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Generalized Sasakian Manifolds: Pseudosymmetry Characterizations Related to Some Important Curvature Tensors

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Abstract

In this study, generalized Sasakian space forms are examined on W_5-, W_6, W_7, and W_9- curvature tensors. Moreover, special curvature conditions with the help of W_5-, W_6, W_7, W_9- pseudosymmetry and W_5-, W_6, W_7, W_9- Ricci pseudosymmetry are defined. The behavior for the generalized Sasakian space form is then represented in accordance with these concepts.

1 Introduction

Contact geometry and its Riemannian counterparts are essential to modern differential geometry, spanning areas from close connections to global analysis, mathematical physics, and topology. Among these structures, Sasakian manifolds constitute an important class, serving as odd-dimensional analogues of Kähler manifolds. Their rich geometric structure has motivated extensive investigations into their curvature properties and symmetry conditions [1, 15, 16].

Assume that $M(\phi, \xi, \eta, g)$ denotes the almost contact metric manifold. If there are functions F_1, F_2, F_3 on M such that

$$\begin{aligned}
&R(X_1, X_2)X_3 = F_1[g(X_2, X_3)X_1 - g(X_1, X_3)X_2]\\
&+F_2[g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1\\
&2g(X_1, \phi X_2)\phi X_3] + F_3[\eta(X_1)\eta(X_3)X_2\\
&-\eta(X_2)\eta(X_3)X_1 + g(X_1, X_3)\eta(X_2)\xi\\
&-g(X_2, X_3)\eta(X_1)\xi], \qquad (1.1)
\end{aligned}$$

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$M = M(\phi, \xi, \eta, g)$ is defined as a generalized Sasakian space form and this kind of manifold is shown by $M^{2n+1}(F_1, F_2, F_3)$. In order to unify and generalize various curvature models arising in contact metric geometry, the notion of generalized Sasakian space forms was introduced as a natural extension of classical Sasakian space forms [3]. These manifolds are characterized by curvature tensors depending on three smooth functions, allowing a broader framework that includes Sasakian, Kenmotsu, and cosymplectic space forms as special cases. Since their introduction, generalized Sasakian space forms have attracted considerable attention, particularly in the study of curvature-restricted geometric structures [4, 7].

A significant direction of research in this area concerns the interaction between generalized Sasakian space forms and various curvature tensors. For instance, investigations involving projective, concircular, conformal, and τ-curvature tensors have revealed notable geometric and topological consequences [2,10]. In particular, conditions imposed on these tensors often lead to characterizations of local symmetry, conformal flatness, or Einstein-like structures [8, 9].

Further developments include the study of generalized Sasakian space forms satisfying special curvature conditions such as those involving the W_0-curvature tensor or concircular curvature tensor, which yield rigidity results and classification theorems [5, 6]. These findings show that the global geometry of such manifolds is mainly determined by curvature constraints.

In recent years, attention has also shifted toward geometric flows and soliton structures on generalized Sasakian space forms. Various types of solitons, including Ricci and other generalized solitons, have been examined in low-dimensional settings, highlighting the dynamical aspects of these manifolds under curvature evolution equations [11]. Parallel to this, new curvature tensors such as pseudo-quasi conformal and contact conformal curvature tensors have been employed to derive further geometric properties and invariance results [12, 13].

Moreover, invariant submanifolds of generalized Sasakian space forms have been examined extensively, especially in relation to different curvature tensors. These investigations provide insight into how ambient curvature influences the intrinsic geometry of submanifolds [14]. The geometric behavior of generalized Sasakian structures and its submanifold theory are better understood thanks to these investigations.

Motivated by the above studies, the present study aims to further investigate some curvature properties of generalized Sasakian space forms under W_5-, W_6, W_7, and W_9- tensorial conditions. Therefore, this study is organized as follows: In Section 2, some properties of W_5-, W_6, W_7, and W_9- curvature tensors and structures of generalized Sasakian space form are given. In Section 3, W_5-, W_6, W_7, and W_9- pseudosymmetry and Ricci pseuosymmetry conditions are discuused. Moreover, being Einstein or η Einstein manifolds conditions are discussed in detail. In Section 4, the effectiveness of W operators as tools for classifying and comprehending almost contact metric manifolds is confirmed by the transformation of geometric meanings under W curvature conditions into new categories, namely Einstein manifold and $\eta-$ Einstein manifold.

2 Preliminaries

In this section, some curvature tensor concepts are introduced with some basic properties of generalized Sasakian manifolds.

If a smooth manifold M^{2n+1} admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η that meet the following conditions, then it has an almost contact structure (ϕ, ξ, η):

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

and

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2) \tag{2.2}$$

for all vector fields X_1, X_2 on M^{2n+1}, then $(M^{2n+1}, \phi, \xi, \eta, g)$ is defined as an almost contact metric manifold. It is clear that

$$g(\xi, X_1) = \eta(X_1). \tag{2.3}$$

The transformation Φ defined by

$$\Phi(X_1, X_2) = g(X_1, \Phi X_2) \tag{2.4}$$

for all $X_1, X_2 \in \chi(M)$, is defined as the fundamental 2- form of the almost contact metric structure (ϕ, ξ, η, g). Here,

$$\eta \wedge \Phi^n \neq 0.$$

The Riemannian curvature tensor for the Sasakian space form is given by

$$\begin{aligned} R(X_1, X_2)X_3 &= (\frac{k+3}{4})[g(X_1, X_3)X_1 - g(X_1, X_3)X_2] \\ &+ (\frac{k-1}{4})[g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 \\ &+ 2g(X_1, \phi X_2)\phi X_3 + \eta(X_1)\eta(X_3)X_2 \\ &- \eta(X_2)\eta(X_3)X_1 + g(X_1, X_3)\eta(X_2)\xi \\ &- g(X_2, X_3)\eta(X_1)\xi]. \end{aligned} \tag{2.5}$$

As we choose $X_1 = \xi, X_2 = \xi$, and $X_3 = \xi$ in (2.5), we get

$$R(\xi, X_2)X_3 = (F_1 - F_3)[g(X_2, X_3)\xi - \eta(X_3)X_2], \tag{2.6}$$

$$R(X_1, \xi)X_3 = (F_1 - F_3)[-g(X_1, X_3)\xi + \eta(X_3)X_1], \tag{2.7}$$

$$R(X_1, X_2)\xi = (F_1 - F_3)[\eta(X_2)X_1 - \eta(X_1)X_2]. \tag{2.8}$$

Moreover, we write

$$\eta(R(X_1, X_2)X_3) = (F_1 - F_3)[g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)].$$

Lemma 2.1. *For the generalized Sasakian space form $M^{2n+1}(F_1, F_2, F_3)$, the relations listed below are true:*

$$S(X_1, X_2) = [2nF_1 + 3F_2 - F_3]g(X_1, X_2) - [3F_2 + (2n-1)F_3]\eta(X_1)\eta(X_2),$$

$$S(X_1, \xi) = 2n(F_1 - F_3)\eta(X_1), \tag{2.9}$$

$$QX_1 = [2nF_1 + 3F_2 - F_3]X_1 - [3F_2 + (2n-1)F_3]\eta(X_1)\xi,$$

$$Q\xi = 2n(F_1 - F_3)\xi, \tag{2.10}$$

$$r = 2n(2n+1)F_1 + 6nF_2 - 4nF_3$$

for all $X_1, X_2 \in \chi(M^{2n+1})$. The Ricci operator, Ricci tensor, and scalar curvature of $M^{2n+1}(F_1, F_2, F_3)$ are denoted by Q, S, and r, respectively, [8].

Definition 2.2. *Assume that M^{2n+1} is a generalized Sasakian space form. The curvature tensor W_5 is given by*

$$W_5(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{n-1}S(X_1, X_3)X_2 - g(X_1, X_3)QX_2 \tag{2.11}$$

for all vector fields X_1, X_2, X_3 on M, [2].

Lemma 2.3. *Assume that M^{2n+1} is a generalized Sasakian space form. The relations listed below are true [2]:*

$$W_5(\xi, X_2)X_3 = (F_1 - F_3)[g(X_2, X_3)\xi - \eta(X_3)X_2 - \frac{2n}{n-1}\eta(X_3X_2) + \frac{1}{n-1}\eta(X_3)QX_2, \tag{2.12}$$

$$W_5(X_1, \xi)X_3 = (F_1 - F_3)[-g(X_1, X_3)\xi + \eta(X_3)X_1 + \frac{2n}{n-1}g(X_1, X_3)\xi] - \frac{1}{n-1}S(X_1, X_3)\xi, \tag{2.13}$$

$$W_5(X_1, X_2)\xi = (F_1 - F_3)[\eta(X_2)X_1 - \eta(X_1)X_2] + \frac{1}{n-1}[\eta(X_1)QX_2 - S(X_1, X_2)\xi]. \tag{2.14}$$

Definition 2.4. *Assume that M^{2n+1} is a generalized Sasakian space form. The curvature tensor W_6 is given by*

$$W_6(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{n-1}S(X_2, X_3)X_1 - g(X_2, X_3)QX_1 \tag{2.15}$$

for all vector fields X_1, X_2, X_3 on M, [2].

