$$

*converges to $x^*$ with respect to the metric $d$.*

Now to conclude this section, we state an open problem and illustrate it.

**Conjecture 2.8.** *Consider a complete metric space $(X,d)$ endowed with a mapping $T: X \mapsto X$ that satisfies a polynomial Chatterjea contractive condition with respect to a collection of functions $a_1, \cdots, a_k : X \times X \mapsto \mathbb{R}_+$. Assume further that the conditions below are in effect*

*(i) For every $1 \leq i \leq k$, there exists $M_i > 0$ such that the coefficient function $a_i$ satisfies $a_i(x,y) \leq M_i$ across all $(x,y)$ in $X \times X$.*

*(ii) There is an index $j \in \{1,2,\cdots,k\}$ and $A_j > 0$ such that the corresponding coefficient function $a_j$ satisfies the inequality $a_j(x,y) \geq A_j$, for all $x,y \in X$.*

*(iii) Each coefficient function $a_i$ is lower semi-continuous in its first argument.*

*Then $T$ possesses exactly one fixed point $x^*$. Moreover, for any starting value $x_0$ in $X$, the Picard sequence $\{x_n\}$ converges to $x^*$.*

**Example 2.9.** We consider $(X,d)$ a metric space with $X = [0,1]$ and the standard Euclidean metric $d(x,y) = |x-y|$. Define $T: X \mapsto X$ by

$$T(x) = \frac{x^2}{3}.$$

$T$ fails to be a classical Chatterjea contraction. For instance, taking $x = 0$, $y = 1$, we observe that

$$d(T0,T1) = \frac{1}{3} \leq \frac{4}{3}\lambda = \lambda[d(0,T1) + d(1,T0)]$$

which implies that $\lambda \geq \frac{1}{4}$, a contradiction to the fact that $\lambda \in [0,\frac{1}{2})$. We show $T$ is a polynomial Chatterjea contraction with $k = 1$ and weight function $a_1(x,y) = 1 + x^2 + y^2$. To verify the contraction condition, we must establish that for some $\lambda \in [0,\frac{1}{2})$, the inequality below holds for every pair $x,y \in X$:

$$\left(1 + \frac{x^4}{9} + \frac{y^4}{9}\right)\left|\frac{x^2}{3} - \frac{y^2}{3}\right| \leq \frac{5}{19}\left[\left(1 + x^2 + \frac{y^4}{9}\right)\left|x - \frac{y^2}{3}\right| + \left(1 + y^2 + \frac{x^4}{9}\right)\left|y - \frac{x^2}{3}\right|\right].$$

Figure 1 shows that the "BlueGreenYellow" surface (RHS of the contraction inequality) dominates the "RustTones" surface (LHS of the contraction inequality). This demonstrates that the inequality is satisfied for all $x,y \in [0,1]$ with $\lambda = \frac{5}{19}$. Since $\frac{5}{19} < \frac{1}{2}$, $T$ satisfies the polynomial Chatterjea contraction condition with the given parameters. Moreover the function $a_1(x,y)$ is bounded above by 3, that is, $a_1(x,y) \leq 3$. It also satisfies the lower bound $a_1(x,y) \geq 1 = A_1 > 0$. Therefore the requirements of the above conjecutre hold, ensuring that $T$ has exactly one fixed point, which is $x = 0$.

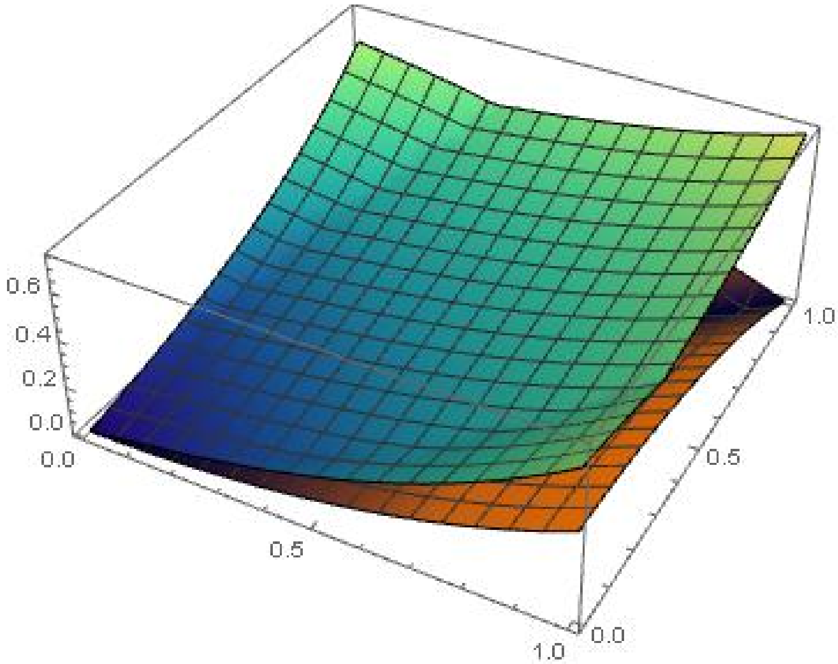


Figure 1: Comparison of both sides of the polynomial Chatterjea inequality in the above example over $[0, 1]$.

