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A Collection of Inequalities Involving Power Exponential and Logarithmic Functions

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Abstract

Power exponential functions and logarithmic functions, are two classes of functions which are ubiquitous in Mathematical Analysis with lots of contemporary applications. In this article, interpolation type inequalities involving power exponential and logarithmic functions are derived, and the techniques applied to derive these inequalities are not the usual that somebody encounters in the literature. All the results are derived using functional estimates and popular integral inequalities such as the Chebyshev integral inequality version. Most of the authors in the literature use monotonicity properties and series expansions, whereas in the current work the inequalities are rigorously proved using predominantly functional estimates, which is a technique more encountered in Functional Analysis and PDEs. To the best of our knowledge, the inequalities are new in the literature and the methods to yield the inequalities is novel and non trivial. This work serves in dual manner, having a research and pedagogical purpose and contributes to the field of Mathematical Analysis and Inequalities.

1 Introduction

Mathematical inequalities play an essential part in Mathematical Analysis, as many times various bounds are required to do appropriate approximations of functions and say whether a function has a specific bound and the graph of the function is below another one. In this article, the main results are interpolation type of inequalities involving power exponential and logarithmic functions. This class of functions has contemporary applications in other parts of Mathematics such as Analytic Number Theory and moreover applications in other fields such as Engineering and Physics. The current stream of work is motivated by these works in the literature ([1], [2], [3], [4], [5], [6], [7], [8]) where the authors obtain various bounds involving power exponential functions. This work could also be seen as an extension of the work in the literature, where novel inequalities involving logarithmic functions ([10]) are obtained. There are many works in the literature of Mathematical Analysis and Inequalities. The most dominant strategies to derive novel inequalities in the literature is to use classical analysis techniques, such as monotonicity properties and series expansions. Recently Chesneau adopted another approach to derive trigonometric inequalities

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using primitive integral methods and he states that this approach is underexplored in the literature ([9]). In this work, more powerful tools are employed, mainly used in Functional Analysis and PDE theory and the results are obtained using functional estimates involving Lebesgue normed quantities. To the best knowledge of the author this approach is not employed at all in classical analysis comparing with the works of Szego and Polya ([12], [13]), Mitrinovic ([14], [15]), Hardy-Littlewood-Polya ([16]), Zygmund ([17]) and other mathematical analysts of the 20th century. This approach has been tested before, generating novel trigonometric and logarithmic inequalities via estimates involving Lebesgue norms ([11]). This machinery opens a new path for deriving novel inequalities for another class of functions such as hyperbolic, inverse trigonometric functions, inequalities involving transcendental functions in two space coordinates, creating a new path of research in Analysis and Inequalities. In terms of the current work, this is organized as follows: In Section 2, there is a paragraph dedicated to motivation and description of the results. In Section 3, the main results are presented as corollaries. In Section 4, some preliminary estimates are established as general theorems and they are rigorously proved. In Section 5, we derive the main results as corollaries of the preliminary estimates and we write down the proofs. Section 6 contains commentary on the main inequalities. Section 7 contains conclusions and remarks. Ultimately, Section 8 contains graphical illustrations of the main inequalities that validate the rigorous mathematical analysis.

2 Motivation and Description of the Results

The work is motivated by the various works on inequalities involving power exponential functions, ([1], [2], [3], [4], [5], [6], [7], [8]) where the authors employ various analytical techniques. In this work explicit inequalities are derived which involve power exponential and logarithmic functions. The machinery used to obtain the results is by employing estimates involving Lebesgue normed quantities. Some inequalities are tight and some inequalities are more coarse in this work. In this research article emphasis is given on the approach to derive the inequalities, rather than the tightness of the inequalities. All the results are graphically verified and there is appropriate commentary for each inequality. In terms of applications, this would motivate further research as power exponential functions and logarithmic functions are useful and can be applied in Analytic Number Theory, Geometric Analysis, Control Theory and various other areas of Analysis.

3 Main Results as Corollaries

In the current section the main results are presented.

Corollary 3.0.1.

$$(x+1)^{\frac{2}{x+1}}\left(1-2\ \frac{\ln(x+1)}{x+1}\right)\leq 1+2^{\frac{4}{3}}\left(\frac{1}{5}\right)^{\frac{1}{5}}\left(\frac{7}{15}\right)^{\frac{7}{15}}\left(\frac{\ln(x+1)}{x+1}\right)^{\frac{4}{3}}\times$$
$$\times\left((x+1)^{\frac{5}{x+1}}-1\right)^{\frac{1}{5}}\left((x+1)^{\frac{15}{7(x+1)}}-1\right)^{\frac{7}{15}}, \tag{1}$$
$$x\ \in[0,+\infty[.$$

Corollary 3.0.2.

$$(x+1)^{\frac{7}{x+1}}\left(1-7\ \frac{\ln(x+1)}{x+1}\right)\leq 1+42\ \left(\frac{1}{6}\right)^{\frac{1}{5}}\ \left(\frac{1}{35}\right)^{\frac{1}{7}}\left(\frac{1}{9}\right)^{\frac{1}{9}}\left(\frac{172}{315}\right)^{\frac{172}{315}}\times$$
$$\times\left(\frac{\ln(x+1)}{x+1}\right)^{\frac{6}{5}}\left((x+1)^{\frac{35}{x+1}}-1\right)^{\frac{1}{7}}\times$$
$$\times\left((x+1)^{\frac{9}{x+1}}-1\right)^{\frac{1}{9}}\left((x+1)^{\frac{315}{172(x+1)}}-1\right)^{\frac{172}{315}}+$$
$$+\frac{\sqrt{42}}{7}\left(\frac{\ln(x+1)}{x+1}\right)^{\frac{3}{2}}\sqrt{(x+1)^{\frac{14}{x+1}}-1}, \tag{2}$$
$$x\ \in[0,+\infty[.$$

Corollary 3.0.3.

$$\frac{\ln(x+1)}{x+1}\leq\left(\frac{1}{5}\right)^{\frac{1}{4}}\left(\frac{1}{6}\right)^{\frac{1}{6}}\left(\frac{1}{12}\right)^{\frac{1}{12}}\left(1-\left(\frac{\ln(x+1)}{x+1}+1\right)^{-1}\right)^{\frac{1}{2}}\times$$
$$\times\left(\left(\frac{\ln(x+1)}{x+1}+1\right)^{5}-1\right)^{\frac{1}{4}}\left((x+1)^{\frac{6}{x+1}}-1\right)^{\frac{1}{6}}\times$$
$$\times\left(1-(x+1)^{\frac{-12}{x+1}}\right)^{\frac{1}{12}}, \tag{3}$$
$$x\ \in[0,+\infty[.$$

Corollary 3.0.4.

$$4(x+1)^{\frac{1}{x+1}}+(x+1)^{\frac{2}{x+1}}+\frac{1}{3}(x+1)^{\frac{3}{x+1}}\leq\frac{16}{3}+2\sqrt{2}\sqrt{\frac{\ln(x+1)}{x+1}}\sqrt{(x+1)^{\frac{2}{x+1}}-1}+$$
$$+\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{2}{3}}\left((x+1)^{\frac{3}{x+1}}-1\right)^{\frac{1}{3}}\left((x+1)^{\frac{3}{2(x+1)}}-1\right)^{\frac{2}{3}}+$$
$$+\left(\frac{1}{4}\right)^{\frac{1}{4}}\left(\frac{3}{4}\right)^{\frac{3}{4}}\left((x+1)^{\frac{4}{x+1}}-1\right)^{\frac{1}{4}}\left((x+1)^{\frac{4}{3(x+1)}}-1\right)^{\frac{3}{4}}+$$
$$+\left(\frac{1}{5}\right)^{\frac{1}{5}}\left(\frac{1}{6}\right)^{\frac{1}{6}}\left(\frac{19}{30}\right)^{\frac{19}{30}}\left((x+1)^{\frac{5}{x+1}}-1\right)^{\frac{1}{5}}\left((x+1)^{\frac{6}{x+1}}-1\right)^{\frac{1}{6}}\left((x+1)^{\frac{30}{19(x+1)}}-1\right)^{\frac{19}{30}}, \tag{4}$$
$$x\ \in[0,+\infty[.$$

Corollary 3.0.5.