Lemma 2.5. *Assume that M^{2n+1} is a generalized Sasakian space form. The relations listed below are true [2]:*

$$W_6(\xi, X_2)X_3 = \eta(X_3)X_2 - 2g(X_2, X_3)\xi - \frac{1}{n-1}S(X_2, X_3)\xi, \tag{2.16}$$

$$W_6(X_1, \xi)X_3 = -g(X_1, X_3)\xi + 2\eta(X_3)X_1 + \frac{1}{n-1}\eta(X_3)QX_1, \tag{2.17}$$

$$W_6(X_1, X_2)\xi = \eta(X_1)X_2 + \frac{1}{n-1}\eta(X_2)QX_1. \tag{2.18}$$

Definition 2.6. *Assume that M^{2n+1} is a generalized Sasakian space form. The curvature tensor W_7 is given by*

$$W_7(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{n-1}S(X_2, X_3)X_1 + \frac{1}{n-1}g(X_2, X_3)QX_1 \tag{2.19}$$

for all vector fields X_1, X_2, X_3 on M, [2].

Lemma 2.7. *Assume that M^{2n+1} is a generalized Sasakian space form. The relations listed below are true [2]:*

$$W_7(\xi, X_2)X_3 = (F_1 - F_3)[g(X_2, X_3)\xi - \eta(X_3)X_2 + \frac{2n}{n-1}g(X_2, X_3)\xi] - \frac{1}{n-1}S(X_2, X_3)\xi, \tag{2.20}$$

$$W_7(X_1, \xi)X_3 = (F_1 - F_3)[-g(X_1, X_3)\xi + \eta(X_3)X_1 - \frac{2n}{n-1}\eta(X_3)X_1] + \frac{1}{n-1}\eta(X_3)QX_1, \tag{2.21}$$

$$W_7(X_1, X_2)\xi = (F_1 - F_3)[\eta(X_2)X_1 - \eta(X_1)X_2 - \frac{2n}{n-1}\eta(X_2)X_1] + \frac{1}{n-1}\eta(X_2)QX_1. \tag{2.22}$$

Definition 2.8. *Assume that M^{2n+1} is a generalized Sasakian space form. The curvature tensor W_9 is given by*

$$W_9(X_1, X_2)X_3 = R(X_1, X_2)X_3 + \frac{1}{n-1}S(X_1, X_2)X_3 - \frac{1}{n-1}g(X_2, X_3)QX_1 \tag{2.23}$$

for all vector fields X_1, X_2, X_3 on M, [2].

Lemma 2.9. *Assume that M^{2n+1} is a generalized Sasakian space form. The relations listed below are true [2]:*

$$W_9(\xi, X_2)X_3 = (F_1 - F_3)[g(X_2, X_3)\xi - \eta(X_3)X_2 - \frac{2n}{n-1}g(X_2, X_3)\xi + \frac{2n}{n-1}\eta(X_2)X_3], \tag{2.24}$$

$$W_9(X_1, \xi)X_3 = (F_1 - F_3)[-g(X_1, X_3)\xi + \eta(X_3)X_1 + \frac{2n}{n-1}\eta(X_1)X_3] - \frac{1}{n-1}\eta(X_3)QX_1, \tag{2.25}$$

$$W_9(X_1, X_2)\xi = (F_1 - F_3)[\eta(X_2)X_1 - \eta(X_1)X_2] + \frac{1}{n-1}[S(X_1, X_2)\xi - \eta(X_2)QX_1]. \tag{2.26}$$

3 On Geometric Properties for Generalized Sasakian Manifolds

In this section, for W_5-, W_6-, W_7-, and W_9- curvature tensors in a generalized Sasakian manifold, we provide the following geometric characterizations:

Definition 3.1. *Assume that $M^{2n+1}(F_1, F_2, F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R \cdot W_5$ and $Q(g, W_5)$ are linearly dependent, where R is the Riemannian curvature tensor, then M^{2n+1} is called a W_5- pseudosymmetric.*

In this specific case, a function λ_0 exists such that

$$R \cdot W_5 = \lambda_0 \cdot Q(g, W_5).$$

If $\lambda_0 = 0$, then the M^{2n+1} is called a W_5- semisymmetric.

Theorem 3.2. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. For $F_1 \neq F_3$, M^{2n+1} is a $\eta-$ Einstein manifold if it is W_5- pseudosymmetric.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_5- pseudosemisymmetric:

$$(R(X_1, X_2) \cdot W_5)(X_4, X_5, X_3) = \lambda_0 \cdot Q(g, W_5)(X_4, X_5, X_3; X_1, X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned} & R(X_1,X_2)W_5(X_4,X_5)X_3 - W_5(R(X_1,X_2)X_4,X_5)X_3 - W_5(X_4,R(X_1,X_2)X_5)X_3 \\ & -W_5(X_4,X_5R(X_1,X_2)X_3 = \lambda_0 \cdot [-g(X_5,X_3)W_5(X_4,X_1)X_2 + g(X_4,X_3)W_6(X_5,X_1)X_2 \\ & -g(X_5,X_1)W_6(X_3,X_4)X_2 + g(X_4,X_1)W_5(X_3,X_5)X_2 g(X_5,X_2)W_5(X_3,X_1)X_4 + g(X_4,X_2)W_5(X_3,X_1)X_5]. \end{aligned} \tag{3.1}$$

Choosing $X_1 = \xi$ in (3.1) and exploiting (2.1), (2.3), (2.6), (2.12), (2.13), and (2.14), we obtain

$$\begin{aligned} & (F_1-F_3)g(X_2,W_5(X_4,X_5)X_3)\xi - \eta(W_5(X_4,X_5)X_3)X_2 - (F_1-F_3)g(X_2,X_4)[(F_1-F_3)g(X_3,X_5)\xi \\ & -\frac{2n(F_1-F_3)}{n-1}\eta(X_3)X_5 + \frac{1}{n-1}\eta(X_3)QX_5] + (F_1-F_3)W_5(X_2,X_5)X_3 + (F_1-F_3)g(X_2,X_5) \\ & [-(F_1-F_3)g(X_3,X_4)\xi + \eta(X_3)X_4 - \frac{1}{n-1}S(X_3,X_4)\xi + \frac{2n(F_1-F_3)}{n-1}g(X_3,X_4)\xi] \\ & +(F_1-F_3)\eta(X_5)W_5(X_2,X_4)X_3 - (F_1-F_3)g(X_2,X_5)[(F_1-F_3)\eta(X_5)X_4 - (F_1-F_3)\eta(X_4)X_5 \\ & -\frac{1}{n-1}S(X_4,X_5)\xi + \frac{1}{n-1}\eta(X_4)QX_5] = \lambda_0[-g(X_5,X_3)[-(F_1-F_3)g(X_4,X_2) + (F_1-F_3)\eta(X_2)X_4\xi \\ & +\frac{2n(F_1-F_3)}{n-1}g(X_2,X_4)\xi] + -g(X_3,X_4)[-(F_1-F_3)g(X_5,X_2)\xi + (F_1-F_3)\eta(X_2)X_4 \\ & +\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi - \frac{1}{n-1}S(X_2,X_5)\xi] - \eta(X_5)W_5(X_3,X_4)X_2 + \eta(X_4)W_5(X_3,X_5)X_2 \\ & -g(X_2,X_5)[-(F_1-F_3)g(X_3,X_4) + \frac{2n(F_1-F_3)}{n-1}g(X_3,X_4)\xi\xi - \frac{1}{n-1}S(X_3,X_4)\xi + (F_1-F_3)\eta(X_4)X_3] \\ & +g(X_2,X_4)[-(F_1-F_3)g(X_3,X_5)\xi + (F_1-F_3)\eta(X_5)X_3 + \frac{2n(F_1-F_3)}{n-1}g(X_3,X_5)\xi - \frac{1}{n-1}S(X_3,X_5)\xi]. \end{aligned} \tag{3.2}$$

If we write $X_4 = \xi$ in (3.2) and make simplifications by considering (2.1), (2.3), (2.9), and (2.12), we calculate

$$\begin{aligned} & (F_1-F_3)[(F_1-F_3)g(X_5,X_3)\eta(X_2)\xi - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)g(X_2,X_5)\xi + \frac{1}{n-1}\eta(X_3)S(X_4,X_2)\xi - g(X_5,X_3)X_2 \\ & +\eta(X_2)\eta(X_5)X_2 - (F_1-F_3)\eta(X_2)g(X_3,X_5)\xi + W_5(X_2,X_5)X_3] - (F_1-F_3)\eta(X_5)\eta(X_3)X_2 \\ & -\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2 - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2 + \frac{1}{n-1}\eta(X_3)\eta(X_5)QX_2 + (F_1-F_3)g(X_2,X_3)X_5 \\ & +\frac{2n(F_1-F_3)}{n-1}g(X_2,X_3)\eta(X_5)\xi - \frac{1}{n-1}g(X_2,X_3)QX_5 = \lambda_0[\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\eta(X_3)\xi \\ & +(F_1-F_3)\eta(X_2)\eta(X_3)X_5 - \frac{1}{n-1}S(X_2,X_5)\eta(X_3)\xi + (F_1-F_3)\eta(X_5)g(X_2,X_3)\xi + \frac{1}{n-1}S(X_2,X_3)\eta(X_5)\xi \\ & -\frac{2n(F_1-F_3)}{n-1}\eta(X_5)g(X_2,X_3)\xi + W_5(X_3,X_5)X_2 - (F_1-F_3)g(X_2,X_5)X_3 - (F_1-F_3)\eta(X_2)g(X_3,X_5)\xi \\ & -\frac{1}{n-1}S(X_3,X_5)\eta(X_2)\xi + \frac{2n(F_1-F_3)}{n-1}g(X_3,X_5)\eta(X_2)\xi]. \end{aligned} \tag{3.3}$$

