Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 2, March 2026, Pages 319-349
https://doi.org/10.34198/ejms.16226.25.319349

An Efficient Block Multistep Method for the Numerical Approximation of General Fourth-Order Ordinary Differential Equations

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Abstract

Fourth-order ordinary differential equations (ODEs) are applied in real-life situations such as analyzing the vibration and stability of structural elements, including beams, plates, and airplane wings. Other applications include modeling fluid flow, such as in the lungs, and the development of surface profiles in material science. In engineering, they are used in areas such as beam theory to predict beam failure and in analyzing the stress and strain in materials like reinforced concrete shells. The purpose of this study is to develop a class of continuous hybrid numerical methods for the direct solution of general fourth-order initial value problems of ordinary differential equations. The technique adopted in this work involves interpolation and collocation of a basis function and its corresponding differential system, respectively. The differential systems and the basis functions are collocated and interpolated, respectively, at selected grid and intra-step grid points. The unknown parameters in the system of linear equations arising from the collocation and interpolation procedures were determined, and the values were substituted into the approximate solution. The required continuous methods were obtained for different step numbers after the necessary simplifications. The derived methods were tested and found to be consistent, convergent, and to possess low error constants. The discrete schemes obtained from the continuous methods were implemented in block mode. The methods were applied to solve linear and nonlinear fourth-order initial value problems directly. The errors in the results obtained were compared with those of existing methods of the same and even higher order of accuracy to establish the superiority of the newly proposed method.

1 Introduction

Ordinary differential equations (ODEs) have important applications and serve as powerful tools in the study of many problems in the natural sciences and technology. They are extensively employed in

Received: December 23, 2025; Revised & Accepted: January 26, 2026; Published: March 16, 2026
2020 Mathematics Subject Classification: Primary 65L05; Secondary 65L06, 65L20, 65D05.
Keywords and phrases: fourth-order, initial value problem, hybrid, block, multi step, interpolation, collocation.
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mechanics, astronomy, physics, and in many areas of chemistry and biology. Mathematical models across this wide range of disciplines describe how quantities change, which naturally leads to the language of ordinary differential equations.

For instance, Newton's laws in mechanics make it possible to describe the motion of mass points or rigid bodies through ordinary differential equations. The computation of radio technical circuits, satellite trajectories, studies of the stability of an aircraft in flight, and the analysis of chemical reactions are all carried out by formulating and solving ordinary differential equations. Some of the most interesting and important applications of these equations arise in the theory of oscillations, ship dynamics, and automatic control theory.

These applied problems, in turn, generate new theoretical challenges involving first-, second-, third-, fourth-, fifth-, and even higher-order derivatives in ODEs. However, explicit solutions to many higher-order ODE problems do not exist. Consequently, there is a need to develop numerical methods, particularly implicit linear multistep methods (LMMs), to solve these problems effectively.

Therefore, in this research paper, we consider the general fourth-order ordinary differential equation of the form

$$\left.\begin{aligned} y^{iv} &= f(x, y, y', y'', y'''), \qquad x \in [a, b], \\ &\text{subject to} \\ y(x_0) &= y_0, \\ y'(x_0) &= y'_0, \\ y''(x_0) &= y''_0, \\ y'''(x_0) &= y'''_0, \qquad x_0 \in [a, b] \end{aligned}\right\} \tag{1}$$

on the partition

$$\Xi : a = x_0 < x_1 < \cdots < x_N = b, \qquad x_n = x_{n-1} + h, \qquad n = 1, \ldots, N.$$

Here, $f : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $N > 0$ is an integer, m denotes the dimension of the system, n represents the grid index, and

$$h = \frac{b-a}{N}$$

is the step size.

[1] developed a new efficient numerical model for directly solving second-, third-, and fourth-order ordinary differential equations.

[2] examined a new special 15-step block method for solving general fourth-order ordinary differential equations. The formulation of the new scheme was achieved through interpolation and collocation techniques. The method is of sixteenth order, and the number of interpolated points corresponds to the order of the problem being solved.

[21] proposed the development and implementation of a four-step predictor–corrector method with

an improvement strategy for fourth-order ordinary differential equations, together with applications. The method is of order ten and exhibits high accuracy.

[18] proposed a block algorithm containing nine intermediate steps with an order of accuracy $p = 11$.

[4] developed an improved Runge–Kutta method in which the problem was first converted into a system of first-order ODEs before being solved.

[13] proposed a block extension of a single-step hybrid multistep method for directly solving fourth-order initial value problems with order of accuracy nine.

[16] developed a Chebyshev-generated block method for directly solving nonlinear and ill-posed fourth-order ODEs. The method was derived by applying interpolation and collocation procedures to a Chebyshev approximate polynomial. The method was shown to be zero-stable, consistent, convergent, and p-stable, with order seven.

Moreover, most of the methods mentioned above for solving higher-order ODEs and implemented in block mode were attempts to overcome early setbacks of predictor–corrector methods. For instance, predictor–corrector methods often combine lower-order predictors with higher-order correctors, which may result in relatively low overall accuracy. It should also be noted that the block method is problem-independent, unlike conventional block methods that are problem-dependent. This observation forms the motivation for the present work. In this paper, an order-eight block method with five inter-steps embedded within a step length of three is presented for the solution of general fourth-order ODEs.

For completeness and readability, a power series is employed as the basis function to derive the method in Section 2. In Section 3, the fundamental properties of the method are examined to determine its applicability. Section 4 provides numerical examples to demonstrate the effectiveness of the proposed method, and Section 5 concludes the study with final remarks.

Definition 1.1. *A numerical method is said to be* A-stable *if its region of absolute stability contains the entire complex left-half plane*

$$\mathrm{Re}(h\lambda) \leq 0$$

(see Dahlquist).

Alternatively, a numerical method is called A-stable *if all solutions of* (1) *tend to zero as* $n \to \infty$ *when the method is applied, with fixed positive* h*, to any differential equation of the form*

$$\frac{dy}{dx} = \lambda y,$$

where λ *is a complex constant with negative real part.*

Definition 1.2. *A numerical method is said to be* $A(\alpha)$*-stable,* $\alpha \in \left(0, \frac{\pi}{2}\right)$*, if its region of absolute stability contains the infinite wedge*

$$W_\alpha = \{\lambda h : -\alpha \leq \pi - \arg(\lambda h) \leq \alpha\}.$$

2 Methodology

The following power series is used as the approximate solution:

$$y(x) = \sum_{j=0}^{(C+I)-1} e_j x^j, \tag{2}$$

where e_j, $j = 0, 1, 2, \cdots, k$, are the coefficients to be determined, and x is continuous and differentiable. Here, C is the number of collocation points and I is the number of interpolation points.

The fourth derivative of (2) is given as

$$y^{iv}(x) = \sum_{j=4}^{(C+I)-1} j(j-1)(j-2)(j-3)e_j x^{j-4}. \tag{3}$$

Equations (2) and (3) form the differential system to be solved using Gaussian elimination in order to determine the unknowns e_j. Interpolating (2) at the points

$$x = x_n,\ x_{n+r},\ x_{n+s},\ x_{n+1},\ x_{n+t},\ x_{n+2},\ x_{n+u},\ x_{n+v},$$

and collocating (3) at the points

$$x = x_n,\ x_{n+1},\ x_{n+2},\ x_{n+3},$$

where $0 < r, s < 1$, $1 < t < 2$, and $2 < u, v < 3$.