$$\begin{aligned}
&\frac{3}{4}(x+1)^{\frac{2}{x+1}}+\frac{1}{8}(x+1)^{\frac{4}{x+1}}\\
&\le \frac{7}{8}+\frac{1}{2}\left((x+1)^{\frac{2}{x+1}}-1\right)^{\frac{1}{2}}\left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}}-(x+1)^{-\frac{2}{x+1}}\right)+\frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}}+\\
&+\frac{1}{2}\left((x+1)^{\frac{2}{x+1}}-1\right)^{\frac{1}{2}}\left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}}-(x+1)^{-\frac{2}{x+1}}\right)-\frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}}+\\
&+\frac{\sqrt{2}}{2}\left(\frac{1}{5}\right)^{\frac{1}{5}}\left(\frac{1}{7}\right)^{\frac{1}{7}}\left(\frac{11}{70}\right)^{\frac{11}{70}}\left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}}-(x+1)^{-\frac{2}{x+1}}\right)+\frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}}\times\\
&\times\left((x+1)^{\frac{5}{x+1}}-1\right)^{\frac{1}{5}}\left((x+1)^{\frac{7}{x+1}}-1\right)^{\frac{1}{7}}\left((x+1)^{\frac{70}{11(x+1)}}-1\right)^{\frac{11}{70}},\\
&x\ \in[0,+\infty[.
\end{aligned} \tag{5}$$

Corollary 3.0.6.

$$\begin{aligned}
&\frac{1}{2}\left((x+1)^{\frac{2}{x+1}}\exp\left(2(x+1)^{\frac{1}{x+1}}\right)-1\right)\\
&\le\left(\frac{\ln(x+1)}{x+1}+(x+1)^{\frac{1}{x+1}}\right)(x+1)^{\frac{2}{x+1}}\exp\left(2(x+1)^{\frac{1}{x+1}}\right)+\\
&+\left(\frac{1}{8}\right)^{\frac{15}{56}}\left(\frac{41}{56}\right)^{\frac{41}{56}}\left(\frac{\ln(x+1)}{x+1}+(x+1)^{\frac{1}{x+1}}\right)^{\frac{8}{7}}\times\\
&\times\left((x+1)^{\frac{8}{x+1}}\exp\left(8(x+1)^{\frac{1}{x+1}}\right)-1\right)^{\frac{1}{8}}\times\\
&\times\left((x+1)^{\frac{56}{41(x+1)}}\exp\left(\frac{56}{41}(x+1)^{\frac{1}{x+1}}\right)-1\right)^{\frac{41}{56}}+\\
&+\left(\frac{1}{12}\right)^{\frac{23}{132}}\left(\frac{109}{132}\right)^{\frac{109}{132}}\left(\frac{\ln(x+1)}{x+1}+(x+1)^{\frac{1}{x+1}}\right)^{\frac{17}{11}}\times\\
&\times\left((x+1)^{\frac{12}{x+1}}\exp\left(12(x+1)^{\frac{1}{x+1}}\right)-1\right)^{\frac{1}{12}}\times\\
&\times\left((x+1)^{\frac{132}{109(x+1)}}\exp\left(\frac{132}{109}(x+1)^{\frac{1}{x+1}}\right)-1\right)^{\frac{109}{132}},\\
&x\ \in[0,+\infty[.
\end{aligned} \tag{6}$$

Corollary 3.0.7.

$$\left((x+1)^{\frac{1}{x+1}}-1\right)^2\le\frac{1}{2}\frac{\ln(x+1)}{x+1}\left((x+1)^{\frac{2}{x+1}}-1\right),\ x\ \in[0,+\infty[. \tag{7}$$

4 Some Preliminary Estimates with Rigorous Proofs

In this section we derive some estimates involving Lebesgue norms, to be employed later on to derive the main results. All the preliminary estimates are stated and are rigorously proved.

Theorem 4.1.

$$\Big|\int_0^{\omega(x)} u^p dt\Big| \leq |\omega(x)\ u^p(\omega(x))| + p\left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}} \omega(x)^{1+\frac{1}{r_1}} ||u^{(p-1)r_2}||_{L_{r_2}(\Omega(x))}\ ||u'||_{L_{r_3}(\Omega(x))}, \tag{8}$$
$$\Omega(x) = (0,\omega(x)),\ \omega(x) > 0,\ t \in\ \Omega(x), u > 0, p \in]1,+\infty[, \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = 1.$$

Proof. The proof follows below

$$\begin{aligned}
\Big|\int_0^{\omega(x)} u^p dt\Big| &= \Big|\int_0^{\omega(x)} (t)'\ u^p\ dt\Big| \\
&= \Big|\ [tu^p]_0^{\omega(x)} - \int_0^{\omega(x)} t\ p\ u^{p-1}\ u' dt\Big| \\
&\leq \Big|\ [tu^p]_0^{\omega(x)}\Big| + \Big|\int_0^{\omega(x)} t\ p\ u^{p-1}\ u' dt\Big| \\
&\leq \Big|\ [tu^p]_0^{\omega(x)}\Big| + p\int_0^{\omega(x)} |\ t\ u^{p-1}\ u'\ |\ dt \\
&\leq \Big|\ [tu^p]_0^{\omega(x)}\Big| + p\left(\int_0^{\omega(x)} t^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} u^{(p-1)r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} u'^{r_3} dt\right)^{\frac{1}{r_3}} \\
&= \Big|\omega(x)u^p(\omega(x))\Big| + p\ \left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}} \omega(x)^{1+\frac{1}{r_1}} ||u^{(p-1)r_2}||_{L_{r_2}(\Omega(x))}\ ||u'||_{L_{r_3}(\Omega(x))}, \\
& p\ \in (1,+\infty),\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = 1
\end{aligned}$$

using integration by parts, employing the triangle inequality and Hölder generalized inequality. □

Theorem 4.2.

$$\Big|\int_0^{\omega(x)} u'^p dt\Big| \leq \Big|\big[\ln(u)\ (u')^{p-1} u\big]_0^{\omega(x)}\Big| +$$
$$+(p-1)||\ln(u)\ ||_{L_{r_1}(\Omega(x))}\ ||\ (u')^{p-2}||_{L_{r_2}(\Omega(x))}\ \ ||\ u''||_{L_{r_3}(\Omega(x))}\ ||u\ ||_{L_{r_4}(\Omega(x))} +$$
$$+||\ln(u)\ ||_{L_2(\Omega(x))}\ \ ||\ (u')^p||_{L_2(\Omega(x))} \tag{9}$$
$$\Omega(x) = (0,\omega(x)),\ \omega(x) > 0,\ t \in\ \Omega(x),$$
$$p\ \in (1,+\infty),\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4} = 1, u > 0.$$

Proof. The proof follows below

$$
\begin{aligned}
&\Big|\int_0^{\omega(x)} u'^p dt\Big| = \Big|\int_0^{\omega(x)} \frac{u'^p}{u} u' \ \frac{u}{u'} \ dt\Big| \\
&= \Big|\int_0^{\omega(x)} \frac{u'}{u} (u')^{p-1} u \ dt \quad \Big| \\
&= \Big| \int_0^{\omega(x)} (\ln(u))' \ \ (u')^{p-1} u \ dt \ \ \Big| \\
&= \Big| \ \big[\ln(u)(u')^{p-1}u\big]_0^{\omega(x)} - (p-1)\int_0^{\omega(x)} \ln(u) \ (u')^{p-2}u'' \ u \ dt - \int_0^{\omega(x)} \ln(u)(u')^{p-1}u' \ dt \ \Big| \\
&= \Big| \ \big[\ln(u)(u')^{p-1}u\big]_0^{\omega(x)} - (p-1)\int_0^{\omega(x)} \ln(u) \ (u')^{p-2}u'' \ u \ dt - \int_0^{\omega(x)} \ln(u)(u')^{p} \ \ dt \ \Big| \\
&\le \Big|\big[\ln(u)(u')^{p-1}u\big]_0^{\omega(x)} \Big| + (p-1)\Big|\int_0^{\omega(x)} \ln(u) \ (u')^{p-2}u'' \ u \ dt\Big| + \\
&+ \Big|\int_0^{\omega(x)} \ln(u)(u')^{p} \ \ dt \ \ \Big| \\
&\le \Big|\big[\ln(u)(u')^{p-1}u\big]_0^{\omega(x)} \Big| + (p-1)\int_0^{\omega(x)} |\ln(u) \ (u')^{p-2}u'' \ u| \ dt + \\
&+ \int_0^{\omega(x)} \ |\ln(u)(u')^{p} \ | \ dt \\
Holder\ and\ Schwarz &\le \Big|\big[\ln(u)(u')^{p-1}u\big]_0^{\omega(x)} \Big| + \\
&+ (p-1) \ \left(\int_0^{\omega(x)} \ \ |\ln(u)|^{r_1} dt\right)^{\frac{1}{r_1}} \ \left(\int_0^{\omega(x)} |(u')^{p-2} \ |^{r_2} dt\right)^{\frac{1}{r_2}} \ \times \\
&\times \left(\int_0^{\omega(x)} |u''|^{r_3} \ dt\right)^{\frac{1}{r_3}} \ \left(\int_0^{\omega(x)} |u|^{r_4} \ dt\right)^{\frac{1}{r_4}} + \\
&+ \left(\int_0^{\omega(x)} |\ln(u)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} |(u')^p|^2 dt\right)^{\frac{1}{2}} \\
&= \Big|\big[\ln(u) \ (u')^{p-1}u\big]_0^{\omega(x)} \Big| + \\
&+ (p-1)||\ln(u) \ ||_{L_{r_1}(\Omega(x))} \ || \ (u')^{p-2}||_{L_{r_2}(\Omega(x))} \ \ || \ u''||_{L_{r_3}(\Omega(x))} \ ||u \ ||_{L_{r_4}(\Omega(x))} + \\
&+ ||\ln(u) \ ||_{L_2(\Omega(x))} \ \ || \ (u')^{p}||_{L_2(\Omega(x))}.
\end{aligned}
$$