If we choose $X_3 = \xi$ in (3.3), we briefly compute

$$\begin{aligned} & (F_1-F_3)[-\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi - \frac{2n(F_1-F_3)}{n-1}\eta(X_5)X_2 + \frac{1}{n-1}\eta(X_5)QX_2 + \frac{2n(F_1-F_3)}{n-1}\eta(X_5)\eta(X_2)\xi] \\ & = \lambda_0[-\frac{1}{n-1}S(X_2,X_5)\xi + \frac{2n(F_1-F_3)}{n-1}g(X_5,X_2)\xi - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)X_5 + \frac{1}{n-1}\eta(X_2)QX_5. \end{aligned} \tag{3.4}$$

Taking inner product both sides of the equation by $X_6 \in \chi(M)$ and choosing $X_5 = \xi$, we acquire

$$S(X_2, X_6) = 2ng(X_2, X_6).$$

□

Corollary 3.3. *Let $M^{2n+1}(F_1, F_2, F_3)$ be a generalized Sasakian space form that is $(2n+1)-$ dimesional. M^{2n+1} is either an Einstein manifold or $F_1 = F_3$ if it is W_5- semisymmetric.*

Definition 3.4. *Assume that $M^{2n+1}(F_1, F_2, F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R \cdot W_5$ and $Q(S, W_5)$ are linearly dependent, then M^{2n+1} is called W_5- Ricci pseudosymmetric.*

In this case, a function λ_{10} exists such that

$$R \cdot W_5 = \lambda_{10} \cdot Q(S, W_5),$$

where S is the Ricci curvature tensor. If $\lambda_{10} = 0$, then the M^{2n+1} is called a W_5- Ricci semisymmetric.

Theorem 3.5. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. For $F_1 \neq F_3$ and $\lambda_{10} \neq 0$, M^{2n+1} is a $\eta-$ Einstein manifold if it is W_5- pseudosymmetric.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_5- Ricci pseudosemisymmetric:

$$(R(X_1, X_2) \cdot W_5)(X_4, X_5, X_3) = \lambda_{10} \cdot Q(S, W_5)(X_4, X_5, X_3; X_1, X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned}
&R(X_1, X_2)W_5(X_4, X_5)X_3 - W_5(R(X_1, X_2)X_4, X_5)X_3 - W_5(X_4, R(X_1, X_2)X_5)X_3 - W_5(X_4, X_5 R(X_1, X_2)X_3 \\
&= \lambda_{10} \cdot [-S(X_5, X_3)W_5(X_4, X_1)X_2 + S(X_4, X_3)W_5(X_5, X_1)X_2 - S(X_5, X_1)W_5(X_3, X_4)X_2 \\
&-g(X_5, X_2)W_5(X_3, X_1)X_4 + S(X_4, X_2)W_5(X_3, X_1)X_5].
\end{aligned} \tag{3.5}$$

Choosing $X_1 = \xi$ in (3.5) and exploiting (2.9) and (2.13), we obtain

$$\begin{aligned}
&(F_1 - F_3)g(X_2, W_5(X_4, X_5)X_3)\xi - \eta(W_5(X_4, X_5)X_3)X_2 - (F_1 - F_3)g(X_2, X_4)[(F_1 - F_3)g(X_3, X_5)\xi \\
&-\frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_5 + \frac{1}{n-1}\eta(X_3)QX_5] + (F_1 - F_3)W_5(X_2, X_5)X_3 + (F_1 - F_3)g(X_2, X_5) \\
&[-(F_1 - F_3)g(X_3, X_4)\xi + \eta(X_3)X_4 - \frac{1}{n-1}S(X_3, X_4)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_3, X_4)\xi] \\
&+(F_1 - F_3)\eta(X_5)W_5(X_2, X_4)X_3 - (F_1 - F_3)g(X_2, X_5)[(F_1 - F_3)\eta(X_5)X_4 - (F_1 - F_3)\eta(X_4)X_5 \\
&-\frac{1}{n-1}S(X_4, X_5)\xi + \frac{1}{n-1}\eta(X_4)QX_5] = \lambda_{10}[-S(X_5, X_3)[-(F_1 - F_3)g(X_4, X_2)\xi + (F_1 - F_3)\eta(X_2)X_4 \\
&-\frac{1}{n-1}S(X_2, X_4)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_4)\xi] + S(X_3, X_4)[-(F_1 - F_3)g(X_5, X_2)\xi + (F_1 - F_3)\eta(X_2)X_5 \\
&+\frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)\xi - \frac{1}{n-1}S(X_2, X_5)\xi] + 2n(F_1 - F_3)\eta(X_5)W_5(X_3, X_4)X_2 + 2n\eta(X_4)W_5(X_3, X_5)X_2 \\
&-S(X_2, X_5)[-(F_1 - F_3)g(X_3, X_4)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_4)\xi - \frac{1}{n-1}S(X_2, X_4)\xi + (F_1 - F_3)\eta(X_2)X_5] \\
&+S(X_2, X_4)[-(F_1 - F_3)g(X_3, X_5)\xi - (F_1 - F_3)\eta(X_5)X_3 + \frac{2n(F_1 - F_3)}{n-1}g(X_3, X_5)\xi - \frac{1}{n-1}S(X_3, X_5)\xi].
\end{aligned} \tag{3.6}$$

If we write $X_4 = \xi$ in (3.6) and make simplifications by considering (2.1), (2.3), (2.9), and (2.13), we calculate

$$
\begin{aligned}
&(F_1-F_3)[(F_1-F_3)g(X_5,X_3)\eta(X_2)\xi-\frac{2n(F_1-F_3)}{n-1}\eta(X_3)g(X_2,X_5)\xi+\frac{1}{n-1}\eta(X_3)S(X_4,X_2)\xi\\
&-g(X_5,X_3)X_2+\eta(X_2)\eta(X_5)X_2-(F_1-F_3)\eta(X_2)g(X_3,X_5)\xi+W_5(X_2,X_5)X_3]-(F_1-F_3)\eta(X_5)\eta(X_3)X_2\\
&-\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2-\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2+\frac{1}{n-1}\eta(X_3)\eta(X_5)QX_2+\\
&(F_1-F_3)g(X_2,X_3)X_5+\frac{2n(F_1-F_3)}{n-1}g(X_2,X_3)\eta(X_5)\xi-\frac{1}{n-1}g(X_2,X_3)QX_5\\
&=\lambda_{10}[2n(F_1-F_3)\eta(X_3)[-(F_1-F_3)g(X_2,X_5)+(F_1-F_3)\eta(X_2)X_5-\frac{1}{n-1}S(X_2,X_5)\xi\\
&+\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi]+2n(F_1-F_3)\eta(X_5)[-(F_1-F_3)g(X_2,X_3)+(F_1-F_3)\eta(X_2)X_3\\
&-\frac{1}{n-1}S(X_2,X_3)\xi+\frac{2n(F_1-F_3)}{n-1}g(X_2,X_3)\xi]+2n(F_1-F_3)W_5(X_3,X_5)X_2-S(X_2,X_5)\\
&[-(F_1-F_3)\eta(X_3)\xi+(F_1-F_3)X_3]+2n(F_1-F_3\eta(X_2))[-(F_1-F_3)g(X_3,X_5)\xi+(F_1-F_3)\eta(X_5)X_3\\
&-\frac{1}{n-1}S(X_3,X_5)\xi+\frac{2n(F_1-F_3)}{n-1}g(X_3,X_5)\xi]].
\end{aligned}\tag{3.7}
$$

If we choose $X_3=\xi$ in (3.7), we briefly compute

$$
\begin{aligned}
&(F_1-F_3)[-\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi-\frac{2n(F_1-F_3)}{n-1}\eta(X_5)X_2+\frac{1}{n-1}\eta(X_5)QX_2+\frac{2n(F_1-F_3)}{n-1}\eta(X_5)\eta(X_2)\xi]\\
&=\lambda_{10}[-\frac{2n(F_1-F_3)}{n-1}S(X_2,X_5)\xi+\frac{4n^2(F_1-F_3)}{n-1}g(X_5,X_2)\xi-\frac{4n^2(F_1-F_3)^2}{n-1}\eta(X_2)X_5].
\end{aligned}\tag{3.8}
$$

Taking inner product both sides of (3.8) by $X_6\in\chi(M)$ and choosing $X_5=\xi$, we acquire

$$S(X_2,X_6)-2ng(X_2,X_6)+4n^2\lambda_{10}(F_1-F_3)\eta(X_2)\eta(X_6)=0.$$

□

Corollary 3.6. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. For $F_1\neq F_3$, M^{2n+1} is an Einstein manifold if it is W_5- semisymmetric.*

Definition 3.7. *Assume that $M^{2n+1}(F_1,F_2,F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R\cdot W_6$ and $Q(g,W_6)$ are linearly dependent, then M^{2n+1} is called a W_6- pseudosymmetric.*

In this specific case, a function λ_1 exists such that

$$R\cdot W_6=\lambda_1\cdot Q(g,W_6),$$

where R is the Riemann curvature tensor. If $\lambda_1=0$, the M^{2n+1} is called a W_6- semisymmetric.