Specifically, the values of r, s, t, u, v are taken as

$$\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{7}{3}, \frac{8}{3},$$

respectively. The values of e_j, $j = 0, 1, \ldots, 11$, are substituted into (2) to obtain a linear hybrid multistep method with continuous coefficients in the form:

$$y(x) = \left\{ \begin{array}{l} \alpha_0(t)y_n + \alpha_{\frac{1}{3}}(t)y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(t)y_{n+\frac{2}{3}} + \alpha_1(t)y_{n+1} + \alpha_{\frac{3}{2}}(t)y_{n+\frac{3}{2}} + \alpha_2(t)y_{n+2} + \alpha_{\frac{7}{3}}(t)y_{n+\frac{7}{3}} \\ + \alpha_{\frac{8}{3}}(t)y_{n+\frac{8}{3}} + h^4(\beta_0(t)f_n + \beta_1(t)f_{n+1} + \beta_2(t)f_{n+2} + \beta_3(t)f_{n+3} \end{array} \right\} \tag{4}$$

Using the transformation in Obarhua [19],

$$t = \frac{x - x_{n+k-1}}{h},$$
$$\frac{dt}{dx} = \frac{1}{h}.$$

The coefficients of y_{n+j} and f_{n+j} are obtained in terms of t as follows:

$$\alpha_0(t) = \left\{ \begin{array}{l} 1 - \frac{53899915}{21730632}t - \frac{1887573}{164626}t^2 + \frac{26936935}{718368}t^3 - \frac{9998964}{82313}t^5 + \frac{18810387}{94072}t^6 \\ - \frac{29810997}{188144}t^7 + \frac{189317655}{2634016}t^8 - \frac{50141835}{2634016}t^9 + \frac{7223661}{2634016}t^{10} - \frac{2407887}{14487088}t^{11} \end{array} \right\} \tag{5}$$

$$\alpha_{\frac{1}{3}}(t) = \left\{ \begin{array}{l} -\dfrac{2514507764247936}{65421448436575}\,t + \dfrac{11865063358905552}{41631830823275}\,t^2 - \dfrac{2212479979458192}{3784711893025}\,t^3 \\ + \dfrac{65498623238043288}{41631830823275}\,t^5 - \dfrac{14965151920137324}{5947404403325}\,t^6 + \dfrac{81637067352375837}{41631830823275}\,t^7 \\ - \dfrac{146597574614397993}{166527323293100}\,t^8 + \dfrac{385648447904316}{1665273232931}\,t^9 - \dfrac{5530470810060393}{166527323293100}\,t^{10} \\ + \dfrac{1837676031505893}{915900278112050}\,t^{11} \end{array} \right\} \tag{6}$$

$$\alpha_{\frac{2}{3}}(t) = \left\{ \begin{array}{l} \dfrac{6802027220144583}{46729606026125}\,t - \dfrac{32176055059768179}{33985168019000}\,t^2 + \dfrac{11389931730591453}{6179121458000}\,t^3 \\ - \dfrac{40662574728650559}{8496292004750}\,t^5 + \dfrac{64605656892252609}{8496292004750}\,t^6 - \dfrac{200606214555836019}{33985168019000}\,t^7 \\ + \dfrac{179673176866579443}{67970336038000}\,t^8 - \dfrac{377498830527063}{543762688304}\,t^9 + \dfrac{6758923323485043}{67970336038000}\,t^{10} \\ - \dfrac{2243841946782453}{373836848209000}\,t^{11} \end{array} \right\} \tag{7}$$

$$\alpha_{1}(t) = \left\{ \begin{array}{l} -\dfrac{1581554948115072}{9345921205225}\,t + \dfrac{922877025926512}{849629200475}\,t^2 - \dfrac{161085515709652}{77239018225}\,t^3 \\ + \dfrac{4537620997303023}{849629200475}\,t^5 - \dfrac{14376961268550531}{1699258400950}\,t^6 + \dfrac{5570064234248337}{849629200475}\,t^7 \\ - \dfrac{19929563434220751}{6797033603800}\,t^8 + \dfrac{104570316262557}{135940672076}\,t^9 - \dfrac{748179033514821}{6797033603800}\,t^{10} \\ + \dfrac{248148260180541}{37383684820900}\,t^{11} \end{array} \right\} \tag{8}$$

$$\alpha_{\frac{3}{2}}(t) = \left\{ \begin{array}{l} \dfrac{86202246039076864}{981321726548625}\,t - \dfrac{116870818864562176}{208159154116375}\,t^2 + \dfrac{60875398908674048}{56770678395375}\,t^3 \\ - \dfrac{566284122573225984}{208159154116375}\,t^5 + \dfrac{127766956598157312}{29737022016625}\,t^6 - \dfrac{691141065551732736}{208159154116375}\,t^7 \\ + \dfrac{308233119097092096}{208159154116375}\,t^8 - \dfrac{644810572781568}{1665273232931}\,t^9 + \dfrac{11490448657360896}{208159154116375}\,t^{10} \\ - \dfrac{7590618972076032}{2289750695280125}\,t^{11} \end{array} \right\} \tag{9}$$

$$\alpha_{2}(t) = \left\{ \begin{array}{l} -\dfrac{18179626299387}{849629200475}\,t + \dfrac{84213149031871}{617912145800}\,t^2 - \dfrac{320004100401977}{1235824291600}\,t^3 \\ + \dfrac{99794844594531}{154478036450}\,t^5 - \dfrac{77818317796233}{77239018225}\,t^6 + \dfrac{473156442280701}{617912145800}\,t^7 \\ - \dfrac{412952299836957}{1235824291600}\,t^8 + \dfrac{4201383172941}{49432971664}\,t^9 - \dfrac{14476133562057}{1235824291600}\,t^{10} \\ + \dfrac{4599059492007}{6797033603800}\,t^{11} \end{array} \right\} \tag{10}$$

$$\alpha_{\frac{7}{3}}(t) = \left\{ \begin{array}{l} -\dfrac{1966787338950144}{327107242182875}\,t + \dfrac{1153062628975584}{29737022016625}\,t^2 - \dfrac{203041069528644}{2703365637875}\,t^3 \\ + \dfrac{5956740515058831}{29737022016625}\,t^5 - \dfrac{2796484795006941}{8496292004750}\,t^6 + \dfrac{1137056023843182}{4248146002375}\,t^7 \\ - \dfrac{3022188720153563 7}{237896176133000}\,t^8 + \dfrac{33967726281045}{951584704532}\,t^9 - \dfrac{1309183742090187}{237896176133000}\,t^{10} \\ + \dfrac{468358311024027}{1308428968731500}\,t^{11} \end{array} \right\} \tag{11}$$

$$\alpha_{\frac{8}{3}}(t)=\left\{\begin{array}{l}\frac{17930294352837}{4325385020600}\,t-\frac{402549707075427}{15138847572100}\,t^2+\frac{6186570289869321}{121110780576800}\,t^3\\ -\frac{501412362975477}{3784711893025}\,t^5+\frac{921809360892393}{4325385020600}\,t^6-\frac{10218156392025333}{60555390288400}\,t^7\\ +\frac{9394715758141521}{121110780576800}\,t^8-\frac{101850500665989}{4844431223072}\,t^9+\frac{377840422089891}{121110780576800}\,t^{10}\\ -\frac{11834122224051}{60555390288400}\,t^{11}\end{array}\right\} \tag{12}$$

$$\beta_0=\left\{\begin{array}{l}-\frac{9192239625323}{856426234078800}\,t^5h^4+\frac{23331901128773}{856426234078800}\,t^7h^4-\frac{193226800501}{13594067207600}\,t^8h^4\\ +\frac{3831038177}{951584704532}\,t^9h^4-\frac{57314583297}{95158470453200}\,t^{10}h^4+\frac{19554783957}{523371587492600}\,t^{11}h^4\\ -\frac{34360917827}{926868218700}\,t^3h^4+\frac{1528959503021}{107053279259850}\,t^2h^4-\frac{55811930068}{28037763615675}\,th^4\\ -\frac{1385163465563}{61173302434200}\,t^6h^4+\frac{1}{24}\,h^4t^4\end{array}\right\} \tag{13}$$

$$\beta_1(t)=\left\{\begin{array}{l}-\frac{135513397800013}{53526639629925}\,t^2h^4+\frac{11147741231848}{28037763615675}\,th^4-\frac{573873132859529}{47579235226600}\,t^5h^4\\ -\frac{4167556353758111}{285475411359600}\,t^7h^4+\frac{88376860194661}{13594067207600}\,t^8h^4-\frac{6473565562749}{3806338818128}\,t^9h^4\\ +\frac{2889997461369}{11894808806650}\,t^{10}h^4-\frac{3831085930311}{261685793746300}\,t^{11}h^4+\frac{2319067856787571}{122346604868400}\,t^6h^4\\ +\frac{2229372477541}{463434109350}\,t^3h^4\end{array}\right\} \tag{14}$$