□

Theorem 4.3.

$$|\omega(x)| \leq \left(\int_0^{\omega(x)} \left(\frac{u'}{u}\right)^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} \left(\frac{u}{u'}\right)^{r_2} dt\right)^{\frac{1}{r_2}} \times$$
$$\times \left(\int_0^{\omega(x)} \exp(r_3 u) dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} \exp(-r_4 u) dt\right)^{\frac{1}{r_4}}, \tag{10}$$
$$\omega(x) > 0,\ u, u' > 0,\ t \in \Omega(x) = (0, \omega(x)),\ \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1.$$

Proof.

$$\begin{aligned}
|\omega(x)| &= \Big|\int_0^{\omega(x)} dt\Big| \\
&= \Big| \int_0^{\omega(x)} \frac{u'}{u}\ \frac{u}{u'}\ \exp(u)\ \exp(-u) dt \Big| \\
&\leq \int_0^{\omega(x)} \Big|\frac{u'}{u}\ \frac{u}{u'}\ \exp(u)\ \exp(-u)\Big| dt \\
&\leq \left(\int_0^{\omega(x)} \Big|\frac{u'}{u}\Big|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} \Big|\frac{u}{u'}\Big|^{r_2} dt\right)^{\frac{1}{r_2}} \times \\
&\times \left(\int_0^{\omega(x)} \Big|\exp(u)\Big|^{r_3} dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} \Big|\exp(-u)\Big|^{r_4} dt\right)^{\frac{1}{r_4}} \\
&= \left(\int_0^{\omega(x)} \left(\frac{u'}{u}\right)^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} \left(\frac{u}{u'}\right)^{r_2} dt\right)^{\frac{1}{r_2}} \times \\
&\times \left(\int_0^{\omega(x)} (\exp(u))^{r_3} dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} (\exp(-u))^{r_4} dt\right)^{\frac{1}{r_4}} \\
&= \left(\int_0^{\omega(x)} \left(\frac{u'}{u}\right)^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} \left(\frac{u}{u'}\right)^{r_2} dt\right)^{\frac{1}{r_2}} \times \\
&\times \left(\int_0^{\omega(x)} (\exp(r_3 u)) dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} (\exp(-r_4 u)) dt\right)^{\frac{1}{r_4}}.
\end{aligned}$$

□

Theorem 4.4.

$$\begin{aligned}
|J| &\leq \sqrt{\omega(x)}\ ||u\ ||_{L_2(\Omega(x))} + 2\sqrt{\omega(x)}\ ||u'\ ||_{L_2(\Omega(x))} + \sqrt{\omega(x)}\ ||u''\ ||_{L_2(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_1}(\Omega(x))}||u'\ ||_{L_{r_2}(\Omega(x))} + ||u'\ ||_{L_{r_3}(\Omega(x))}\ ||u''\ ||_{L_{r_4}(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_5}(\Omega(x))}\ ||u'\ ||_{L_{r_6}(\Omega(x))}\ ||u''\ ||_{L_{r_7}(\Omega(x))}, \qquad (11)\\
J &= \int_0^{\omega(x)} u\ dt + 2\int_0^{\omega(x)} u' dt + \int_0^{\omega(x)} u'' dt + \int_0^{\omega(x)} u\ u' dt + \\
&+ \int_0^{\omega(x)} u'\ u'' dt + \int_0^{\omega(x)} u\ u'\ u''\ dt, \\
&t\ \in\ \Omega(x) = (0, \omega(x)),\ \omega(x) > 0,\ x > 0, \\
&\frac{1}{r_1} + \frac{1}{r_2} = 1,\ \frac{1}{r_3} + \frac{1}{r_4} = 1, \\
&\frac{1}{r_5} + \frac{1}{r_6} + \frac{1}{r_7} = 1, \\
&r_i\ \in (1, +\infty),\ i = 1, 2, ..., 7.
\end{aligned}$$

Proof.

$$\begin{aligned}
|J| &= \Big| \int_0^{\omega(x)} u\ dt + 2\int_0^{\omega(x)} u' dt + \int_0^{\omega(x)} u'' dt + \int_0^{\omega(x)} u\ u' dt + \\
&+ \int_0^{\omega(x)} u'\ u'' dt + \int_0^{\omega(x)} u\ u'\ u''\ dt \Big| \\
&\leq \Big| \int_0^{\omega(x)} u\ dt \Big| + 2\Big| \int_0^{\omega(x)} u' dt \Big| + \Big| \int_0^{\omega(x)} u'' dt \Big| + \Big| \int_0^{\omega(x)} u\ u' dt \Big| + \\
&+ \Big| \int_0^{\omega(x)} u'\ u'' dt \Big| + \Big| \int_0^{\omega(x)} u\ u'\ u''\ dt \Big| \\
&\leq \int_0^{\omega(x)} |u|\ dt + 2\int_0^{\omega(x)} |u'| dt + \int_0^{\omega(x)} |u''| dt + \int_0^{\omega(x)} |u\ u'| dt + \\
&+ \int_0^{\omega(x)} |u'\ u''| dt + \int_0^{\omega(x)} |u\ u'\ u''\ | dt \\
\text{Schwarz and Holder} &\leq \left(\int_0^{\omega(x)} dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} u^2 dt\right)^{\frac{1}{2}} + 2\left(\int_0^{\omega(x)} dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} u'^2 dt\right)^{\frac{1}{2}} + \\
&+ \left(\int_0^{\omega(x)} dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} u''^2 dt\right)^{\frac{1}{2}} + \left(\int_0^{\omega(x)} |u|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u'|^{r_2} dt\right)^{\frac{1}{r_2}} + \\
&+ \left(\int_0^{\omega(x)} |u'|^{r_3} dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} |u''|^{r_4} dt\right)^{\frac{1}{r_4}} + \\
&+ \left(\int_0^{\omega(x)} |u|^{r_5} dt\right)^{\frac{1}{r_5}} \left(\int_0^{\omega(x)} |u'|^{r_6} dt\right)^{\frac{1}{r_6}} \left(\int_0^{\omega(x)} |u''|^{r_7} dt\right)^{\frac{1}{r_7}} \\
&= \sqrt{\omega(x)}\ ||u\ ||_{L_2(\Omega(x))} + 2\sqrt{\omega(x)}\ ||u'\ ||_{L_2(\Omega(x))} + \sqrt{\omega(x)}\ ||u''\ ||_{L_2(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_1}(\Omega(x))}||u'\ ||_{L_{r_2}(\Omega(x))} + ||u'\ ||_{L_{r_3}(\Omega(x))}\ ||u''\ ||_{L_{r_4}(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_5}(\Omega(x))}\ ||u'\ ||_{L_{r_6}(\Omega(x))}\ ||u''\ ||_{L_{r_7}(\Omega(x))}.
\end{aligned}$$

□

Theorem 4.5.

$$|J| \leq \left(\int_0^{\omega(x)} u^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} \cosh^2(t) dt\right)^{\frac{1}{2}} + \left(\int_0^{\omega(x)} u'^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} \sinh^2(t) dt\right)^{\frac{1}{2}} +$$
$$+ \left(\int_0^{\omega(x)} |\cosh(t)|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} |u'|^{r_3} dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} |u''|^{r_4} dt\right)^{\frac{1}{r_4}}, \quad (12)$$
$$\sum_{i=1}^{4} \frac{1}{r_i} = 1, \ r_i \in (1, +\infty),$$
$$J = \int_0^{\omega(x)} u \ \cosh(t) \ dt + \int_0^{\omega(x)} u' \ \sinh(t) \ dt \ + \int_0^{\omega(x)} u \ \cosh(t) \ u' \ u'' \ dt.$$

Proof. Pick the quantity

$$J = \int_0^{\omega(x)} u \ \cosh(t) \ dt + \int_0^{\omega(x)} u' \ \sinh(t) \ dt \ + \int_0^{\omega(x)} u \ \cosh(t) \ u' \ u'' \ dt,$$
$$t \ \in \ \Omega(x) = (0, \omega(x)), \ x > 0, \ \omega(x) > 0.$$