Theorem 3.8. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. M^{2n+1} is either an Eintein manifold if $\lambda_1=\frac{(F_1-F_3)(n+1)}{8n(n-1)}$ or a $\eta-$ Einstein manifold if $F_1\neq F_3$ if it is W_6- pseudosymmetric.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_6- pseudosemisymmetric:

$$(R(X_1,X_2)\cdot W_6)(X_4,X_5,X_3)=\lambda_1\cdot Q(g,W_6)(X_4,X_5,X_3;X_1,X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned}
&R(X_1,X_2)W_6(X_4,X_5)X_3 - W_6(R(X_1,X_2)X_4,X_5)X_3 - W_6(X_4,R(X_1,X_2)X_5)X_3\\
&-W_6(X_4,X_5R(X_1,X_2)X_3 = \lambda_1\cdot[-g(X_5,X_3)W_6(X_4,X_1)X_2 + g(X_4,X_3)W_6(X_5,X_1)X_2\\
&-g(X_5,X_1)W_6(X_3,X_4)X_2 + g(X_4,X_1)W_6(X_3,X_5)X_2 - g(X_5,X_2)W_6(X_3,X_1)X_4 + g(X_4,X_2)W_6(X_3,X_1)X_5].
\end{aligned} \tag{3.9}$$

Choosing $X_1 = \xi$ in (3.9) and exploiting (2.1), (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned}
&(F_1-F_3)[g(X_2,W_6(X_4,X_5)X_3)\xi - \eta(W_6(X_4,X_5)X_3)X_2 - g(X_2,X_4)[(F_1-F_3)g(X_5,X_3)\xi - \eta(X_3)X_5\\
&-\frac{1}{n-1}S(X_5,X_3)\xi + \frac{1}{n-1}\eta(X_5)QX_3] + \eta(X_4)W_6(X_2,X_5)X_3 + g(X_2,X_5)[-(F_1-F_3)g(X_4,X_3)\xi\\
&+(F_1-F_3)\eta(X_3)X_4 - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_4)QX_3] + \eta(X_5)W_6(X_4,X_2)X_3\\
&-(F_1-F_3)g(X_2,X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{2n}{n-1}g(X_4,X_5)\xi] + \eta(X_5)W_6(X_4,X_5)X_2]\\
&= \lambda_1[-g(X_5,X_3)[-(F_1-F_3)g(X_4,X_2) + (F_1-F_3)\eta(X_2)X_4 - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)X_4\\
&+\frac{1}{n-1}\eta(X_4)QX_2] + g(X_4,X_2)[-(F_1-F_3)g(X_5,X_2)\xi + (F_1-F_3)\eta(X_2)X_5 - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)X_5\\
&+\frac{1}{n-1}\eta(X_5)QX_2] - \eta(X_5)W_6(X_3,X_4)X_2 + \eta(X_4)W_6(X_3,X_5)X_2 + g(X_5,X_2)[-(F_1-F_3)g(X_3,X_4)\xi\\
&+(F_1-F_3)\eta(X_4)X_3 - \frac{2n(F_1-F_3)}{n-1}\eta(X_4)X_3 + \frac{1}{n-1}\eta(X_3)QX_4] + g(X_2,X_4)\\
&[-(F_1-F_3)g(X_3,X_5)\xi + (F_1-F_3)\eta(X_5)X_3] - \frac{2n(F_1-F_3)}{n-1}\eta(X_5)X_3 + \frac{1}{n-1}\eta(X_3)QX_5]
\end{aligned} \tag{3.10}$$

If we write $X_4 = \xi$ in (3.10) and make simplifications by considering (2.16), we calculate

$$\begin{aligned}
&(F_1-F_3)[(F_1-F_3)g(X_5,X_3)\eta(X_2)\xi - (F_1-F_3)\eta(X_3)g(X_2,X_5)\xi - (F_1-F_3)g(X_5,X_3)X_2 + \frac{1}{n-1}S(X_4,X_3)X_2\\
&-\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2 - (F_1-F_3)\eta(X_2)g(X_5,X_3)\xi + (F_1-F_3)\eta(X_2)\eta(X_3)X_5 + W_6(X_2,X_5)X_3]\\
&+\frac{2n(F_1-F_3)}{n-1}\eta(X_3)g(X_2,X_5)\xi - \frac{1}{n-1}g(X_2,X_5)QX_3 + (F_1-F_3)\eta(X_5)g(X_2,X_3)\xi - (F_1-F_3)g(X_2,X_3)\eta(X_5)\xi\\
&+(F_1-F_3)g(X_2,X_3)X_5 + \frac{2n(F_1-F_3)}{n-1}g(X_2,X_3)\eta(X_5)\xi - \frac{2n(F_1-F_3)}{n-1}\eta(X_5)g(X_2,X_3)\xi\\
&= \lambda_1[\frac{2n(F_1-F_3)}{n-1}g(X_5,X_3) + \eta(X_2)\xi - \frac{(F_1-F_3)}{n-1}g(X_5,X_3)QX_2 - (F_1-F_3)\eta(X_3)g(X_5,X_2)\xi\\
&+(F_1-F_3)\eta(X_2)\eta(X_3)X_5 + \frac{1}{n-1}\eta(X_5)\eta(X_3)QX_2 + (F_1-F_3)\eta(X_5)g(X_3,X_2)\xi - \frac{1}{n-1}\eta(X_5)\eta(X_3)QX_2\\
&+W_6(X_3,X_5)X_2 - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)\eta(X_3)X_5 - (F_1-F_3)g(X_5,X_2)\eta(X_3)\xi + (F_1-F_3)g(X_5,X_2)X_3\\
&-\frac{2n(F_1-F_3)}{n-1}g(X_5,X_2)X_3 + \frac{2n}{n-1}\eta(X_3)g(X_5,X_2)\xi - (F_1-F_3)\eta(X_2)g(X_3,X_5)\xi + \frac{1}{n-1}\eta(X_2)\eta(X_3)QX_5].
\end{aligned} \tag{3.11}$$

If we choose $X_3 = \xi$ in (3.11), we briefly compute

$$\begin{aligned}
&(F_1-F_3)^2[(\frac{n+1}{n-1})g(X_2,X_5)\xi + \eta(X_2)X_5] = \lambda_1[\frac{2n(F_1-F_3)}{n-1}\eta(X_5)\eta(X_2)\xi - \frac{(F_1-F_3)}{n-1}\eta(X_5)QX_2\\
&+\frac{2n(F_1-F_3)}{n-1}\eta(X_2)QX_5 + \frac{1}{n-1}S(X_2,X_5)\xi - \frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi + \frac{2n}{n-1}g(X_2,X_5)\xi + \frac{1}{n-1}\eta(X_2)QX_5].
\end{aligned}$$

Taking inner product both sides of the equation by $X_6 \in \chi(M)$ and choosing $X_5 = \xi$, we acquire

$$(F_1-F_3)[-2n(F_1-F_3)g(X_2,X_6) + [(F_1-F_3)(n+1) - 8\lambda_1 n(n-1)]\eta(X_2)\eta(X_6) + \lambda_1 S(X_2,X_6)] = 0.$$

□

Definition 3.9. *Assume that $M^{2n+1}(F_1, F_2, F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R \cdot W_6$ and $Q(S, W_6)$ are linearly dependent, then M^{2n+1} is called a W_6- Ricci pseudosymmetric.*

In this specific case, a function λ_1 exists such that

$$R \cdot W_6 = \lambda_2 \cdot Q(S, W_6),$$

where S is the Rici curvature tensor. If $\lambda_2 = 0$, then the M^{2n+1} is said to be a W_6- Ricci semisymmetric.