$$\beta_2=\left\{\begin{array}{l}\frac{287524127417567}{95158470453200}\,t^5h^4+\frac{1086222244158869}{285475411359600}\,t^7h^4-\frac{23582144398589}{13594067207600}\,t^8h^4\\ +\frac{221443397229}{475792352266}\,t^9h^4-\frac{6495020572653}{95158470453200}\,t^{10}h^4+\frac{2209494893913}{523371587492600}\,t^{11}h^4\\ +\frac{32716955528042}{53526639629925}\,t^2h^4-\frac{362146175091}{308956072900}\,t^3h^4-\frac{2674647114752}{28037763615675}\,th^4\\ -\frac{36985455609899}{7646662804275}\,t^6h^4\end{array}\right\} \tag{15}$$

$$\beta_3(t)=\left\{\begin{array}{l}-\frac{6390518968}{28037763615675}\,th^4-\frac{435453157}{154478036450}\,t^3h^4+\frac{3156575039551}{428213117039400}\,t^5h^4\\ -\frac{1461136410241}{122346604868400}\,t^6h^4+\frac{8181261781393}{856426234078800}\,t^7h^4-\frac{60569328601}{13594067207600}\,t^8h^4\\ +\frac{4691323969}{3806338818128}\,t^9h^4-\frac{8953755381}{47579235226600}\,t^{10}h^4+\frac{802386639}{65421448436575}\,t^{11}h^4\\ +\frac{78367503463}{53526639629925}\,t^2h^4\end{array}\right\} \tag{16}$$

Evaluating at the non-interpolation point gives the main scheme of the method.

$$y_{n+3}=\left\{\begin{array}{l}\frac{5510839779}{505774675}\,y_{n+\frac{8}{3}}-\frac{12268164582}{361267625}\,y_{n+\frac{7}{3}}+\frac{3179719998}{72253525}\,y_{n+2}-\frac{100876746752}{2528873375}\,y_{n+\frac{3}{2}}\\ +\frac{3179719998}{72253525}\,y_{n+1}-\frac{12268164582}{361267625}\,y_{n+\frac{2}{3}}+\frac{5510839779}{505774675}\,y_{n+\frac{1}{3}}-y_n\\ +\frac{1}{2528873375}\left(755090\,f_{n+3}-158515490\,f_{n+2}-158515490\,f_{n+1}+755090\,f_n\right)h^4\end{array}\right\} \tag{17}$$

Putting $t = 1$ in (6) and evaluating its first, second, and third derivatives at the points

$$x = x_n,\ x_{n+\frac{1}{3}},\ x_{n+1},\ x_{n+\frac{3}{2}},\ x_{n+2},\ x_{n+\frac{7}{3}},\ x_{n+\frac{8}{3}},\ x_{n+3}.$$

While the fourth derivative of (6) is evaluated at the points

$$x = x_{n+\frac{1}{3}},\ x_{n+\frac{2}{3}},\ x_{n+\frac{3}{2}},\ x_{n+\frac{7}{3}},$$

and represented below.

Represented in matrix form:

$$Y_m = A_i y_i + h^4 b_i f_i. \tag{18}$$

Where

$$Y_m = \begin{bmatrix} y_{n+3} \\ hy'_n \\ hy'_{n+\frac{1}{3}} \\ hy'_{n+\frac{2}{3}} \\ hy'_{n+1} \\ hy'_{n+\frac{3}{2}} \\ hy'_{n+2} \\ hy'_{n+\frac{7}{3}} \\ hy'_{n+\frac{8}{3}} \\ hy'_{n+3} \\ h^2y''_n \\ h^2y''_{n+\frac{1}{3}} \\ h^2y''_{n+\frac{2}{3}} \\ h^2y''_{n+1} \\ h^2y''_{n+\frac{3}{2}} \\ h^2y''_{n+2} \\ h^2y''_{n+\frac{7}{3}} \\ h^2y''_{n+\frac{8}{3}} \\ h^2y''_{n+3} \\ h^3y'''_n \\ h^3y'''_{n+\frac{1}{3}} \\ h^3y'''_{n+\frac{2}{3}} \\ h^3y'''_{n+1} \\ h^3y'''_{n+\frac{3}{2}} \\ h^3y'''_{n+2} \\ h^3y'''_{n+\frac{7}{3}} \\ h^3y'''_{n+\frac{8}{3}} \\ h^3y'''_{n+3} \\ h^4y^{iv}_{n+\frac{1}{3}} \\ h^4y^{iv}_{n+\frac{2}{3}} \\ h^4y^{iv}_{n+\frac{3}{2}} \\ h^4y^{iv}_{n+\frac{7}{3}} \end{bmatrix}, y_i = \begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{7}{3}} \\ y_{n+\frac{8}{3}} \end{bmatrix}, f_i = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}$$

Adopting the matrix inversion method to solve (18), the quantities

$$y_{n+\frac{1}{3}},\ y_{n+\frac{2}{3}},\ y_{n+1},\ y_{n+\frac{3}{2}},\ y_{n+2},\ y_{n+\frac{7}{3}},\ y_{n+\frac{8}{3}},$$

$$y'_{n+\frac{1}{3}},\ y'_{n+\frac{2}{3}},\ y'_{n+1},\ y'_{n+\frac{3}{2}},\ y'_{n+2},\ y'_{n+\frac{7}{3}},\ y'_{n+\frac{8}{3}},\ y'_{n+3},$$

$$y''_{n+\frac{1}{3}},\ y''_{n+\frac{2}{3}},\ y''_{n+1},\ y''_{n+\frac{3}{2}},\ y''_{n+2},\ y''_{n+\frac{7}{3}},\ y''_{n+\frac{8}{3}},\ y''_{n+3},$$

$$y'''_{n+\frac{1}{3}},\ y'''_{n+\frac{2}{3}},\ y'''_{n+1},\ y'''_{n+\frac{3}{2}},\ y'''_{n+2},\ y'''_{n+\frac{7}{3}},\ y'''_{n+\frac{8}{3}},\ y'''_{n+3}$$

are determined and expressed as given below.

$$y_{n+\frac{1}{3}} = \left\{ \begin{array}{l} \frac{1}{54997766208000}\Big(-894614706\, f_{n+\frac{7}{3}} - 5990371328\, f_{n+\frac{3}{2}} - 20824471116\, f_{n+\frac{2}{3}} \\ +\, 21092333175\, f_{n+\frac{1}{3}} + 35497225\, f_{n+3} + 2723030100\, f_{n+2} + 14258711250\, f_{n+1} \\ +\, 17890917400\, f_n)h^4) + \frac{1}{162}\, y'''{}_n h^3 + \frac{1}{18}\, y''_n h^2 + \frac{1}{3}\, y'_n h + y_n \end{array} \right\} \tag{19}$$

$$y_{n+\frac{2}{3}} = \left\{ \begin{array}{l} \frac{1}{214835024250}\Big(-52580826\, f_{n+\frac{7}{3}} - 354377728\, f_{n+\frac{3}{2}} - 1257396696\, f_{n+\frac{2}{3}} \\ +\, 1649769525\, f_{n+\frac{1}{3}} + 2081075\, f_{n+3} + 160353900\, f_{n+2} + 851404050\, f_{n+1} \\ +\, 768936200\, f_n)h^4 + \frac{4}{81}\, y'''_n h^3 + \frac{2}{9}\, y''_n h^2 + \frac{2}{3}\, y'_n h + y_n \end{array} \right\} \tag{20}$$

$$y_{n+1} = \left\{ \begin{array}{l} \frac{1}{2794176000}\Big(-2744442\, f_{n+\frac{7}{3}} - 18436096\, f_{n+\frac{3}{2}} - 58486212\, f_{n+c} \\ +\, 105895755\, f_{n+\frac{1}{3}} + 108725\, f_{n+3} + 8362620\, f_{n+2} + 43983450\, f_{n+1} \\ +\, 37740200\, f_n)h^4 + \frac{1}{6}\, y'''_n h^3 + \frac{1}{2}\, y''_n h^2 + y'{}_n h + y_n \end{array} \right\} \tag{21}$$