Then, it follows

$$|J| = \Big| \int_0^{\omega(x)} u \ \cosh(t) \ dt + \int_0^{\omega(x)} u' \ \sinh(t) \ dt \ + \int_0^{\omega(x)} u \ \cosh(t) \ u' \ u'' \ dt \ \Big|$$
$$\leq \ \Big| \int_0^{\omega(x)} u \ \cosh(t) \ dt \Big| + \Big| \int_0^{\omega(x)} u' \ \sinh(t) \ dt \Big| + \Big| \int_0^{\omega(x)} u \ \cosh(t) \ u' \ u'' \ dt \Big|$$
$$\leq \int_0^{\omega(x)} |u \ \cosh(t)| \ dt + \int_0^{\omega(x)} |u' \ \sinh(t)| \ dt + \int_0^{\omega(x)} |u \ \cosh(t) \ u' \ u'' \ | dt$$
$$\text{Schwarz and Holder} \leq \left(\int_0^{\omega(x)} u^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} \cosh^2(t) dt\right)^{\frac{1}{2}} + \left(\int_0^{\omega(x)} u'^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\omega(x)} \sinh^2(t) dt\right)^{\frac{1}{2}} +$$
$$+ \left(\int_0^{\omega(x)} |\cosh(t)|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} |u'|^{r_3} dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} |u''|^{r_4} dt\right)^{\frac{1}{r_4}},$$
$$\sum_{i=1}^{4} \frac{1}{r_i} = 1, \ r_i \in (1, +\infty).$$

□

Theorem 4.6.

$$|J| \leq \left|[\ln(u)\ u\ u'']_0^{\omega(x)}\right| + \left(\int_0^{\omega(x)} |\ln(u)|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u'|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} |u''|^{r_3} dt\right)^{\frac{1}{r_3}} +$$
$$+ \left(\int_0^{\omega(x)} |\ln(u)|^{r_4} dt\right)^{\frac{1}{r_4}} \left(\int_0^{\omega(x)} |u|^{r_5} dt\right)^{\frac{1}{r_5}} \left(\int_0^{\omega(x)} |u'''|^{r_6} dt\right)^{\frac{1}{r_6}}, \tag{13}$$
$$\sum_{i=1}^{3} \frac{1}{r_i} = 1,\ \sum_{j=4}^{6} \frac{1}{r_j} = 1,$$
$$J = \int_0^{\omega(x)} u'\ u''\ dt,\ t\ \in \Omega(x) = (0, \omega(x)),\ x > 0,\ \omega(x) > 0,\ u > 0.$$

Proof. Let J be the functional below

$$J = \int_0^{\omega(x)} u'\ u''\ dt,\ t\ \in \Omega(x) = (0, \omega(x)),\ x > 0,\ \omega(x) > 0,\ u > 0.$$

Then, this gives the bound

$$\begin{aligned}
|J| &= \left| \int_0^{\omega(x)} u'\ u''\ dt \right| \\
&= \left| \int_0^{\omega(x)} u'\ u''\ \frac{u}{u}\ dt \right| \\
&= \left| \int_0^{\omega(x)} \frac{u'}{u}\ u\ u'' dt \right| \\
&= \left| [\ln(u)\ u\ u'']_0^{\omega(x)} - \int_0^{\omega(x)} \ln(u)\ u'\ u''\ dt - \int_0^{\omega(x)} \ln(u)\ u\ u'''\ dt \right| \\
&\leq \left|[\ln(u)\ u\ u'']_0^{\omega(x)}\right| + \left|\int_0^{\omega(x)} \ln(u)\ u'\ u''\ dt\right| + \left|\int_0^{\omega(x)} \ln(u)\ u\ u'''\ dt\right| \\
&\leq \left|[\ln(u)\ u\ u'']_0^{\omega(x)}\right| + \int_0^{\omega(x)} |\ln(u)\ u'\ u''\ | dt + \int_0^{\omega(x)} |\ln(u)\ u\ u'''\ | dt \\
&\leq \left|[\ln(u)\ u\ u'']_0^{\omega(x)}\right| + \left(\int_0^{\omega(x)} |\ln(u)|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u'|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} |u''|^{r_3} dt\right)^{\frac{1}{r_3}} + \\
&+ \left(\int_0^{\omega(x)} |\ln(u)|^{r_4} dt\right)^{\frac{1}{r_4}} \left(\int_0^{\omega(x)} |u|^{r_5} dt\right)^{\frac{1}{r_5}} \left(\int_0^{\omega(x)} |u'''|^{r_6} dt\right)^{\frac{1}{r_6}}, \\
&\sum_{i=1}^{3} \frac{1}{r_i} = 1,\ \sum_{j=4}^{6} \frac{1}{r_j} = 1.
\end{aligned}$$

□

Theorem 4.7.

$$
\begin{gathered}
Q \leq \omega(x) \int_0^{\omega(x)} u_1 \ u_2 \ dt + \left(\left(\left(\int_0^{\omega(x)} |u_1 + u_2|^p dt \right)^{\frac{1}{p}} + \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \right)^p + \right. \\
\left. + |u_1 + u_2|^p dt)^{\frac{1}{p}} - \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \right|^p, \qquad (14) \\
Q = \int_0^{\omega(x)} u_1 \ dt \ \int_0^{\omega(x)} u_2 \ dt + 2^p \left(\int_0^{\omega(x)} |u_1|^p dt + \int_0^{\omega(x)} |u_2|^p dt \right), \ p \ \in (2, +\infty), \\
u_1' > 0, u_2' > 0, \ t \ \in \ \Omega(x) = (0, \omega(x)), \ x > 0, \ \omega(x) > 0.
\end{gathered}
$$

Proof. Let the quantity Q be

$$
\begin{gathered}
Q = \int_0^{\omega(x)} u_1 \ dt \ \int_0^{\omega(x)} u_2 \ dt + 2^p \left(\int_0^{\omega(x)} |u_1|^p dt + \int_0^{\omega(x)} |u_2|^p dt \right), \ p \ \in (2, +\infty), \\
u_1' > 0, u_2' > 0, \ t \ \in \ \Omega(x) = (0, \omega(x)), \ x > 0, \ \omega(x) > 0.
\end{gathered}
$$

Then, it follows

$$
\begin{gathered}
Q = \int_0^{\omega(x)} u_1 \ dt \ \int_0^{\omega(x)} u_2 \ dt + 2^p \left(\int_0^{\omega(x)} |u_1|^p dt + \int_0^{\omega(x)} |u_2|^p dt \right) \leq \ \textit{Chebyshev and Hanner inequality} \\
\omega(x) \int_0^{\omega(x)} u_1 \ u_2 \ dt + \left(\left(\left(\int_0^{\omega(x)} |u_1 + u_2|^p dt \right)^{\frac{1}{p}} + \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \right)^p + \right. \\
\left. + \left| \ \left(\int_0^{\omega(x)} |u_1 + u_2|^p dt \right)^{\frac{1}{p}} - \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \ \right|^p. \right.
\end{gathered}
$$

□

5 Proof of Corollaries via Preliminary Estimates

In this section, we provide proof of the main results by employing the preliminary estimates in the previous section. All the results in the main section are immediate corollaries-applications of the rigorously yielded preliminary estimates.

Proof. **Proof of Corollary 3.0.1.** Pick $u(t) = \exp(t)$, $\omega(x) = \frac{\ln(x+1)}{x+1} > 0$ in (8). The left and right hand side of the inequality (8) are given by the following functions

$$L(x) = \Big|\int_0^{\omega(x)} u^p dt\Big| = \frac{(x+1)^{\frac{p}{x+1}} - 1}{p},$$

$$R(x) = \Big|\ [tu^p]_0^{\omega(x)}\ \Big| + p\left(\int_0^{\omega(x)} t^{r_1} dt\right)^{\frac{1}{r_1}}\left(\int_0^{\omega(x)} u^{(p-1)r_2} dt\right)^{\frac{1}{r_2}}\left(\int_0^{\omega(x)} u'^{r_3} dt\right)^{\frac{1}{r_3}}$$
$$= \frac{\ln(x+1)}{x+1}\ (x+1)^{\frac{p}{x+1}}$$
$$+ p\ \left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}}\left(\frac{\ln(x+1)}{x+1}\right)^{1+\frac{1}{r_1}}\left(\frac{1}{(p-1)r_2}\right)^{\frac{1}{r_2}}\left((x+1)^{\frac{(p-1)r_2}{x+1}} - 1\right)^{\frac{1}{r_2}} \times$$
$$\times\left(\frac{1}{r_3}\right)^{\frac{1}{r_3}}\left((x+1)^{\frac{r_3}{x+1}} - 1\right)^{\frac{1}{r_3}}$$

taking into account the Lebesgue norms below

$$||u^{(p-1)r_2}||_{L_{r_2}(\Omega(x))} = \left(\int_0^{\omega(x)} \exp((p-1)r_2)\ dt\right)^{\frac{1}{r_2}}$$
$$= \left(\frac{1}{(p-1)r_2}\right)^{\frac{1}{r_2}}(\exp((p-1)r_2\ \omega(x)) - 1)^{\frac{1}{r_2}},$$
$$||u'||_{L_{r_3}(\Omega(x))} = \left(\int_0^{\omega(x)} \exp(r_3\ t)\ dt\right)^{\frac{1}{r_3}}$$
$$= \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}}(\exp(r_3\omega(x)) - 1)^{\frac{1}{r_3}}.$$