Theorem 3.10. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. Given $F_1 - F_3 - 1 \neq 0$ and $(F_1 - F_3)^2 = \frac{2n}{n-1}$, M^{2n+1} is an Eintein manifold if it is W_6- Ricci semisymmetric.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_6- pseudosemisymmetric:

$$(R(X_1, X_2) \cdot W_6)(X_4, X_5, X_3) = \lambda_2 \cdot Q(S, W_6)(X_4, X_5, X_3; X_1, X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned}
&R(X_1, X_2)W_6(X_4, X_5)X_3 - W_6(R(X_1, X_2)X_4, X_5)X_3 - W_6(X_4, R(X_1, X_2)X_5)X_3 \\
&-W_6(X_4, X_5 R(X_1, X_2)X_3 = \lambda_2 \cdot [-S(X_5, X_3)W_6(X_4, X_1)X_2 + S(X_4, X_3)W_6(X_5, X_1)X_2 \\
&-S(X_5, X_1)W_6(X_3, X_4)X_2 + S(X_4, X_1)W_6(X_3, X_5)X_2 - S(X_5, X_2)W_6(X_3, X_1)X_4 + S(X_4, X_2)W_6(X_3, X_1)X_5].
\end{aligned} \tag{3.12}$$

If we write $X_1 = \xi$ in (3.12) and make simplifications by using (2.9) and (2.17), we calculate

$$\begin{aligned}
&(F_1 - F_3)[(F_1 - F_3)g(X_5, X_3)\eta(X_2)\xi - (F_1 - F_3)\eta(X_3)g(X_2, X_5)\xi - (F_1 - F_3)g(X_5, X_3)X_2 + \frac{1}{n-1}S(X_4, X_3)X_2 \\
&-\frac{2n(F_1 - F_3)}{n-1}\eta(X_3)\eta(X_5)X_2 - (F_1 - F_3)\eta(X_2)g(X_5, X_3)\xi + (F_1 - F_3)\eta(X_2)\eta(X_3)X_5 + W_6(X_2, X_5)X_3] \\
&+\frac{2n(F_1 - F_3)}{n-1}\eta(X_3)g(X_2, X_5)\xi - \frac{1}{n-1}g(X_2, X_5)QX_3 + (F_1 - F_3)\eta(X_5)g(X_2, X_3)\xi - (F_1 - F_3)g(X_2, X_3)\eta(X_5)\xi \\
&+(F_1 - F_3)g(X_2, X_3)X_5 + \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_3)\eta(X_5)\xi - \frac{2n(F_1 - F_3)}{n-1}\eta(X_5)g(X_2, X_3)\xi = \lambda_2[-S(X_5, X_3) \\
&[-(F_1 - F_3)g(X_4, X_2)\xi + (F_1 - F_3)\eta(X_2)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_2)X_4 + \frac{1}{n-1}\eta(X_4)QX_2] \\
&S(X_4, X_3)[-(F_1 - F_3)g(X_5, X_2)\xi + (F_1 - F_3)\eta(X_2)X_5 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_2)X_5 + \frac{1}{n-1}\eta(X_5)QX_2] \\
&-2n(F_1 - F_3)\eta(X_5)W_6(X_4, X_3)X_2 + 2n(F_1 - F_3)\eta(X_4)W_6(X_3, X_5)X_2 - S(X_4, X_2)[-(F_1 - F_3) \\
&g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_4)X_3 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_4)X_3 + \frac{1}{n-1}\eta(X_3)QX_4]].
\end{aligned} \tag{3.13}$$

If we choose $X_4=\xi$ in (3.13), we easily calculate by using (2.3), (2.10), and (2.17)

$$\begin{aligned}
&(F_1-F_3)[(F_1-F_3)g(X_5,X_3)\eta(X_2)\xi-(F_1-F_3)\eta(X_3)g(X_2,X_5)\xi-(F_1-F_3)g(X_5,X_3)X_2+\frac{1}{n-1}S(X_4,X_3)X_2\\
&-\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)X_2-(F_1-F_3)\eta(X_2)g(X_5,X_3)\xi+(F_1-F_3)\eta(X_2)\eta(X_3)X_5+W_6(X_2,X_5)X_3]\\
&+\frac{2n(F_1-F_3)}{n-1}\eta(X_3)g(X_2,X_5)\xi-\frac{1}{n-1}g(X_2,X_5)QX_3+(F_1-F_3)\eta(X_5)g(X_2,X_3)\xi-(F_1-F_3)g(X_2,X_3)\eta(X_5)\xi\\
&+(F_1-F_3)g(X_2,X_3)X_5+\frac{2n(F_1-F_3)}{n-1}g(X_2,X_3)\eta(X_5)\xi-\frac{2n(F_1-F_3)}{n-1}\eta(X_5)g(X_2,X_3)\xi\\
&=\lambda_2[\frac{2n(F_1-F_3)}{n-1}S(X_5,X_3)+\eta(X_2)\xi-\frac{1}{n-1}QX_2S(X_5,X_3)-2n(F_1-F_3)^2\eta(X_3)g(X_2,X_5)\xi\\
&+2n(F_1-F_3)^2\eta(X_2)\eta(X_3)X_5-\frac{4n^2(F_1-F_3)^2}{n-1}\eta(X_3)\eta(X_2)X_5+\frac{2n(F_1-F_3)}{n-1}\eta(X_3)\eta(X_5)QX_2\\
&+2n(F_1-F_3)^2\eta(X_3)g(X_2,X_5)\xi+2n(F_1-F_3)^2\eta(X_5)g(X_2,X_3)\xi-2n(F_1-F_3)^2\eta(X_5)\eta(X_2)X_3\\
&+\frac{4n^2(F_1-F_3)^2}{n-1}\eta(X_5)\eta(X_2)X_3-\frac{2n}{n-1}\eta(X_3)\eta(X_5)QX_2+2n(F_1-F_3)W_6(X_3,X_5)X_2\\
&+(F_1-F_3)S(X_2,X_5)\eta(X_3)-(F_1-F_3)S(X_2,X_5)X_3+\frac{2n(F_1-F_3)}{n-1}S(X_2,X_5)X_3\\
&-\frac{2n(F_1-F_3)}{n-1}S(X_2,X_5)\eta(X_3)\xi]. \qquad (3.14)
\end{aligned}$$

If we choose $X_3=\xi$ in (3.14), we compute

$$\begin{aligned}
&(F_1-F_3)^2[-g(X_2,X_5)\xi+\frac{2n}{n-1}g(X_2,X_5)\xi+\eta(X_2)X_5]=\lambda_2[\frac{8n^2(F_1-F_3)^2}{n-1}\eta(X_5)\eta(X_2)\xi\\
&-\frac{4n^2(F_1-F_3)^2}{n-1}\eta(X_2)X_5-\frac{2n(F_1-F_3)}{n-1}\eta(X_5)QX_2-\frac{2n(F_1-F_3)}{n-1}S(X_2,X_5)\xi+\frac{2n(F_1-F_3)}{n-1}\eta(X_5)QX_2].
\end{aligned}$$

Taking inner product both sides of the equation by $X_6\in\chi(M)$ and choosing $X_5=\xi$, we acquire

$$-\frac{2n}{n-1}g(X_2,X_6)+(-(F_1-F_3)^2+\frac{2n}{n-1})\eta(X_2)\eta(X_6)=\frac{\lambda_2 2n(F_1-F_3-1)}{n-1}S(X_2,X_6).$$

□

Definition 3.11. *Assume that $M^{2n+1}(F_1,F_2,F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form, R is the Riemann curvature tensor. If $R\cdot W_7$ and $Q(g,W_7)$ are linearly dependent, then M^{2n+1} is called a W_7- pseudosymmetric.*

In this specific case, a function λ_1 exists such that

$$R\cdot W_7=\lambda_3\cdot Q(g,W_7).$$

If $\lambda_3=0$, then the M^{2n+1} is called a W_7- semisymmetric.

Theorem 3.12. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. If M^{2n+1} is W_7- pseudosymmetric, then it is either an Eintein manifold if $F_1\neq F_3$ or $F_1=F_3$.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_7- pseudosemisymmetric:

$$(R(X_1,X_2)\cdot W_7)(X_4,X_5,X_3)=\lambda_3\cdot Q(g,W_7)(X_4,X_5,X_3;X_1,X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned} & R(X_1,X_2)W_7(X_4,X_5)X_3 - W_7(R(X_1,X_2)X_4,X_5)X_3 - W_7(X_4,R(X_1,X_2)X_5)X_3 \\ & -W_7(X_4,X_5R(X_1,X_2)X_3 = \lambda_1\cdot[-g(X_5,X_3)W_7(X_4,X_1)X_2 + g(X_4,X_3)W_7(X_5,X_1)X_2 \\ & -g(X_5,X_1)W_7(X_3,X_4)X_2 + g(X_4,X_1)W_7(X_3,X_5)X_2 - g(X_5,X_2)W_7(X_3,X_1)X_4 + g(X_4,X_2)W_7(X_3,X_1)X_5]. \end{aligned} \tag{3.15}$$