$$y_{n+\frac{3}{2}} = \left\{ \begin{array}{l} \frac{1}{2207744000}\Big(-8089713\, f_{n+\frac{7}{3}} - 53921664\, f_{n+\frac{3}{2}} - 116739873\, f_{n+\frac{2}{3}} \\ +\, 366103800\, f_{n+\frac{1}{3}} + 320850\, f_{n+3} + 24621975\, f_{n+2} + 145052775\, f_{n+1} \\ +\, 108347850\, f_n)h^4 + \frac{9}{16}\, y'''_n h^3 + \frac{9}{8}\, y''_n h^2 + \frac{3}{2}\, y'{}_n h + y_n \end{array} \right\} \tag{22}$$

$$y_{n+2} = \left\{ \begin{array}{l} \frac{1}{1212750}\Big(-10638\, f_{n+\frac{7}{3}} - 57344\, f_{n+\frac{3}{2}} - 99468\, f_{n+\frac{2}{3}} \\ +\, 543375\, f_{n+\frac{1}{3}} + 425\, f_{n+3} + 32200\, f_{n+2} + 253750\, f_{n+1} \\ +\, 146200\, f_n)h^4 + \frac{4}{3}\, y'''{}_n h^3 + 2\, y''_n h^2 + 2\, y'{}_n h + y_n \end{array} \right\} \tag{23}$$

$$y_{n+\frac{7}{3}} = \left\{ \begin{array}{l} \frac{1}{1122403392000}\Big(-15880151574\, f_{n+\frac{7}{3}} - 48257871872\, f_{n+\frac{3}{2}} - 103752501804\, f_{n+\frac{2}{3}} \\ +\, 844694189325\, f_{n+\frac{1}{3}} + 630682675\, f_{n+3} + 51353788500\, f_{n+2} + 439366072950\, f_{n+1} \\ +\, 218106359800\, f_n)h^4 + \frac{343}{162}\, y'''{}_n h^3 + \frac{49}{18}\, y''_n h^2 + \frac{7}{3}\, y'{}_n h + y_n \end{array} \right\} \tag{24}$$

$$y_{n+\frac{8}{3}} = \left\{ \begin{array}{l} \frac{1}{107417512125} \left(-2127264768\, f_{n+\frac{7}{3}} - 140509184\, f_{n+\frac{3}{2}} - 9602178048\, f_{n+\frac{2}{3}} \right. \\ + 125821278720\, f_{n+\frac{1}{3}} + 91686400\, f_{n+3} + 9293921280\, f_{n+2} + 71480908800\, f_{n+1} \\ \left. + 31510412800\, f_n \right) h^4 + \frac{256}{81}\, y'''_n h^3 + \frac{32}{9}\, y''_n h^2 + \frac{8}{3}\, y'_n h + y_n \end{array} \right\} \tag{25}$$

$$y_{n+3} = \left\{ \begin{array}{l} \frac{1}{34496000} \left(-354294\, f_{n+\frac{7}{3}} + 3575808\, f_{n+\frac{3}{2}} - 2125764\, f_{n+\frac{2}{3}} \right. \\ + 59344245\, f_{n+\frac{1}{3}} + 52875\, f_{n+3} + 5409180\, f_{n+2} + 35976150\, f_{n+1} \\ \left. + 14545800\, f_n \right) h^4 + \frac{9}{2}\, y'''_n h^3 + \frac{9}{2}\, y''_n h^2 + 3\, y'_n h + y_n \end{array} \right\} \tag{26}$$

$$y'_{n+\frac{1}{3}} = \left\{ \begin{array}{l} \frac{1}{185177664000} \left(-38092518\, f_{n+\frac{7}{3}} - 255954944\, f_{n+\frac{3}{2}} - 902943288\, f_{n+\frac{2}{3}} \right. \\ + 962627085\, f_{n+\frac{1}{3}} + 1509475\, f_{n+3} + 116063640\, f_{n+2} + 612501750\, f_{n+1} \\ \left. + 647360800\, f_n \right) h^3 + \frac{1}{18}\, y'''_n h^2 + \frac{1}{3}\, y''_n h + y'_n \end{array} \right\} \tag{27}$$

$$y'_{n+\frac{2}{3}} = \left\{ \begin{array}{l} \frac{1}{2170050750} \left(-2854278\, f_{n+\frac{7}{3}} - 19226624\, f_{n+\frac{3}{2}} - 66876273\, f_{n+\frac{2}{3}} \right. \\ + 102350385\, f_{n+\frac{1}{3}} + 112975\, f_{n+3} + 8703765\, f_{n+2} + 46081350\, f_{n+1} \\ \left. + 38871700\, f_n \right) h^3 + \frac{2}{9}\, y'''_n h^2 + \frac{2}{3}\, y''_n h + y'_n \end{array} \right\} \tag{28}$$

$$y'_{n+1} = \left\{ \begin{array}{l} \frac{1}{84672000} \left(-275562\, f_{n+\frac{7}{3}} - 1847296\, f_{n+\frac{3}{2}} - 4846392\, f_{n+\frac{2}{3}} \right. \\ + 12126915\, f_{n+1/3} + 10925\, f_{n+3} + 839160\, f_{n+2} + 4419450\, f_{n+1} \\ \left. + 3684800\, f_n \right) h^3 + \frac{1}{2}\, y'''_n h^2 + y''_n h + y'_n \end{array} \right\} \tag{29}$$

$$y'_{n+\frac{3}{2}} = \left\{ \begin{array}{l} \frac{1}{1605632000} \left(-12321558\, f_{n+\frac{7}{3}} - 78833664\, f_{n+\frac{3}{2}} - 104634828\, f_{n+\frac{2}{3}} \right. \\ + 624836835\, f_{n+\frac{1}{3}} + 490725\, f_{n+3} + 37319940\, f_{n+2} + 271054350\, f_{n+1} \\ \left. + 165256200\, f_n \right) h^3 + \frac{9}{8}\, y'''_n h^2 + \frac{3}{2}\, y''_n h + y'_n \end{array} \right\} \tag{30}$$

$$y'_{n+2} = \left\{ \begin{array}{l} \frac{1}{330750} \left(-4374\, f_{n+\frac{7}{3}} - 8192\, f_{n+\frac{3}{2}} - 15309\, f_{n+\frac{2}{3}} \right. \\ + 251505\, f_{n+\frac{1}{3}} + 175\, f_{n+3} + 13545\, f_{n+2} + 141750\, f_{n+1} \\ \left. + 61900\, f_n \right) h^3 + 2\, y'''_n h^2 + 2\, y''_n h + y'_n \end{array} \right\} \tag{31}$$

$$y'_{n+\frac{7}{3}} = \left\{ \begin{array}{l} \frac{1}{3779136000} \left(-71624574\, f_{n+7/3} + 226193408\, f_{n+\frac{3}{2}} - 51342984\, f_{n+\frac{2}{3}} \right. \\ + 4065625305\, f_{n+\frac{1}{3}} + 2821175\, f_{n+3} + 308576520\, f_{n+2} + 2548781550\, f_{n+1} \\ \left. + 972473600\, f_n \right) h^3 + \frac{49}{18}\, y'''_n h^2 + \frac{7}{3}\, y''_n h + y'_n \end{array} \right\} \tag{32}$$

$$y'_{n+\frac{8}{3}} = \left\{ \begin{array}{l} \frac{1}{1085025375} \left(-8989056\, f_{n+\frac{7}{3}} + 218365952\, f_{n+\frac{3}{2}} + 38911104\, f_{n+\frac{2}{3}} \right. \\ + 1568185920\, f_{n+\frac{1}{3}} + 1150400\, f_{n+3} + 183415680\, f_{n+2} + 1060617600\, f_{n+1} \\ \left. + 367558400\, f_n \right) h^3 + \frac{32}{9}\, y'''_n h^2 + \frac{8}{3}\, y'' h + y'_n \end{array} \right\} \tag{33}$$

$$y'_{n+3}=\left\{\begin{array}{l}\frac{1}{3136000}\Big(275562\,f_{n+\frac{7}{3}}+1437696\,f_{n+\frac{3}{2}}+472392\,f_{n+\frac{2}{3}}\\ +5806485\,f_{n+\frac{1}{3}}+13275\,f_{n+3}+748440\,f_{n+2}+3997350\,f_{n+1}\\ +1360800\,f_n)h^3+\frac{9}{2}\,y'''_n h^2+3\,y''_n h+\,y'_n\end{array}\right\} \quad (34)$$