Pick

$$p = 2,\ r_1 = 3,\ r_2 = 5,\ r_3 = \frac{15}{7}.$$

Choosing the above exponents, combining the functions $L(x), R(x)$ in (8) and working algebraically on both sides of the estimate, this yields (1). □

Proof. **Proof of Corollary 3.0.2.** Pick $u = u(t) = \exp(t)$, $\omega(x) = \frac{\ln(x+1)}{x+1}$ in (9). The two functions expressing the left and right hand side of the inequality (9) are

$$L(x) = \Big|\int_0^{\omega(x)} u'^p dt\Big| = \frac{(x+1)^{\frac{p}{x+1}} - 1}{p},$$
$$R(x) = \ |\omega(x)\ \exp(p\ \omega(x))| +$$
$$+(p-1)\ \left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}}(\omega(x))^{1+\frac{1}{r_1}}\left(\frac{1}{(p-2)r_2}\right)^{\frac{1}{r_2}}(\exp((p-2)r_2\ \omega(x)) - 1)^{\frac{1}{r_2}} \times$$
$$\times\left(\frac{1}{r_3}\right)^{\frac{1}{r_3}}(\exp(r_3\omega(x)) - 1)^{\frac{1}{r_3}}\ \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}}(\exp(r_4\omega(x)) - 1)^{\frac{1}{r_4}} +$$
$$+\frac{\sqrt{3}}{3}\ \omega(x)^{\frac{3}{2}}\ \frac{\sqrt{2p}}{2p}\sqrt{\exp(2p\ \omega(x)) - 1}.$$

Pick $p = 7, r_1 = 5, r_2 = 7, r_3 = 9, r_4 = \frac{315}{172}$. Combining the functions $L(x)$ and $R(x)$, forming the inequality (9) and producing the necessary computations on both sides, yields the inequality (2). □

Proof. **Proof of Corollary 3.0.3.** Pick $\omega(x) = \frac{\ln(x+1)}{x+1}$, $u(t) = t+1$ in (10). Then the left and right part of the estimate (10), are expressed by the functions below

$$
\begin{aligned}
L(x) &= |\omega(x)| = \frac{\ln(x+1)}{x+1},\\
R(x) &= \left(\int_0^{\omega(x)} \left(\frac{u'}{u}\right)^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} \left(\frac{u}{u'}\right)^{r_2} dt\right)^{\frac{1}{r_2}} \times\\
&\times \left(\int_0^{\omega(x)} (\exp(r_3 u))\, dt\right)^{\frac{1}{r_3}} \left(\int_0^{\omega(x)} (\exp(-r_4 u)) dt\right)^{\frac{1}{r_4}}\\
&= \left(\frac{1}{r_1-1}\right)^{\frac{1}{r_1}} \left(\frac{1}{r_2+1}\right)^{\frac{1}{r_2}} \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}} \times\\
&\times \left(1-(\omega(x)+1)^{1-r_1}\right)^{\frac{1}{r_1}} \left((\omega(x)+1)^{r_2+1}-1\right)^{\frac{1}{r_2}} \times\\
&\times (\exp(r_3\omega(x))-1)^{\frac{1}{r_3}} (1-\exp(-r_4\omega(x)))^{\frac{1}{r_4}}\\
&= \left(\frac{1}{r_1-1}\right)^{\frac{1}{r_1}} \left(\frac{1}{r_2+1}\right)^{\frac{1}{r_2}} \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}} \times\\
&\times \left(1-\left(\frac{\ln(x+1)}{x+1}+1\right)^{1-r_1}\right)^{\frac{1}{r_1}} \left(\left(\frac{\ln(x+1)}{x+1}+1\right)^{r_2+1}-1\right)^{\frac{1}{r_2}} \times\\
&\times \left(\exp\left(r_3\frac{\ln(x+1)}{x+1}\right)-1\right)^{\frac{1}{r_3}} \left(1-\exp\left(-r_4\frac{\ln(x+1)}{x+1}\right)\right)^{\frac{1}{r_4}}.
\end{aligned}
$$

Consequently, this yields

$$
\begin{aligned}
\frac{\ln(x+1)}{x+1} \leq & \left(\frac{1}{r_1-1}\right)^{\frac{1}{r_1}} \left(\frac{1}{r_2+1}\right)^{\frac{1}{r_2}} \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}} \times\\
&\times \left(1-\left(\frac{\ln(x+1)}{x+1}+1\right)^{1-r_1}\right)^{\frac{1}{r_1}} \left(\left(\frac{\ln(x+1)}{x+1}+1\right)^{r_2+1}-1\right)^{\frac{1}{r_2}} \times\\
&\times \left(\exp\left(r_3\frac{\ln(x+1)}{x+1}\right)-1\right)^{\frac{1}{r_3}} \left(1-\exp\left(-r_4\frac{\ln(x+1)}{x+1}\right)\right)^{\frac{1}{r_4}},\\
& x \in [0,+\infty[.
\end{aligned}
$$

Picking Holder exponents $r_1 = 2, r_2 = 4, r_3 = 6, r_4 = 12$ yields the desired result. □

Proof. **Proof of Corollary 3.0.4.** Pick $u(t) = \exp(t)$, $\omega(x) = \frac{\ln(x+1)}{x+1}, x > 0$ (11). The functions that define the left and right member of the inequality (11) are given below

$$
\begin{aligned}
L(x) &= J \\
&= \int_0^{\omega(x)} u\ dt + 2\int_0^{\omega(x)} u' dt + \int_0^{\omega(x)} u'' dt + \int_0^{\omega(x)} u\ u' dt + \\
&+ \int_0^{\omega(x)} u'\ u'' dt + \int_0^{\omega(x)} u\ u'\ u''\ dt \\
&= 4(x+1)^{\frac{1}{x+1}} + (x+1)^{\frac{2}{x+1}} + \frac{1}{3}\ (x+1)^{\frac{3}{x+1}} - \frac{16}{3}, \\
R(x) &= \sqrt{\omega(x)}\ ||u\ ||_{L_2(\Omega(x))} + 2\sqrt{\omega(x)}\ ||u'\ ||_{L_2(\Omega(x))} + \sqrt{\omega(x)}\ ||u''\ ||_{L_2(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_1}(\Omega(x))} ||u'\ ||_{L_{r_2}(\Omega(x))} + ||u'\ ||_{L_{r_3}(\Omega(x))}\ ||u''\ ||_{L_{r_4}(\Omega(x))} + \\
&+ ||u\ ||_{L_{r_5}(\Omega(x))}\ ||u'\ ||_{L_{r_6}(\Omega(x))}\ ||u''\ ||_{L_{r_7}(\Omega(x))} \\
&= 2\sqrt{2}\sqrt{\frac{\ln(x+1)}{x+1}}\sqrt{(x+1)^{\frac{2}{x+1}} - 1} + \\
&+ \left(\frac{1}{r_1}\right)^{\frac{1}{r_1}} \left(\frac{1}{r_2}\right)^{\frac{1}{r_2}} \left((x+1)^{\frac{r_1}{x+1}} - 1\right)^{\frac{1}{r_1}} \left((x+1)^{\frac{r_2}{x+1}} - 1\right)^{\frac{1}{r_2}} + \\
&+ \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}} \left((x+1)^{\frac{r_3}{x+1}} - 1\right)^{\frac{1}{r_3}} \left((x+1)^{\frac{r_4}{x+1}} - 1\right)^{\frac{1}{r_4}} + \\
&+ \left(\frac{1}{r_5}\right)^{\frac{1}{r_5}} \left(\frac{1}{r_6}\right)^{\frac{1}{r_6}} \left(\frac{1}{r_7}\right)^{\frac{1}{r_7}} \left((x+1)^{\frac{r_5}{x+1}} - 1\right)^{\frac{1}{r_5}} \left((x+1)^{\frac{r_6}{x+1}} - 1\right)^{\frac{1}{r_6}} \times \\
&\times \left((x+1)^{\frac{r_7}{x+1}} - 1\right)^{\frac{1}{r_7}}.
\end{aligned}
$$