Choosing $X_1 = \xi$ in (3.15) and using (2.1), (2.3), (2.17), (2.20), (2.21), and (2.22), we obtain

$$\begin{aligned} & (F_1-F_3)[g(X_2,W_7(X_4,X_5)X_3)\xi - \eta(W_7(X_4,X_5)X_3)X_2 - g(X_2,X_4)[(F_1-F_3)g(X_5,X_3)\xi \\ & -\eta(X_3)X_5 - \frac{1}{n-1}S(X_5,X_3)\xi + \frac{2n(F_1-F_3)}{n-1}g(X_5,X_3)\xi] + \eta(X_4)W_7(X_2,X_5)X_3 \\ & -g(X_2,X_5)[-(F_1-F_3)g(X_4,X_3)\xi + (F_1-F_3)\eta(X_3)X_4 - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_3)QX_4] \\ & \eta(X_5)W_7(X_4,X_2)X_3 - g(X_2,X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{1}{n-1}\eta(X_5)QX_4] \\ & +\eta(X_3)W_7(X_4,X_5)X_2] = \lambda_3[-g(X_5,X_3)[-(F_1-F_3)g(X_4,X_3)\xi + (F_1-F_3)\eta(X_3)X_4 - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)X_4 \\ & +\frac{1}{n-1}\eta(X_3)QX_4] + g(X_4,X_3)[-(F_1-F_3)g(X_5,X_2)\xi + (F_1-F_3)\eta(X_2)X_5 - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)X_5 \\ & +\frac{1}{n-1}\eta(X_2)QX_5] - \eta(X_5)W_7(X_3,X_4)X_2 + \eta(X_4)W_7(X_3,X_5)X_2 - g(X_5,X_2)[-(F_1-F_3)g(X_3,X_4)\xi \\ & +(F_1-F_3)\eta(X_4)X_3 - \frac{2n(F_1-F_3)}{n-1}\eta(X_4)X_3 + \frac{1}{n-1}\eta(X_4)QX_3] \\ & +g(X_2,X_4)[-(F_1-F_3)g(X_3,X_5)\xi + (F_1-F_3)\eta(X_5)X_3] - \frac{2n(F_1-F_3)}{n-1}\eta(X_3)X_5 + \frac{1}{n-1}\eta(X_5)QX_3] \end{aligned} \tag{3.16}$$

If we write $X_4 = \xi$ in (3.16) and make simplifications by considering (2.20) and (2.21), we calculate

$$\begin{aligned} & (F_1-F_3)[(F_1-F_3)g(X_5,X_3)X_2 + \frac{1}{n-1}S(X_5,X_2)X_2 - \frac{2n(F_1-F_3)}{n-1}g(X_5,X_3)X_2 + W_7(X_2,X_5)X_3 \\ & -\frac{1}{n-1}S(X_2,X_3)\eta(X_5)\xi + \frac{2n}{n-1}g(X_2,X_3)\eta(X_5)\xi + g(X_2,X_3)X_5 - \frac{1}{n-1}S(X_5,X_2)\eta(X_3)\xi \\ & +\frac{2n}{n-1}\eta(X_3)g(X_2,X_5)\xi = \lambda_3[(F_1-F_3)\eta(X_3)\eta(X_2)X_5 - \frac{2n(F_1-F_3)}{n-1}\eta(X_2)\eta(X_3)X_5 \\ & +\frac{1}{n-1}\eta(X_2)\eta(X_3)QX_5 + \eta(X_5)g(X_2,X_3)\xi W_7(X_3,X_5)X_2 - (F_1-F_3)g(X_2,X_5)X_3 \\ & +\frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)X_3 - \frac{1}{n-1}g(X_2,X_5)QX_3 - (F_1-F_3)g(X_5,X_3)\eta(X_2)\xi]. \end{aligned} \tag{3.17}$$

If we choose $X_3 = \xi$ in (3.17), we compute

$$\begin{aligned} & -\frac{2n(F_1-F_3)^2}{n-1}\eta(X_5)X_2 + \frac{(F_1-F_3)}{n-1}\eta(X_5)QX_2 - \frac{(F_1-F_3)}{n-1}S(X_2,X_5)\xi + \frac{2n(F_1-F_3)^2}{n-1}g(X_2,X_5)\xi \\ & = \lambda_3[-\frac{2n(F_1-F_3)}{n-1}\eta(X_2)X_5 + \frac{1}{n-1}\eta(X_2)QX_5 - \frac{1}{n-1}S(X_2,X_5)\xi + \frac{2n(F_1-F_3)}{n-1}g(X_2,X_5)\xi]. \end{aligned} \tag{3.18}$$

Taking inner product both sides of the equation by $X_6 \in \chi(M)$ and choosing $X_5 = \xi$, we acquire

$$\frac{(F_1-F_3)}{n-1}[S(X_2,X_6) - 2n(F_1-F_3)g(X_2,X_6)] = 0.$$

□

Corollary 3.13. *Let $M^{2n+1}(F_1,F_2,F_3)$ be a generalized Sasakian space form that is $(2n+1)-$ dimesional. M^{2n+1} is either an Einstein manifold or $F_1 = F_3$ if it is W_6- semisymmetric.*

Definition 3.14. *Assume that $M^{2n+1}(F_1,F_2,F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R\cdot W_7$ and $Q(S,W_7)$ are linearly dependent, then M^{2n+1} is called a W_7- Ricci pseudosymmetric.*

In this specific case, a function λ_4 exists such that

$$R \cdot W_7 = \lambda_3 \cdot Q(S, W_7),$$

where S is the Ricci curvature tensor. If $\lambda_4 = 0$, then the M^{2n+1} is called a W_7- Ricci semisymmetric.

Theorem 3.15. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. M^{2n+1} is either an Eintein manifold if $F_1 \neq F_3$ or $F_1 = F_3$ if it is W_7- Ricci pseudosymmetric.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_7- Ricci pseudosemisymmetric:

$$(R(X_1, X_2) \cdot W_7)(X_4, X_5, X_3) = \lambda_4 \cdot Q(S, W_7)(X_4, X_5, X_3; X_1, X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned}
&R(X_1, X_2)W_7(X_4, X_5)X_3 - W_7(R(X_1, X_2)X_4, X_5)X_3 - W_7(X_4, R(X_1, X_2)X_5)X_3 \\
&-W_7(X_4, X_5R(X_1, X_2)X_3 = \lambda_4 \cdot [-S(X_5, X_3)W_7(X_4, X_1)X_2 + S(X_4, X_3)W_7(X_5, X_1)X_2 \\
&-S(X_5, X_1)W_7(X_3, X_4)X_2 + S(X_4, X_1)W_7(X_3, X_5)X_2 - S(X_5, X_2)W_7(X_3, X_1)X_4 + S(X_4, X_2)W_7(X_3, X_1)X_5].
\end{aligned} \tag{3.19}$$

Choosing $X_1 = \xi$ in (3.19) and using (2.1), (2.3), (2.6), (2.9), (2.21), and (2.22), we obtain

$$\begin{aligned}
&(F_1 - F_3)[g(X_2, W_7(X_4, X_5)X_3)\xi - \eta(W_7(X_4, X_5)X_3)X_2 - S(X_2, X_4)[(F_1 - F_3)g(X_5, X_3)\xi \\
&-\eta(X_3)X_5 - \frac{1}{n-1}S(X_5, X_3)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_5, X_3)\xi] + \eta(X_4)W_7(X_2, X_5)X_3 \\
&-S(X_2, X_5)[-(F_1 - F_3)g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_3)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_3)QX_4] \\
&\eta(X_5)W_7(X_4, X_2)X_3 - S(X_2, X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{1}{n-1}\eta(X_5)QX_4] \\
&+\eta(X_3)W_7(X_4, X_5)X_2] = \lambda_4[-S(X_5, X_3)[-(F_1 - F_3)g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_3)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 \\
&+\frac{1}{n-1}\eta(X_3)QX_4] + S(X_4, X_3)[-(F_1 - F_3)g(X_5, X_2)\xi + (F_1 - F_3)\eta(X_2)X_5 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_2)X_5 \\
&+\frac{1}{n-1}\eta(X_2)QX_5] - 2n(F_1 - F_3)\eta(X_4)W_7(X_3, X_5)X_2 - S(X_5, X_2)[-(F_1 - F_3)g(X_3, X_4)\xi \\
&+(F_1 - F_3)\eta(X_4)X_3 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_4)X_3 + \frac{1}{n-1}\eta(X_4)QX_3] \\
&+S(X_2, X_4)[-(F_1 - F_3)g(X_3, X_5)\xi + (F_1 - F_3)\eta(X_5)X_3] - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_5 + \frac{1}{n-1}\eta(X_5)QX_3].
\end{aligned} \tag{3.20}$$

If we write $X_4 = \xi$ in (3.20) and make simplifications by considering (2.3), (2.9), and (2.20), we calculate

$$\begin{aligned}
&(F_1 - F_3)[g(X_2, W_7(X_4, X_5)X_3)\xi - \eta(W_7(X_4, X_5)X_3)X_2 - S(X_2, X_4)[(F_1 - F_3)g(X_5, X_3)\xi \\
&-\eta(X_3)X_5 - \frac{1}{n-1}S(X_5, X_3)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_5, X_3)\xi] + \eta(X_4)W_7(X_2, X_5)X_3 \\
&-S(X_2, X_5)[-(F_1 - F_3)g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_3)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_3)QX_4] \\
&\eta(X_5)W_7(X_4, X_2)X_3 - S(X_2, X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{1}{n-1}\eta(X_5)QX_4] \\
&+\eta(X_3)W_7(X_4, X_5)X_2] = \lambda_4(F_1 - F_3)[-2n(F_1 - F_3)\eta(X_3)g(X_2, X_5)\xi + 2n(F_1 - F_3)\eta(X_2)\eta(X_3)X_5 \\
&-\frac{4n^2(F_1 - F_3)}{n-1}\eta(X_2)\eta(X_3)X_5 + \frac{2n}{n-1}\eta(X_2)\eta(X_3)QX_5 + 2n\eta(X_5)g(X_2, X_3)\xi - 2n(F_1 - F_3)\eta(X_2)\eta(X_5)X_3 \\
&-\frac{2n}{n-1}\eta(X_2)\eta(X_5)QX_3 + 2nW_7(X_3, X_5)X_2 + S(X_2, X_5)\eta(X_3)\xi - S(X_2, X_5)X_3 + \frac{2n}{n-1}S(X_2, X_5)X_3 \\
&-\frac{1}{n-1}S(X_2, X_5)QX_3 - 2n(F_1 - F_3)\eta(X_2)g(X_3, X_5)\xi + 2n\eta(X_2)\eta(X_5)X_3 + \frac{2n}{n-1}\eta(X_2)\eta(X_5)QX_3].
\end{aligned} \tag{3.21}$$