$$y''_{n+\frac{1}{3}}=\left\{\begin{array}{l}\frac{1}{92588832000}\,(2417453800\,f_n+2824797150\,f_{n+1}+529329780\,f_{n+2}\\ +6861625\,f_{n+3}+4932356895\,f_{n+\frac{1}{3}}-4222085148\,f_{n+\frac{2}{3}}-1171410944\,f_{n+\frac{3}{2}}\\ -173479158\,f_{n+\frac{7}{3}})h^2+\frac{1}{3}\,y'''_n h+\,y''_n\end{array}\right\} \quad (35)$$

$$y''_{n+\frac{2}{3}}=\left\{\begin{array}{l}\frac{1}{1446700500}\,(86917100\,f_n+104646150\,f_{n+1}+20193390\,f_{n+2}+263075\,f_{n+3}\\ +297941085\,f_{n+\frac{1}{3}}-137450034\,f_{n+\frac{2}{3}}-44388352\,f_{n+\frac{3}{2}}-6633414\,f_{n+\frac{7}{3}})\Big)\,h^2\\ +\frac{2}{3}\,y'''_n h+\,y''_n\end{array}\right\} \quad (36)$$

$$y''_{n+1}=\left\{\begin{array}{l}\frac{1}{14112000}\,\Big(1321800\,f_n+1750350\,f_{n+1}+309540\,f_{n+2}+4025\,f_{n+3}+5211135\,f_{n+\frac{1}{3}}\\ -755244\,f_{n+\frac{2}{3}}-684032\,f_{n+\frac{3}{2}}-101574\,f_{n+\frac{7}{3}}\Big)\,h^2+\,y'''_n h+\,y''_n\end{array}\right\} \quad (37)$$

$$y''_{n+\frac{3}{2}}=\left\{\begin{array}{l}\frac{1}{50176000}\,\Big(7206450\,f_n+18725175\,f_{n+1}+1397655\,f_{n+2}+19650\,f_{n+3}+30967920\,f_{n+\frac{1}{3}}\\ +570807\,f_{n+\frac{2}{3}}-1960704\,f_{n+\frac{3}{2}}-478953\,f_{n+\frac{7}{3}}\Big)\,h^2+\frac{3}{2}\,y'''_n h+\,y''_n\end{array}\right\} \quad (38)$$

$$y''_{n+2}=\left\{\begin{array}{l}\frac{1}{220500}\,\Big(42700\,f_n+143850\,f_{n+1}+14070\,f_{n+2}+125\,f_{n+3}+190755\,f_{n+\frac{1}{3}}\\ +16038\,f_{n+\frac{2}{3}}+36864\,f_{n+\frac{3}{2}}-3402\,f_{n+\frac{7}{3}}\Big)\,h^2+2\,y'''_n h+\,y''_n\end{array}\right\} \quad (39)$$

$$y''_{n+\frac{7}{3}}=\left\{\begin{array}{l}\frac{1}{1889568000}\,(429406600\,f_n+1563411150\,f_{n+1}+368649540\,f_{n+2}+1294825\,f_{n+3}\\ +1943281935\,f_{n+\frac{1}{3}}+227042676\,f_{n+\frac{2}{3}}+633622528\,f_{n+\frac{3}{2}}-22885254\,f_{n+\frac{7}{3}}\Big)\,h^2\\ +7/3\,y'''_n h+\,y''_n\end{array}\right\} \quad (40)$$

$$y''_{n+\frac{8}{3}}=\left\{\begin{array}{l}\frac{1}{361675125}\,(94896800\,f_n+348465600\,f_{n+1}+107634240\,f_{n+2}+706400\,f_{n+3}\\ +427174560\,f_{n+\frac{1}{3}}+71430336\,f_{n+\frac{2}{3}}+195166208\,f_{n+\frac{3}{2}}+40481856\,f_{n+\frac{7}{3}}\Big)\,h^2\\ +\frac{8}{3}\,y'''_n h+\,y''_n\end{array}\right\} \quad (41)$$

$$y''_{n+3}=\left\{\begin{array}{l}\frac{1}{1568000}\,\Big(489000\,f_n+1162350\,f_{n+1}+79380\,f_{n+2}+39825\,f_{n+3}+1935495\,f_{n+\frac{1}{3}}\\ +866052\,f_{n+\frac{2}{3}}+1683456\,f_{n+\frac{3}{2}}+800442\,f_{n+7/3}\Big)\,h^2+3\,y'''_n h+\,y''_n\end{array}\right\} \quad (42)$$

$$y'''_{n+\frac{1}{3}}=\left\{\begin{array}{l}\frac{1}{51438240}\,(5382274\,f_n+8302329\,f_{n+1}+1528569\,f_{n+2}+19714\,f_{n+3}\\ +18506880\,f_{n+\frac{1}{3}}-12692619\,f_{n+\frac{2}{3}}-3401216\,f_{n+\frac{3}{2}}-499851\,f_{n+\frac{7}{3}}\Big)\,h+\,y'''_n\end{array}\right\} \quad (43)$$

$$y'''_{n+\frac{2}{3}} = \left\{ \begin{array}{l} \frac{1}{64297800}\ (6422960\, f_n + 5844510\, f_{n+1} + 1306620\, f_{n+2} + 17435\, f_{n+3} \\ +32089365\, f_{n+\frac{1}{3}} + 399492\, f_{n+\frac{2}{3}} - 2781184\, f_{n+\frac{3}{2}} - 433998\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{44}$$

$$y'''_{n+1} = \left\{ \begin{array}{l} \frac{1}{1058400}\ \Big(107470\, f_n + 293895\, f_{n+1} + 29295\, f_{n+2} + 370\, f_{n+3} + 510300\, f_{n+\frac{1}{3}} \\ +194643\, f_{n+\frac{2}{3}} - 68096\, f_{n+\frac{3}{2}} - 9477\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{45}$$

$$y'''_{n+\frac{3}{2}} = \left\{ \begin{array}{l} \frac{1}{40140800}\ (3945560\, f_n + 25135110\, f_{n+1} - 306180\, f_{n+2} + 2285\, f_{n+3} \\ +20437515\, f_{n+\frac{1}{3}} + 3228012\, f_{n+\frac{2}{3}} + 7755776\, f_{n+\frac{3}{2}} + 13122\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{46}$$

$$y'''_{n+2} = \left\{ \begin{array}{l} \frac{1}{52920}\ (5392\, f_n + 26082\, f_{n+1} + 12852\, f_{n+2} + 37\, f_{n+3} \\ +25515\, f_{n+\frac{1}{3}} + 8748\, f_{n+\frac{2}{3}} + 28672\, f_{n+\frac{3}{2}} - 1458\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{47}$$

$$y'''_{n+\frac{7}{3}} = \left\{ \begin{array}{l} \frac{1}{5248800}\ \Big(522410\, f_n + 2948085\, f_{n+1} + 2577645\, f_{n+2} - 490\, f_{n+3} + 2619540\, f_{n+\frac{1}{3}} \\ +607257\, f_{n+\frac{2}{3}} + 2433536\, f_{n+\frac{3}{2}} + 539217\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{48}$$

$$y'''_{n+\frac{8}{3}} = \left\{ \begin{array}{l} \frac{1}{8037225}\ (960280\, f_n + 375480\, f_{n+1} - 682920\, f_{n+2} + 122380\, f_{n+3} \\ +2891700\, f_{n+\frac{1}{3}} + 4088232\, f_{n+\frac{2}{3}} + 7684096\, f_{n+\frac{3}{2}} + 5993352\, f_{n+\frac{7}{3}}\Big)\, h + y'''_n \end{array} \right\} \tag{49}$$

$$y'''_{n+3} = \left\{ \begin{array}{l} \frac{1}{39200}\ \Big(6730\, f_n - 48195\, f_{n+1} - 48195\, f_{n+2} + 6730\, f_{n+3} + 59049\, f_{n+\frac{2}{3}} \\ +82432\, f_{n+\frac{3}{2}} + 59049\, f_{n+\frac{7}{3}})\Big)\, h + y'''_n \end{array} \right\} \tag{50}$$

3 Properties of the Method

In this section, the analysis of the derived schemes such as order and error constant, consistency, zero-stability, convergence and the stability domain was carried out.