Consequently, this yields

$$
\begin{aligned}
&4(x+1)^{\frac{1}{x+1}} + (x+1)^{\frac{2}{x+1}} + \frac{1}{3}\ (x+1)^{\frac{3}{x+1}} - \frac{16}{3} \\
&\leq 2\sqrt{2}\sqrt{\frac{\ln(x+1)}{x+1}}\sqrt{(x+1)^{\frac{2}{x+1}} - 1} + \\
&+ \left(\frac{1}{r_1}\right)^{\frac{1}{r_1}} \left(\frac{1}{r_2}\right)^{\frac{1}{r_2}} \left((x+1)^{\frac{r_1}{x+1}} - 1\right)^{\frac{1}{r_1}} \left((x+1)^{\frac{r_2}{x+1}} - 1\right)^{\frac{1}{r_2}} + \\
&+ \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left(\frac{1}{r_4}\right)^{\frac{1}{r_4}} \left((x+1)^{\frac{r_3}{x+1}} - 1\right)^{\frac{1}{r_3}} \left((x+1)^{\frac{r_4}{x+1}} - 1\right)^{\frac{1}{r_4}} + \\
&+ \left(\frac{1}{r_5}\right)^{\frac{1}{r_5}} \left(\frac{1}{r_6}\right)^{\frac{1}{r_6}} \left(\frac{1}{r_7}\right)^{\frac{1}{r_7}} \left((x+1)^{\frac{r_5}{x+1}} - 1\right)^{\frac{1}{r_5}} \left((x+1)^{\frac{r_6}{x+1}} - 1\right)^{\frac{1}{r_6}} \times \\
&\times \left((x+1)^{\frac{r_7}{x+1}} - 1\right)^{\frac{1}{r_7}}.
\end{aligned}
$$

Pick Holder exponents $r_1 = 3, r_2 = \frac{3}{2}, r_3 = 4, r_4 = \frac{4}{3}, r_5 = 5, r_6 = 6, r_7 = \frac{30}{19}$ in the above inequality, and by trivial manipulations the desired bound is obtained. □

Proof. **Proof of Corollary 3.0.5.** Pick $u(t) = \exp(t)$, $\omega(x) = \frac{\ln(x+1)}{x+1}$ and Holder exponents $r_1 = 2, r_2 = 5, r_3 = 7, r_4 = \frac{70}{11}$ in (12). Then the left and right parts of the inequality are

$$\begin{aligned}
L(x) &= \int_0^{\omega(x)} u\ \cosh(t)\ dt + \int_0^{\omega(x)} u'\ \sinh(t)\ dt\ + \int_0^{\omega(x)} u\ \cosh(t)\ u'\ u''\ dt \\
&= \frac{3}{4}(x+1)^{\frac{2}{x+1}} + \frac{1}{8}(x+1)^{\frac{4}{x+1}} - \frac{7}{8}, \\
R(x) &= \frac{1}{2}\left((x+1)^{\frac{2}{x+1}} - 1\right)^{\frac{1}{2}} \left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}} - (x+1)^{\frac{-2}{x+1}}\right) + \frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}} + \\
&+ \frac{1}{2}\left((x+1)^{\frac{2}{x+1}} - 1\right)^{\frac{1}{2}} \left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}} - (x+1)^{\frac{-2}{x+1}}\right) - \frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}} + \\
&+ \frac{\sqrt{2}}{2}\left(\frac{1}{5}\right)^{\frac{1}{5}}\left(\frac{1}{7}\right)^{\frac{1}{7}}\left(\frac{11}{70}\right)^{\frac{11}{70}} \left(\frac{1}{4}\left((x+1)^{\frac{2}{x+1}} - (x+1)^{\frac{-2}{x+1}}\right) + \frac{\ln(x+1)}{x+1}\right)^{\frac{1}{2}} \times \\
&\times \left((x+1)^{\frac{5}{x+1}} - 1\right)^{\frac{1}{5}} \left((x+1)^{\frac{7}{x+1}} - 1\right)^{\frac{1}{7}} \left((x+1)^{\frac{70}{11(x+1)}} - 1\right)^{\frac{11}{70}}.
\end{aligned}$$

Assembling the functions above, and performing routine algebraic manipulations , this yields the desired inequality. □

Proof. **Proof of Corollary 3.0.6.** Pick $\omega(x) = \frac{\ln(x+1)}{x+1} + (x+1)^{\frac{1}{x+1}}$. Then, the left and right side in (13) are represented by the functions

$$\begin{aligned}
L(x) &= J \\
&= \int_0^{\omega(x)} u'\ u''\ dt \\
&= \frac{1}{2}\left((x+1)^{\frac{2}{x+1}} \exp\left(2(x+1)^{\frac{1}{x+1}}\right) - 1\right), \\
R(x) &= \left|[\ln(u)\ u\ u'']_0^{\omega(x)}\right| + \left(\int_0^{\omega(x)} |\ln(u)|^{r_1} dt\right)^{\frac{1}{r_1}} \left(\int_0^{\omega(x)} |u'|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_0^{\omega(x)} |u''|^{r_3} dt\right)^{\frac{1}{r_3}} + \\
&+ \left(\int_0^{\omega(x)} |\ln(u)|^{r_4} dt\right)^{\frac{1}{r_4}} \left(\int_0^{\omega(x)} |u|^{r_5} dt\right)^{\frac{1}{r_5}} \left(\int_0^{\omega(x)} |u'''|^{r_6} dt\right)^{\frac{1}{r_6}} \\
&= \left(\frac{\ln(x+1)}{x+1} + (x+1)^{\frac{1}{x+1}}\right)\ (x+1)^{\frac{2}{x+1}} \exp\left(2(x+1)^{\frac{1}{x+1}}\right) + \\
&+ \left(\frac{1}{r_1+1}\right)^{\frac{1}{r_1}} \left(\frac{\ln(x+1)}{x+1} + (x+1)^{\frac{1}{x+1}}\right)^{\frac{1}{r_1}+1} \times \left(\frac{1}{r_2}\right)^{\frac{1}{r_2}} \left((x+1)^{\frac{r_2}{x+1}} \exp\left(r_2(x+1)^{\frac{1}{x+1}}\right) - 1\right)^{\frac{1}{r_2}} \times \\
&\times \left(\frac{1}{r_3}\right)^{\frac{1}{r_3}} \left((x+1)^{\frac{r_3}{x+1}} \exp\left(r_3(x+1)^{\frac{1}{x+1}}\right) - 1\right)^{\frac{1}{r_3}} + \\
&+ \left(\frac{1}{r_4+1}\right)^{\frac{1}{r_4}} \left(\frac{\ln(x+1)}{x+1} + (x+1)^{\frac{1}{x+1}}\right)^{\frac{1}{r_4}+1} \times \left(\frac{1}{r_5}\right)^{\frac{1}{r_5}} \left((x+1)^{\frac{r_5}{x+1}} \exp\left(r_5(x+1)^{\frac{1}{x+1}}\right) - 1\right)^{\frac{1}{r_5}} \times \\
&\times \left(\frac{1}{r_6}\right)^{\frac{1}{r_6}} \left((x+1)^{\frac{r_6}{x+1}} \exp\left(r_6(x+1)^{\frac{1}{x+1}}\right) - 1\right)^{\frac{1}{r_6}}.
\end{aligned}$$

Pick Holder exponents $r_1 = 7, r_2 = 8, r_3 = \frac{56}{41}, r_4 = 11, r_5 = 12, r_6 = \frac{132}{109}$. Then the desired bound follows. □

Proof. **Proof of Corollary 3.0.7.** Let $\omega(x) = \frac{\ln(x+1)}{x+1}, x > 0$, $u(t) = \exp(t)$. Then, the left part and the right part of inequality (14) are given by the functions

$$\begin{aligned}
L(x) &= Q \\
&= \int_0^{\omega(x)} u_1 \ dt \ \int_0^{\omega(x)} u_2 \ dt + 2^p \left(\int_0^{\omega(x)} |u_1|^p dt + \int_0^{\omega(x)} |u_2|^p dt \right) \\
&= \left((x+1)^{\frac{1}{x+1}} - 1 \right)^2 + \frac{2^{p+1}}{p} \left((x+1)^{\frac{p}{x+1}} - 1 \right), \\
R(x) &= \omega(x) \int_0^{\omega(x)} u_1 \ u_2 \ dt + \left(\left(\int_0^{\omega(x)} |u_1 + u_2|^p dt \right)^{\frac{1}{p}} + \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \right)^p + \\
&+ \left| \left(\int_0^{\omega(x)} |u_1 + u_2|^p dt \right)^{\frac{1}{p}} - \left(\int_0^{\omega(x)} |u_1 - u_2|^p dt \right)^{\frac{1}{p}} \right|^p \\
&= \frac{\ln(x+1)}{x+1} \frac{1}{2} \left((x+1)^{\frac{2}{x+1}} - 1 \right) + \frac{2^{p+1}}{p} \left((x+1)^{\frac{p}{x+1}} - 1 \right).
\end{aligned}$$

Assembling the functions, this yields the inequality

$$L(x) \leq R(x)$$

$$\left((x+1)^{\frac{1}{x+1}} - 1 \right)^2 + \frac{2^{p+1}}{p} \left((x+1)^{\frac{p}{x+1}} - 1 \right) \leq \frac{\ln(x+1)}{x+1} \frac{1}{2} \left((x+1)^{\frac{2}{x+1}} - 1 \right) + \frac{2^{p+1}}{p} \left((x+1)^{\frac{p}{x+1}} - 1 \right)$$

$$\left((x+1)^{\frac{1}{x+1}} - 1 \right)^2 \leq \frac{\ln(x+1)}{x+1} \frac{1}{2} \left((x+1)^{\frac{2}{x+1}} - 1 \right)$$

which is the desired result. □

6 Commentary on the Inequalities

In this section, we provide remarks on each of the inequality derived. There are specific comments for each inequality.