If we choose $X_3 = \xi$ in (3.21), we compute

$$
\begin{aligned}
&-\frac{2n(F_1-F_3)^2}{n-1}\eta(X_5)X_2 + \frac{(F_1-F_3)}{n-1}\eta(X_5)QX_2 - \frac{(F_1-F_3)}{n-1}S(X_2,X_5)\xi + \frac{2n(F_1-F_3)^2}{n-1}g(X_2,X_5)\xi \\
&= \lambda_4[2n(F_1-F_3)^2\eta(X_2)X_5 - \frac{4n^2(F_1-F_3)^2}{n-1}\eta(X_2)X_5 + \frac{2n(F_1-F_3)}{n-1}\eta(X_2)QX_5 \\
&-2n(F_1-F_3)^2\eta(X_2)X_5 + \frac{4n^2(F_1-F_3)^2}{n-1}g(X_2,X_5)\xi - \frac{2n(F_1-F_3)}{n-1}S(X_2,X_5)\xi]. \qquad (3.22)
\end{aligned}
$$

Taking inner product both sides of the equation by $X_6 \in \chi(M)$ and choosing $X_5 = \xi$, we acquire

$$\frac{(F_1-F_3)}{n-1}[S(X_2,X_6) - 2n(F_1-F_3)g(X_2,X_6)] = 0.$$

□

Corollary 3.16. *Assume that $M^{2n+1}(F_1,F_2,F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If M^{2n+1} is W_7- semisymmetric, M^{2n+1} is either an Einstein manifold or $F_1 = F_3$.*

Definition 3.17. *Assume that $M^{2n+1}(F_1,F_2,F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R\cdot W_9$ and $Q(g,W_9)$ are linearly dependent, then M^{2n+1} is called a W_9- pseudosymmetric.*

In this specific case, a function λ_1 exists such that

$$R\cdot W_9 = \lambda_5 \cdot Q(g,W_6),$$

where R is the Riemann curvature tensor. If $\lambda_5 = 0$, then the M^{2n+1} is called a W_9- semisymmetric.

Theorem 3.18. *Let M^{2n+1} be a $2n+1-$dimensional generalized Sasakian space form. If M^{2n+1} is W_9- pseudosymmetric, then M^{2n+1} is either an Eintein manifold if $(F_1-F_3) = \frac{2n}{n-1}$ or a $\eta-$ Einstein manifold if $(F_1-F_3) \neq \frac{2n}{n-1}$ if $F_1 \neq F_3$.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_9- pseudosemisymmetric:

$$(R(X_1,X_2)\cdot W_9)(X_4,X_5,X_3) = \lambda_1 \cdot Q(g,W_6)(X_4,X_5,X_3;X_1,X_2)$$

for all $X_1,X_2,X_3,X_4,X_5 \in \chi(M)$. Hence, we have

$$
\begin{aligned}
&R(X_1,X_2)W_9(X_4,X_5)X_3 - W_9(R(X_1,X_2)X_4,X_5)X_3 - W_9(X_4,R(X_1,X_2)X_5)X_3 \\
&-W_9(X_4,X_5R(X_1,X_2)X_3 = \lambda_5\cdot[-g(X_5,X_3)W_9(X_4,X_1)X_2 + g(X_4,X_3)W_9(X_5,X_1)X_2 \qquad (3.23)\\
&-g(X_5,X_1)W_9(X_3,X_4)X_2 + g(X_4,X_1)W_9(X_3,X_5)X_2 - g(X_5,X_2)W_9(X_3,X_1)X_4 + g(X_4,X_2)W_9(X_3,X_1)X_5].
\end{aligned}
$$

Choosing $X_1 = \xi$ in (3.23) and exploiting (2.1), (2.6), (2.8), (2.24), (2.25), and (2.26), we obtain

$$\begin{aligned}
&(F_1 - F_3)[g(X_2, W_9(X_4, X_5)X_3)\xi - \eta(W_9(X_4, X_5)X_3)X_2 - g(X_2, X_4)[(F_1 - F_3)g(X_5, X_3)\xi \\
&-\eta(X_3)X_5 - \frac{1}{n-1}S(X_5, X_3)\xi + \frac{1}{n-1}\eta(X_5)QX_3] + \eta(X_4)W_9(X_2, X_5)X_3 \\
&g(X_2, X_5)[-(F_1 - F_3)g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_3)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_4)QX_3] \\
&+\eta(X_5)W_9(X_4, X_2)X_3 - (F_1 - F_3)g(X_2, X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{2n}{n-1}g(X_4, X_5)\xi] \\
&+\eta(X_5)W_9(X_4, X_5)X_2] = \lambda_5[-g(X_5, X_3)[-(F_1 - F_3)g(X_4, X_2)\xi + (F_1 - F_3)\eta(X_2)X_4 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_4)X_2 \\
&-\frac{1}{n-1}\eta(X_2)QX_4] + g(X_4, X_2)[-(F_1 - F_3)g(X_5, X_2)\xi + (F_1 - F_3)\eta(X_2)X_5 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_5)X_2 \\
&-\frac{1}{n-1}\eta(X_2)QX_5] - \eta(X_5)W_9(X_3, X_4)X_2 + \eta(X_4)W_9(X_3, X_5)X_2 - g(X_5, X_2)[-(F_1 - F_3)g(X_3, X_4)\xi \\
&+(F_1 - F_3)\eta(X_4)X_3 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 - \frac{1}{n-1}\eta(X_4)QX_3] + g(X_2, X_4) \\
&[-(F_1 - F_3)g(X_3, X_5)\xi + (F_1 - F_3)\eta(X_5)X_3 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_5 - \frac{1}{n-1}\eta(X_5)QX_3]]
\end{aligned} \tag{3.24}$$

If we write $X_4 = \xi$ in (3.24) and make simplifications by considering (2.1), (2.9), and (2.10) we calculate

$$\begin{aligned}
&(F_1 - F_3)[\frac{2n(F_1 - F_3)}{n-1}g(X_3, X_5)X_2 + W_9(X_2, X_5)X_3 - \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)X_3 - (F_1 - F_3)g(X_2, X_3)X_5 \\
&= \lambda_5[-(F_1 - F_3)\eta(X_2)g(X_3, X_5)\xi - \frac{2n(F_1 - F_3)}{n-1}g(X_3, X_5)\xi + \frac{2n(F_1 - F_3)}{n-1}g(X_3, X_5)\eta(X_2)\xi \\
&+(F_1 - F_3)\eta(X_2)\eta(X_3)X_5 + (F_1 - F_3)\eta(X_2)\eta(X_3)X_5 - \frac{1}{n-1}\eta(X_2)\eta(X_3)QX_5 \\
&+(F_1 - F_3)\eta(X_5)g(X_2, X_3)\xi + \frac{1}{n-1}\eta(X_2)\eta(X_5)QX_3 + W_9(X_3, X_5)X_2 - (F_1 - F_3)g(X_2, X_5)X_3 \\
&-\frac{2n(F_1 - F_3)}{n-1}\eta(X_3)g(X_2, X_5)\xi + \frac{1}{n-1}g(X_2, X_5)QX_3] + \frac{2n(F_1 - F_3)}{n-1}\eta(X_2)\eta(X_3)X_5 \\
&-\frac{1}{n-1}\eta(X_5)\eta(X_2)QX_3].
\end{aligned} \tag{3.25}$$

If we choose $X_3 = \xi$ in (3.25), we briefly compute

$$\begin{aligned}
&(F_1 - F_3)[\frac{2n(F_1 - F_3)}{n-1}\eta(X_5)X_2 - (F_1 - F_3)\eta(X_2)X_5 + \frac{1}{n-1}S(X_2, X_5)\xi - \frac{1}{n-1}\eta(X_5)QX_2 \\
&-\frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)\xi - (F_1 - F_3)\eta(X_2)X_5] = \lambda_5[\frac{2n(F_1 - F_3)}{n-1}\eta(X_2)\eta(X_5)\xi \\
&-\frac{1}{n-1}\eta(X_2)QX_5 - \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)\xi + \frac{2n(F_1 - F_3)}{n-1}\eta(X_2)X_5].
\end{aligned} \tag{3.26}$$

By selecting $X_5 = \xi$ and taking the inner product of both sides of (3.26) by $X_6 \in \chi(M)$, we obtain

$$(F_1 - F_3)[\frac{1}{n-1}S(X_2, X_6) + (-2(F_1 - F_3) - \frac{2n(F_1 - F_3)}{n-1} + \frac{2n}{n-1})\eta(X_2)\eta(X_6) + \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_6)] = 0.$$

□

Corollary 3.19. *Let $M^{2n+1}(F_1, F_2, F_3)$ be a generalized Sasakian space form that is $(2n+1)-$ dimesional. If M^{2n+1} is W_9 semisymmetric, it is either a $\eta-$ Einstein manifold if $F_1 \neq F_3$ or $F_1 = F_3$.*

Definition 3.20. *Assume that $M^{2n+1}(F_1, F_2, F_3)$ is a $(2n+1)-$ dimesional generalized Sasakian space form. If $R \cdot W_9$ and $Q(S, W_9)$ are linearly dependent, then M^{2n+1} is called a W_9- Ricci pseudosymmetric.*

In this specific case, a function λ_6 exists such that

$$R \cdot W_9 = \lambda_6 \cdot Q(S, W_9),$$

where S is the Ricci curvature tensor. If $\lambda_6 = 0$, then the M^{2n+1} is called a W_9- Ricci semisymmetric.