Suppose the linear operator defined on the method (7) is given as

$$L[y(x);h] = \sum_{j=0}^{k} \left\{ \alpha_j y(x_n + jh) - h^4 \beta_j y^{iv}(x_n + jh) \right\}. \tag{51}$$

where $y(x)$ is any continuously differentiable test function on the interval $[a, b]$.

Expanding $y(x_n + jh)$, $y'(x_n + jh)$, $y''(x_n + jh)$, $y'''(x_n + jh)$, $j = 0, 1, \ldots, k$ in (52) in Taylor series about x_n and collecting like terms in h and derivatives of y gives

$$L[y(x);h] = I_0 y(x) + I_1 h y'(x) + \cdots + I_p h^p y^{(p)}(x) + I_{p+1} h^{p+1} y^{(p+1)}(x) + I_{p+2} h^{p+2} y^{(p+2)}(x) + \cdots .$$

Definition 3.1. *[20] The term I_{p+4} is called the error constant, meaning that the local truncation error is given by*

$$T_{n+k} = I_{p+4}h^{p+4}y^{(p+4)}(x) + O(h^{p+5}).$$

Definition 3.2. *The difference operator L associated with the hybrid block method with step number three* (52) *is said to be of order p if*

$$I_0 = I_1 = I_2 = \cdots = I_{p+3} = 0, \qquad I_{p+4} \neq 0$$

(see [22]).

Definition 3.3. *[19] A linear multistep method (LMM) is a computational method for determining the sequence $\{y_n\}$, which takes the form of a linear relationship between y_{n+j} and f_{n+j}, $j = 0(1)k$. The general form of a linear k-step method for an mth-order general ODE may be written as*

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{52}$$

where α_j and β_j are the coefficients of the method, and

$$f_{n+j} = f\big(x_{n+j}, y_{n+j}, y_{n+j}^{(i)}, y_{n+j}^{(ii)}, \ldots, y_{n+j}^{(m-1)}\big), \qquad j = 0(1)k.$$

Here, h is the step length, m is the order of the ODE to be solved, $\alpha_k \neq 0$, and α_0 and β_0 are not both zero (see [11], [22].

Definition 3.4. *A multistep method is said to be P-stable if its interval of periodicity is $(0, \infty)$ (see [19]).*

3.1 Order and Error constant of the Method

Apply the linear operator L in (51) to determine the order and error constant of the derived method. Expanding the method (17) by Taylor series and combining coefficients of like terms in h^n gives

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = C_{10} = 0, \quad C_{11} = 0, \quad C_{12} \neq 0.$$

Since $C_0 = C_1 = \cdots = C_{p+3} = 0$ and $C_{p+4} \neq 0$ (see Yahaya and Badmus [22]), the method has order

$$p = 8$$

with error constant

$$C_{12} = -\frac{15037031}{151697720808000} = -9.91249632486 \times 10^{-8}.$$

Similarly, the additional methods and their derivatives were also analyzed and summarized below:

Table 1: The order and error constants

Equation number	Order	Error Constant
(19)	8	$-\frac{21913}{12570917990400}$
(20)	8	$-\frac{67}{2582803260}$
(21)	8	$-\frac{37819}{362125209600}$
(22)	8	$-\frac{1263}{3229614080}$
(23)	8	$-\frac{169}{176818950}$
(24)	8	$-\frac{991613}{646504353792}$
(25)	8	$-\frac{575296}{248594813775}$
(26)	8	$-\frac{27}{7884800}$
(27)	8	$-\frac{3471989}{158393566679040}$
(28)	8	$-\frac{215603}{1546812174600}$
(29)	8	$-\frac{13963}{40236134400}$
(30)	8	$-\frac{46807}{56518246400}$
(31)	8	$-\frac{23}{15717240}$
(32)	8	$-\frac{32675209}{16162608844800}$
(33)	8	$-\frac{521704}{193351521825}$
(34)	8	$-\frac{47}{11038720}$
(35)	8	$-\frac{339551}{1714216089600}$
(36)	8	$-\frac{10193}{20832487200}$
(37)	8	$-\frac{1403}{1828915200}$
(38)	8	$-\frac{611}{550502400}$
(39)	8	$-\frac{43}{28576800}$
(40)	8	$-\frac{442127}{244888012800}$

Table 2: The order and error constants

Equation number	Order	Error Constant
(41)	8	$-\frac{716}{279006525}$
(42)	8	$-\frac{59}{7526400}$
(43)	8	$-\frac{4059553}{3999837542400}$
(44)	8	$-\frac{94273}{124994923200}$
(45)	8	$-\frac{1667}{1828915200}$
(46)	8	$-\frac{851}{1926758400}$
(47)	8	$-\frac{67}{57153600}$
(48)	8	$-\frac{47089}{81629337600}$
(49)	8	$-\frac{25421}{3906091350}$
(50)	8	$-\frac{59}{2508800}$

3.2 Zero-stability of the Method

Suppose the first characteristic polynomial of equation (52) is

$$\rho(r) = \left\{ \begin{array}{l} r^3 - \dfrac{5510839779}{505774675} r^{\frac{8}{3}} + \dfrac{12268164582}{361267625} r^{\frac{7}{3}} - \dfrac{3179719998}{72253525} r^2 \\ + \dfrac{100876746752}{2528873375} r^{\frac{3}{2}} - \dfrac{3179719998}{72253525} r + \dfrac{12268164582}{361267625} r^{\frac{2}{3}} \\ - \dfrac{5510839779}{505774675} r^{\frac{1}{3}} + 1 \end{array} \right\}.$$

Solving $\rho(r) = 0$ gives the roots $r = 0, 1, 1$, which satisfy

$$|R_j| \leq 1, \qquad j = 1, \ldots, k.$$

The roots lie inside the unit circle, and the repeated root at $r = 1$ has multiplicity not exceeding the order of the method. Therefore, the method is zero-stable.

3.3 Consistency of the Method

According to [12], a sufficient condition for a method to be consistent is that its order p is greater than or equal to one. Consequently, since our method is of order $p = 8$, it is consistent.

3.4 Convergence of the Method

A numerical method is said to be convergent if it is consistent and zero-stable. Thus, the method is convergent since it satisfies the conditions established in **Section 3.2** and **Section 3.3**.

3.5 Stability Domain of the Method

The region of absolute stability of the method is examined via the procedure discussed in Lambert [17] and Obarhua [19]. The stability matrix is expressed as

$$M(z) = zB(I - zA)^{-1}U + V, \tag{53}$$

together with the stability function

$$p(\eta, z) = \det\big(-M(z) + \eta I\big), \tag{54}$$

where η denotes the eigenvalues.

For the stability analysis, the method (19)-(50) is formulated as a general linear method of the form

$$\begin{bmatrix} Y \\ --- \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ --- & --- & --- \\ B & & V \end{bmatrix} \begin{bmatrix} h^4 f(u) \\ --- \\ Y_{i-1} \end{bmatrix} \tag{55}$$

where η represents the roots of the first characteristic polynomial.