- Inequality (1) : This is an interpolation inequality where at the left side there is the product of power exponential type function with a quantity in brackets that involves the natural logarithm divided by its argument. At the right side of the inequality there are various terms involving quantities in fractional powers, power exponential functions minus constant functions in fractional powers, quotient of logarithmic function with linear function in fractional powers. Having a look at Figure

1, we observe that the gap between two curves closes as x grows large. There is a noticeable gap for small values of x but the curves asymptotically approach as x grows large enough. The inequality can be optimized by choosing a suitable exponent.

- Inequality (2) : This inequality has similar structure to (1), but there are some extra terms at the right hand side. This inequality is not strict as Figure 2 suggests. The inequality is not strict for a certain interval, but as x grows sufficiently large the functions asymptotically approach each other. The inequality may be considered not significant for a certain region in the graph, but it performs well for sufficiently large x.
- Inequality (3) : This inequality involves the quotient of logarithmic function with a linear function at the left side. At the right side we have a product of some coefficients with quantities that involve the function at the left hand side in fractional powers and terms that involve power exponential functions in fractional powers. According to Figure 3, this inequality is tight for all $x \geq 0$.
- Inequality (4) : At the left side we have sums of power exponential functions with different coefficients. At the right hand side there are various terms, where the logarithmic function is square rooted and other quantities involving power exponential functions, and these quantities are in fractional powers. As Figure 4 demonstrates, this is a tight estimate all $x \geq 0$.
- Inequality (5) : This is an interpolation inequality where at the left side we have sum of power exponential inequalities. At the right side we have various power exponential functions and logarithmic functions show up. As Figure 5 demonstrates, this is a tight estimate.
- Inequality (6) : At the left side we have power exponential function and exponential function where the power is a power exponential function. At the right side we have various quantities in fractional powers. This inequality is not tight for all $x \geq 0$ as Figure 6 suggests. The vertical gap is sufficiently large between the two curves. A research question could be how to make the inequality more tight.
- Inequality (7) : This estimate is tight, as Figure 7 indicates. It follows by making certain adjustments and employing Hanner functional inequality and Chebyshev integral inequality.

7 Conclusion and Remarks

Inequalities have been derived involving power exponential and logarithmic functions, using estimates involving Lebesgue norms and not classical analysis techniques such as monotonicity properties or power series expansions. This work contributes to the current literature in Mathematical Analysis and Inequalities, especially concerning the class of power-exponential and logarithmic functions. The approach is original and non trivial and the results are new in the literature to the best knowledge of the author. There are also graphical illustrations to verify the rigorous mathematical analysis of the main findings of this article. This article will fuel more intense research and exploration of functional analytic methods employed to derive and optimize classical analysis results. There is also motivation for future work. Instead of using Lebesque norms, there are other estimates that could be employed using Sobolev norms, Orlicz norms etc. Or a combination of primitive integral inequalities such as Chebyshev integral inequality, with normed space inequalities in Lebesgue space, or Sobolev space or Orlicz space.

8 Graphical Illustrations

In this section, graphical illustrations are provided to validate the inequalities (1), (2), (3), (4), (5), (6), (7). The blue dotted curve stands for the right side of each inequality, and the red solid curve stands for the left side of each inequality.

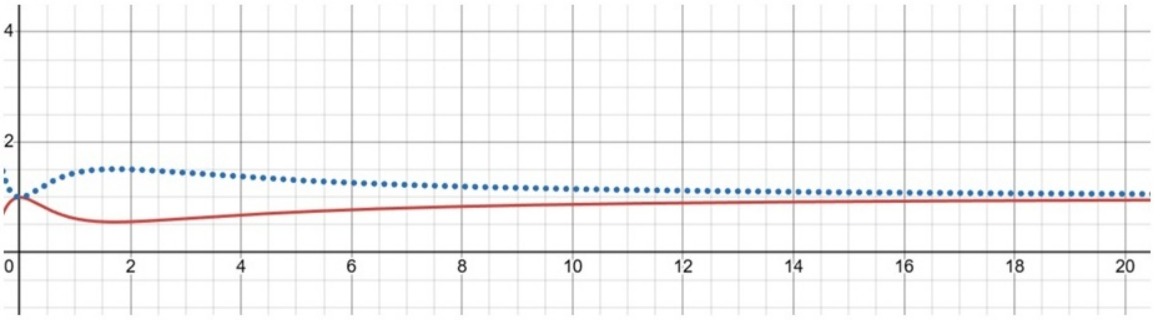

Figure 1: Graphical illustration of analytic inequality (1).

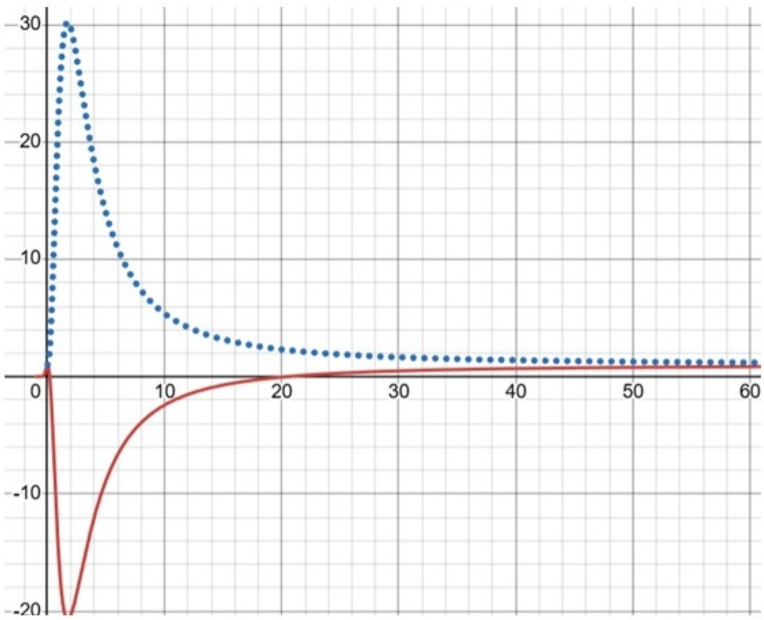

Figure 2: Graphical illustration of analytic inequality (2).

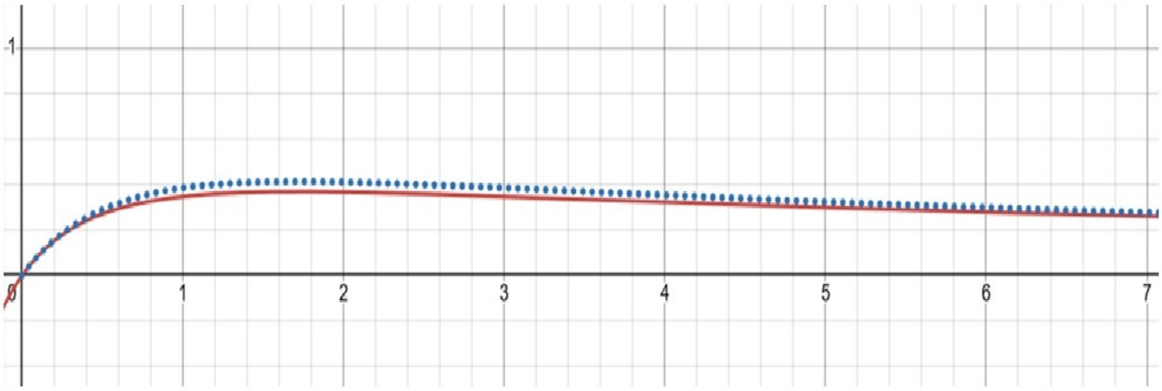

Figure 3: Graphical illustration of analytic inequality (3).