Theorem 3.21. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. If M^{2n+1} is W_9- Ricci pseudosymmetric, it is either an Eintein manifold if $\lambda_8 = \frac{2n(F_1-F_3)-F_1+F_3-n}{2n(F_1-F_3)(n+1)}$ or an Einstein manifold with $\eta-$ if $\lambda_8 = \frac{1}{2n}$ and $\lambda_8 \neq \frac{2n(F_1-F_3)-F_1+F_3-n}{2n(F_1-F_3)(n+1)}$.*

Proof. Let M^{2n+1} be the generalized Sasakian space form that provides W_9- Ricci pseudosemisymmetric:

$$(R(X_1, X_2) \cdot W_9)(X_4, X_5, X_3) = \lambda_1 \cdot Q(S, W_6)(X_4, X_5, X_3; X_1, X_2)$$

for all $X_1, X_2, X_3, X_4, X_5 \in \chi(M)$. Hence, we have

$$\begin{aligned}
&R(X_1, X_2)W_9(X_4, X_5)X_3 - W_9(R(X_1, X_2)X_4, X_5)X_3 - W_9(X_4, R(X_1, X_2)X_5)X_3 \\
&-W_9(X_4, X_5R(X_1, X_2)X_3 = \lambda_6 \cdot [-S(X_5, X_3)W_9(X_4, X_1)X_2 + S(X_4, X_3)W_9(X_5, X_1)X_2 \\
&-S(X_5, X_1)W_9(X_3, X_4)X_2 + S(X_4, X_1)W_9(X_3, X_5)X_2 - S(X_5, X_2)W_9(X_3, X_1)X_4 \\
&+S(X_4, X_2)W_9(X_3, X_1)X_5].
\end{aligned} \tag{3.27}$$

Choosing $X_1 = \xi$ in (3.27) and exploiting (2.1), (2.3), (2.6), (2.9) (2.10), (2.25), we obtain

$$\begin{aligned}
&(F_1 - F_3)[g(X_2, W_9(X_4, X_5)X_3)\xi - \eta(W_9(X_4, X_5)X_3)X_2 - g(X_2, X_4)[(F_1 - F_3)g(X_5, X_3)\xi \\
&-\eta(X_3)X_5 - \frac{1}{n-1}S(X_5, X_3)\xi + \frac{1}{n-1}\eta(X_5)QX_3] + \eta(X_4)W_9(X_2, X_5)X_3 \\
&g(X_2, X_5)[-(F_1 - F_3)g(X_4, X_3)\xi + (F_1 - F_3)\eta(X_3)X_4 - \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 + \frac{1}{n-1}\eta(X_4)QX_3] \\
&+\eta(X_5)W_9(X_4, X_2)X_3 - (F_1 - F_3)g(X_2, X_3)[\eta(X_5)X_4 - \eta(X_4)X_5 - \frac{2n}{n-1}\eta(X_5)X_4 + \frac{2n}{n-1}g(X_4, X_5)\xi] \\
&+\eta(X_5)W_9(X_4, X_5)X_2] = \lambda_6[-S(X_5, X_3)[-(F_1 - F_3)g(X_4, X_2)\xi + (F_1 - F_3)\eta(X_2)X_4 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_4)X_2 \\
&-\frac{1}{n-1}\eta(X_2)QX_4] + S(X_3, X_4)[-g(X_5, X_2)\xi + (F_1 - F_3)\eta(X_2)X_5 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_5)X_2 \\
&-\frac{1}{n-1}\eta(X_2)QX_5] + 2n(F_1 - F_3)\eta(X_4)W_9(X_3, X_5)X_2 - S(X_5, X_2)[-(F_1 - F_3)g(X_3, X_4)\xi \\
&+(F_1 - F_3)\eta(X_4)X_3 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_4 - \frac{1}{n-1}\eta(X_4)QX_3] \\
&+S(X_2, X_4)[-(F_1 - F_3)g(X_3, X_5)\xi + (F_1 - F_3)\eta(X_5)X_3 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_5 - \frac{1}{n-1}\eta(X_5)QX_3]]
\end{aligned} \tag{3.28}$$

If we write $X_4 = \xi$ in (3.28) and make simplifications by considering (2.1), (2.3), (2.10), and (2.25), we calculate

$$\begin{aligned}
&(F_1 - F_3)[\frac{2n(F_1 - F_3)}{n-1}g(X_3, X_5)X_2 + W_9(X_2, X_5)X_3 - \frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)X_3 - (F_1 - F_3)g(X_2, X_3)X_5 \\
&= \lambda_6[-(F_1 - F_3)[\frac{2n}{n-1}X_2 - \frac{2n}{n-1}\eta(X_2)\xi] + 2n(F_1 - F_3)\eta(X_3)[-g(X_2, X_5)\xi + (F_1 - F_3)\eta(X_2)X_5\xi \\
&+\frac{2n(F_1 - F_3)}{n-1}\eta(X_5)X_2 - \frac{1}{n-1}\eta(X_2)QX_5] + 2n(F_1 - F_3)\eta(X_5)[-g(X_2, X_3)\xi + (F_1 - F_3)\eta(X_2)X_3\xi \\
&+\frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_2 - \frac{1}{n-1}\eta(X_2)QX_3] + 2n(F_1 - F_3)W_9(X_3, X_5)X_2 - S(X_2, X_5)[-(F_1 - F_3)\eta(X_3)\xi \\
&(F_1 - F_3)X_3 + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)\xi - \frac{1}{n-1}QX_3] + 2n(F_1 - F_3)\eta(X_2)[-(F_1 - F_3)g(X_3, X_5)\xi \\
&+(F_1 - F_3)\eta(X_5)X_3\xi + \frac{2n(F_1 - F_3)}{n-1}\eta(X_3)X_5 - \frac{1}{n-1}\eta(X_5)QX_3].
\end{aligned} \tag{3.29}$$

If we choose $X_3 = \xi$ in (3.29), we briefly compute

$$
\begin{aligned}
&(F_1 - F_3)[\frac{2n(F_1 - F_3)}{n-1}\eta(X_5)X_2 - (F_1 - F_3)\eta(X_2)X_5 + \frac{1}{n-1}S(X_2, X_5)\xi - \frac{1}{n-1}\eta(X_5)QX_2 \\
&-\frac{2n(F_1 - F_3)}{n-1}g(X_2, X_5)\xi - (F_1 - F_3)\eta(X_2)X_5] = \lambda_6(F_1 - F_3)[-2ng(X_2, X_5)\xi - \frac{2n}{n-1}\eta(X_2)QX_5 \\
&+\frac{4n(F_1 - F_3)}{n-1}\eta(X_5)X_2 + 2n(F_1 - F_3)g(X_2, X_5)\xi + \frac{4n^2(F_1 - F_3)}{n-1}\eta(X_5)X_2 - \frac{4n^2(F_1 - F_3)}{n-1}g(X_2, X_5)\xi \\
&-2n\eta(X_2)\eta(X_5)\xi + 2n(F_1 - F_3\eta(X_2)\eta(X_5)\xi + \frac{4n^2(F_1 - F_3)}{n-1}\eta(X_2)X_5 - \frac{4n^2(F_1 - F_3)}{n-1}\eta(X_2)\eta(X_5)\xi \qquad (3.30)
\end{aligned}
$$

Using the inner product of $X_6 \in \chi(M)$ on both sides of (3.30) and choosing $X_5 = \xi$, we obtain

$$S(X_2, X_6) + 2(2(F_1 - F_3)n + F_1 + F_3 - n - 2n(F_1 - F_3)(n+2))\eta(X_2)\eta(X_6) + 2n(F_1 - F_3)(2\lambda_6 n - 1)g(X_2, X_6).$$

□

Corollary 3.22. *Let M^{2n+1} be a $2n+1-$ dimensional generalized Sasakian space form. If $F_1 - F_3 \neq 0$, M^{2n+1} is an $\eta-$ Einstein manifold if it is W_9- Ricci semisymmetric.*

4 Conclusion

The main results of the paper conclude that W curvature conditions play an important role in defining geometric structures for generalized Sasakian space forms. Under these conditions, geometric meanings are transformed into new categories, specifically Einstein manifold and $\eta-$ Einstein manifold, confirming the efficacy of W operators as tools for classification and understanding of almost contact metric manifolds.

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