$$Y_{i-1} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_n \end{bmatrix}, Y_{i+1} = \begin{bmatrix} y_{n+\frac{1}{3}} \\ y_{n+3} \end{bmatrix},$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{89454587}{274988831040} & \frac{3471989}{9053130240} & -\frac{7141451}{18860688000} & \frac{905315}{3491921664} & -\frac{11699944}{107417512125} & \frac{432227}{8729804160} & -\frac{1840771}{113164128000} & 0 & \frac{1419889}{2199910648320} \\ \frac{15378724}{4296700485} & \frac{271567}{35363790} & -\frac{862412}{147349125} & \frac{270287}{68201595} & -\frac{177188864}{107417512125} & \frac{50906}{68201595} & -\frac{108191}{442047375} & 0 & \frac{83243}{8593400970} \\ \frac{188701}{13970880} & \frac{37353}{985600} & -\frac{180513}{8624000} & \frac{13963}{887040} & -\frac{5144}{779625} & \frac{6637}{2217600} & -\frac{16941}{17248000} & 0 & \frac{4349}{111767040} \\ \frac{2166957}{44154880} & \frac{1830519}{11038720} & -\frac{116739873}{2207744000} & \frac{828873}{12615680} & -\frac{421263}{17248000} & \frac{140697}{12615680} & -\frac{8089713}{2207744000} & 0 & \frac{6417}{44154880} \\ \frac{2924}{24255} & \frac{69}{154} & -\frac{5526}{67375} & \frac{145}{693} & -\frac{4096}{86625} & \frac{92}{3465} & -\frac{591}{67375} & 0 & \frac{17}{48510} \\ \frac{1090531799}{5612016960} & \frac{139044311}{184757760} & -\frac{35580419}{384912000} & \frac{976369051}{2494229760} & -\frac{94253656}{2192194125} & \frac{11411953}{249422976} & -\frac{32675209}{2309472000} & 0 & \frac{25227307}{44896135680} \\ \frac{1260416512}{4296700485} & \frac{103556608}{88409475} & -\frac{13171712}{147349125} & \frac{45384704}{68201595} & -\frac{140509184}{107417512125} & \frac{29504512}{341007975} & -\frac{8754176}{442047375} & 0 & \frac{3667456}{4296700485} \\ \frac{72729}{172480} & \frac{11868849}{6899200} & -\frac{531441}{8624000} & \frac{102789}{98560} & \frac{6984}{67375} & \frac{38637}{246400} & -\frac{177147}{17248000} & 0 & \frac{423}{275968} \end{bmatrix}$$

$$B=\begin{bmatrix} \frac{1260416512}{4296700485} & \frac{103556608}{88409475} & -\frac{13171712}{147349125} & \frac{45384704}{68201595} & -\frac{140509184}{107417512125} & \frac{29504512}{341007975} & -\frac{8754176}{442047375} & 0 & \frac{3667456}{4296700485} \\ \frac{72729}{172480} & \frac{11868849}{6899200} & -\frac{531441}{8624000} & \frac{102789}{98560} & \frac{6984}{67375} & \frac{38637}{246400} & -\frac{177147}{17248000} & 0 & \frac{423}{275968} \end{bmatrix}$$

$$V=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad U=\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad Y=\begin{bmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{7}{3}} \\ y_{n+\frac{8}{3}} \\ y_{n+3} \end{bmatrix}, I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad f(y)=\begin{bmatrix} f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{7}{3}} \\ f_{n+\frac{8}{3}} \\ f_{n+3} \end{bmatrix}$$

$$-zA+M$$

$$K=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{89454587}{274988831040}z & -\frac{3471989}{9053130240}z+1 & \frac{7141451}{18860688000}z & -\frac{905315}{3491921664}z & \frac{11699944}{107417512125}z & -\frac{432227}{8729804160}z & \frac{1840771}{113164128000}z & 0 & -\frac{1419889}{2199910648320}z \\ -\frac{15378724}{4296700485}z & -\frac{271567}{35363790}z & \frac{862412}{147349125}z+1 & -\frac{270287}{68201595}z & \frac{177188864}{107417512125}z & -\frac{50906}{68201595}z & \frac{108191}{442047375}z & 0 & -\frac{83243}{8593400970}z \\ -\frac{188701}{13970880}z & -\frac{37353}{985600}z & \frac{180513}{8624000}z & -\frac{13963}{887040}z+1 & \frac{5144}{779625}z & -\frac{6637}{2217600}z & \frac{16941}{17248000}z & 0 & -\frac{4349}{111767040}z \\ -\frac{2166957}{44154880}z & -\frac{1830519}{11038720}z & \frac{116739873}{2207744000}z & -\frac{828873}{12615680}z & \frac{421263}{17248000}z+1 & -\frac{140697}{12615680}z & \frac{8089713}{2207744000}z & 0 & -\frac{6417}{44154880}z \\ -\frac{2924}{24255}z & -\frac{69}{154}z & \frac{5526}{67375}z & -\frac{145}{693}z & \frac{4096}{86625}z & -\frac{92}{3465}z+1 & \frac{591}{67375}z & 0 & -\frac{17}{48510}z \\ -\frac{1090531799}{5612016960}z & -\frac{139044311}{184757760}z & \frac{35580419}{384912000}z & -\frac{976369051}{2494229760}z & \frac{94253656}{2192194125}z & -\frac{11411953}{249422976}z & \frac{32675209}{2309472000}z+1 & 0 & -\frac{25227307}{44896135680}z \\ -\frac{1260416512}{4296700485}z & -\frac{103556608}{88409475}z & \frac{13171712}{147349125}z & -\frac{45384704}{68201595}z & \frac{140509184}{107417512125}z & -\frac{29504512}{341007975}z & \frac{8754176}{442047375}z & 1 & -\frac{3667456}{4296700485}z \\ -\frac{72729}{172480}z & -\frac{11868849}{6899200}z & \frac{531441}{8624000}z & -\frac{102789}{98560}z & -\frac{6984}{67375}z & -\frac{38637}{246400}z & \frac{177147}{17248000}z & 0 & -\frac{423}{275968}z+1 \end{bmatrix}$$

Now, substituting the values of the matrices A, B, U, V, M, and I into equations (53) and (54), we obtain the stability function.

$$T=\frac{G}{2H} \tag{56}$$

where

$$G = \left\{ \begin{array}{l} \eta\ \big(4667544\,\eta\,z^7 + 37707737040\,\eta\,z^6 + 20762735994\,z^7 + 106139067996290\,\eta\,z^5 \\ +219722788039545\,z^6 + 98334058642570800\,\eta\,z^4 - 18349682103018590\,z^5 \\ +17720873567407584000\,\eta\,z^3 + 1607578015765513932000\,z^4 \\ -114117422614272000\,\eta\,z^2 + 8313528182439535977600\,z^3 - 15674099380002201600000\,\eta\,z \\ +118376581923083311027200000\,z^2 - 72422100420377444352000000\,\eta \\ +244440263018153876889600000\,z + 72422100420377444352000000\big) \end{array} \right\}$$

and

$$H = \left\{ \begin{array}{l} 2333772\,z^7 + 18853868520\,z^6 + 53069533998145\,z^5 + 49167029321285400\,z^4 \\ +\,8860436783703792000\,z^3 - 570587113071360000\,z^2 - 78370496900011008000000\,z \\ -\,36211050210188722176000000 \end{array} \right\}$$

$$Z = \frac{135W}{2L} \tag{57}$$

where

$$W = \left\{ \begin{array}{l} \eta\ \big(898703673646500036\,z^{12} - 16958410341170634393372\,z^{11} \\ -103285618101515061792141055\,z^{10} \\ -1148460228326179013070352212000\,z^9 + 62470204323935944900563388224000\,z^8 \\ +744201254381201957278328639558400000\,z^7 \\ +3396054284097042889200674368385280000\,z^6 \\ +935769682612547623441756735830835200000\,z^5 \\ +1047896104523024780871098396933652480000000\,z^4 \\ +21432477097363235703838258313287434240000000\,z^3 \\ +67598559322029920094693049680687267840000000000\,z^2 \\ +635042953851276504514669352823501619200000000000000\,z \\ +6556200786624043502408942649898570874880000000000000\big) \end{array} \right\}$$

and

$$L = \left\{ \begin{array}{l} (2333772\,z^7 + 18853868520\,z^6 + 53069533998145\,z^5 + 49167029321285400\,z^4 \\ +\,8860436783703792000\,z^3 - 570587113071360000\,z^2 - 78370496900011008000000\,z \\ -\,36211050210188722176000000)^2 \end{array} \right\}$$

The stability polynomial and its first derivative are then plotted in the MATLAB (R2012a) environment. It should be noted that M is a 9×9 identity matrix. The region of absolute stability (RAS) of the method is displayed in Figure 1 below.