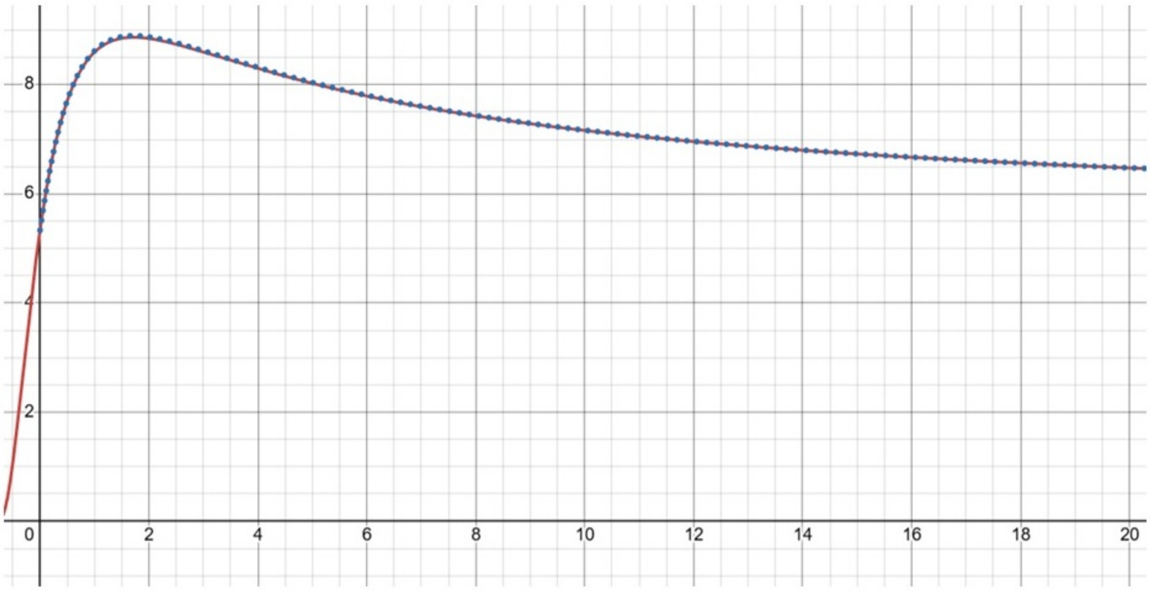

Figure 4: Graphical illustration of analytic inequality (4).

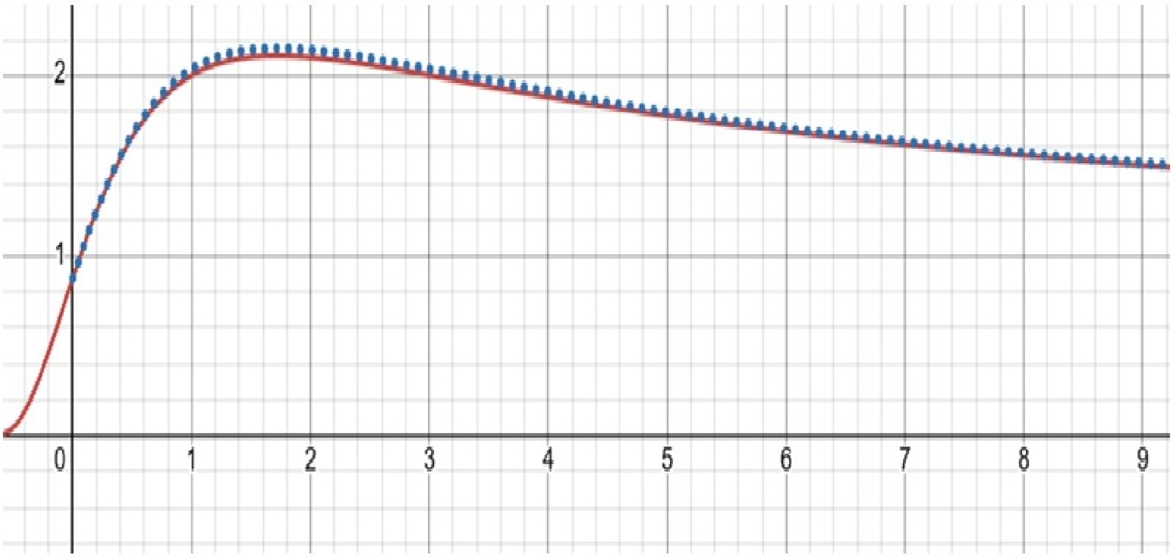

Figure 5: Graphical illustration of analytic inequality (5).

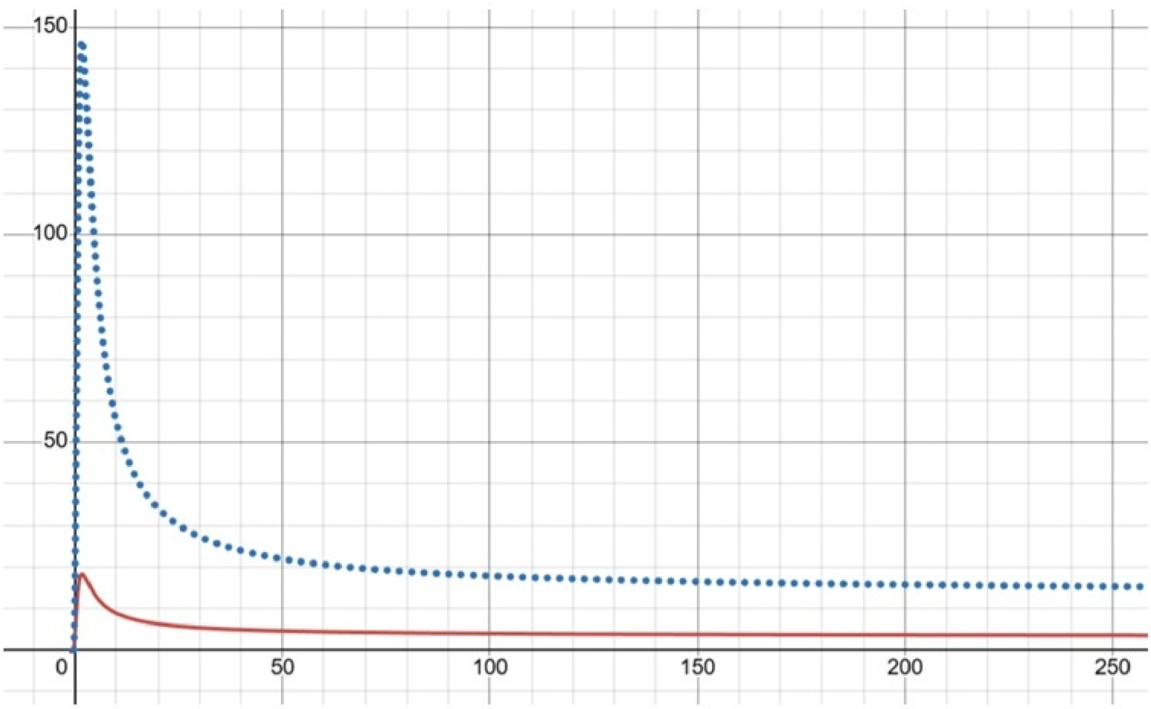

Figure 6: Graphical illustration of analytic inequality (6).

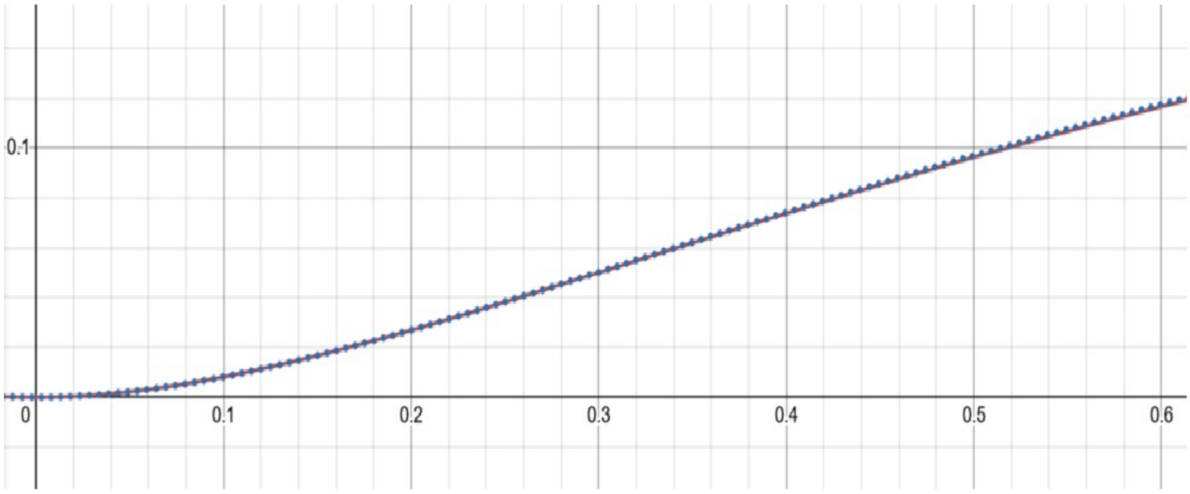

Figure 7: Graphical illustration of analytic inequality (7).

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