## 3 Application

This section is devoted to applying the conjecture to prove the existence and uniqueness of solutions for a certain class of nonlinear PDEs involving the Caputo derivative. Let us focus on the following initial value problem (IVP) involving a Caputo fractional derivative

$$\begin{cases} D^{\alpha}y(t) = f(t, y(t)), \ t \in [0, L] \\ y(0) = y_0 \end{cases}.$$

In this context, $f : [0, L] \times \mathbb{R} \mapsto \mathbb{R}$ denotes a specified nonlinear function and the operator $D^{\alpha}$ is defined as the Caputo derivative of fractional order $\alpha \in (0, 1)$. It is well established that the above IVP problem can be reformulated as the following integral equation of the Volterra type:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s)) ds.$$

We aim to demonstrate that a continuous solution to the IVP problem exists and is unique. We consider the space $C([0, L])$ of continuou functions $y : [0, L] \mapsto \mathbb{R}$, which when equipped with the following supremum norm $\|\cdot\|_\infty$, forms a complete metric space:

$$d(y,z) = \|y-z\|_\infty = \sup_{t\in[0,L]} |y(t)-z(t)|.$$

Define $T : C([0,L]) \mapsto C([0,L])$ by

$$(Ty)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))ds.$$

We proceed to establish the following theorem, which guarantees the unique existence of a solution to the above IVP problem.

**Theorem 3.1.** *Let $\alpha \in (0,1)$, $L > 0$, and suppose the nonlinear function $f : [0,L]\times\mathbb{R} \mapsto \mathbb{R}$ satisfies*

(*A1*) *$f$ is continuous on $[0,L]\times\mathbb{R}$.*

(*A2*) *There exists a constant $\mu > 0$ such that for all $t\in[0,L]$ and all $u,v\in\mathbb{R}$,*

$$|f(t,u)-f(t,v)| \le \mu(|u-Tv| + |v-Tu|)$$

*where $T$ is defined as above and $2\mu L^\alpha < \Gamma(\alpha+1)$.*

*Then the Capuro fractional IVP problem possesses a unique solution in $C([0,L])$.*

*Proof.* Let $y,z\in C([0,L])$. Given any $t\in[0,L]$, we estimate

$$\begin{aligned} |Ty(t)-Tz(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,y(s)) - f(s,z(s))]ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y(s)) - f(s,z(s))|ds. \end{aligned}$$

By assumption (A2), we have

$$|f(s,y(s)) - f(s,z(s))| \le \mu(|y(s)-Tz(s)| + |z(s)-Ty(s)|).$$

Substituting into the integral above we have

$$\begin{aligned} |Ty(t)-Tz(t)| &\le \frac{\mu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(|y(s)-Tz(s)| + |z(s)-Ty(s)|)ds \\ &\le \frac{\mu}{\Gamma(\alpha)}(\|y-Tz\|_\infty + \|z-Ty\|_\infty)\int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{\mu t^\alpha}{\Gamma(\alpha+1)}(\|y-Tz\|_\infty + \|z-Ty\|_\infty). \end{aligned}$$

Evaluating the supremum for each $t\in[0,L]$, we conclude that

$$\|Ty-Tz\|_\infty \le \lambda(\|y-Tz\|_\infty + \|z-Ty\|_\infty)$$

where $\lambda = \frac{\mu L^\alpha}{\Gamma(\alpha+1)}$. Now set $k = 1$ and define the coefficeint function $a_1 : C([0, L]) \times C([0, L]) \mapsto \mathbb{R}_+$ by $a_1(x, y) = C$, where $C$ is a positive real number. Thus the above inequality becomes

$$a_1(Ty, Tz)d(Ty, Tz) \leq \lambda[a_1(y, Tz)d(y, Tz) + a_1(z, Ty)d(z, Ty)]$$

which shows that $T$ satisfies the condition of a polynomial Chatterjea contraction with a bounded coefficient function $a_1$ and $k = 1$. Since the metric space $(C([0, L]), d)$ is complete and all the assumptions of the conjecture are fulfilled, it follows that $T$ possesses a fixed point $y^* \in C([0, L])$ uniquely. Moreover, the sequence $\{y_n\}$ generated by the Picard iteration

$$y_{n+1} = Ty_n$$

converges to $y^*$. Hence the fractional IVP problem possesses a unique solution in $C([0, L])$. □

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