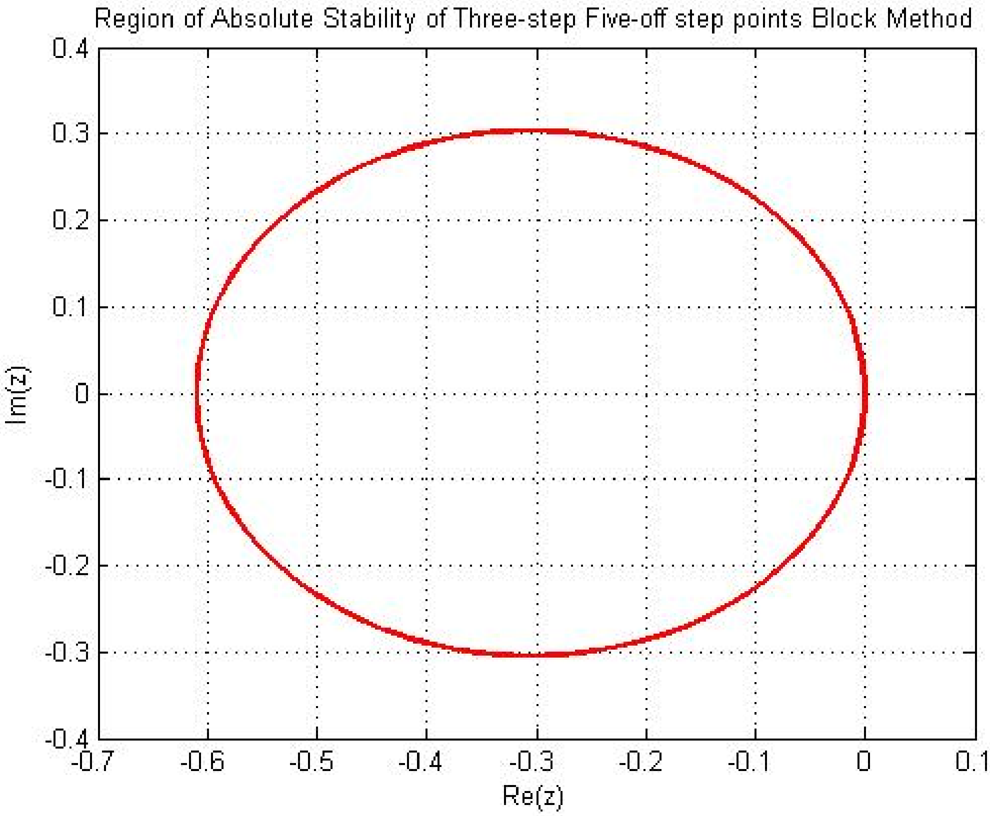

Figure 1: Region of absolute stability of the method.

The inside of the curve represents the unstable region, while the outer segment of the curve represents the stable region. Therefore, the newly generated method is absolutely stable.

4 Numerical Examples and Results

Test problems

Problem 1

Homogeneous linear fourth order problem

$$y^{iv} = (x^4 + 14x^3 + 49x^2 + 32x + 12)\exp(x)$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 2, \quad y'''(0) = -6, \quad h = 0.1$$

Exact solution:

$$y(x) = x^2(x-1)^2\exp(x)$$

Source: Ahamad and Charan [3], and Jena *et al.* [10].

Table 3: Numerical Results for Problem 1 using $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.100000	0.00895188443641274820	0.00895188443641274820	0.000000e+000	0.0073
0.200000	0.03126791060890032900	0.03126791060890036400	3.469447e-017	0.0139
0.300000	0.05952877341410175300	0.05952877341410176700	1.387779e-017	0.0203
0.400000	0.08592910258413723200	0.08592910258413720400	2.775558e-017	0.0273
0.500000	0.10304507941875807000	0.10304507941875804000	2.775558e-017	0.0338
0.600000	0.10495404290249327000	0.10495404290249336000	8.326673e-017	0.0409
0.700000	0.08880649439944778000	0.08880649439944805800	2.775558e-016	0.0474
0.800000	0.05697384776940670600	0.05697384776940727500	5.689893e-016	0.0544
0.900000	0.01992278520037080200	0.01992278520037158200	7.806256e-016	0.0611
1.000000	0.00000000000000000000	0.00000000000000073254	7.325412e-016	0.0677

Problem 1 is solved using the newly developed method of order $p = 8$, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 3.

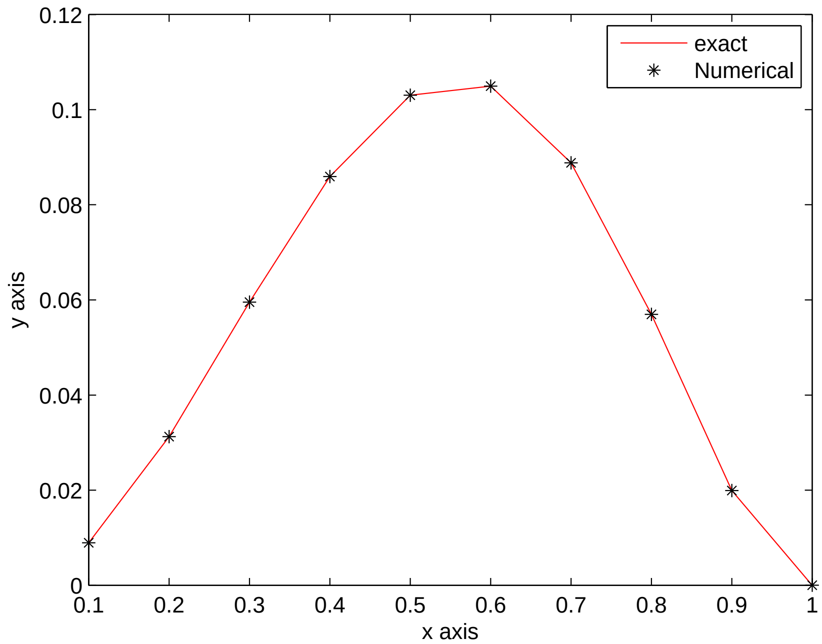

Figure 2: Graph of exact value and numerical value for problem 1.

It was also compared with existing methods, namely: Ahamad and Charan [3], who developed an improved Runge–Kutta method in which the problem was first converted into a system of first-order ODEs before being solved. Likewise, Jena *et al.* [10] proposed a block algorithm containing nine intermediate steps with an order of accuracy $p = 11$. The comparison is presented in Table 4.

Table 4: Comparison of the newly proposed method with Ahamad and Charan [3] and Jena *et al.* [10] for Problem 1

x-value	Error in New Results, $p = 8, h = 0.1$	Error in Ahamad and Charan [3], Runge-kutta, $h = 0.1$	Error in Jena *et al* [10], $p = 11, h = 0.1$
0.1	0.000000e+000	6.23665e-08	1.5370e-14
0.2	3.469447e-017	4.71836e-05	8.2021e-14
0.3	1.387779e-017	4.40097e-05	3.6666e-13
0.4	2.775558e-017	7.68036e-05	6.3424e-13
0.5	2.775558e-017	2.17730e-04	6.7024e-13
0.6	8.326673e-017	2.537301e-03	5.2608e-13
0.7	2.775558e-016	7.777177e-03	3.3906e-13
0.8	5.689893e-016	8.830157e-03	1.9011e-14
0.9	7.806256e-016	1.52112e-03	9.6152e-14
1.0	7.325412e-016	0.00000e-00	4.4983e-14

Problem 2

Linear fourth order problem:

$$y^{iv} = -y''$$

$$y(0) = 0, \quad y'(0) = \frac{1.1}{72 - 50\pi}, \quad y''(0) = \frac{1}{144 - 100\pi}, \quad y'''(0) = \frac{1.2}{144 - 100\pi}, \quad h = \frac{1}{320}$$

Exact solution:

$$y(x) = \frac{1 - x - \cos(x) - 1.2\sin(x)}{144 - 100\pi}$$

Source: Atabo and Adee [1].

The tables displayed below are the numerical results when the newly proposed block method with step-length 3 with four hybrid points were applied to fourth order differential equations above.

The comparison of the generated numerical results is made with the existing methods in terms of error.

Here problem 2 is solved using the newly developed method of order $p = 8$, The results of the problem 1 which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 5.

Table 5: Numerical Results for Problem 2 using $h = \frac{1}{320}$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.003125	0.000040374593022299736	0.00004037459302299726	9.486769e-020	0.0444
0.006250	0.00008069158007106986	0.00008069158007106989	2.710505e-020	0.0522
0.009375	0.00012095074677095987	0.00012095074677096074	8.673617e-019	0.0593
0.012500	0.00016115187931405831	0.00016115187931407801	1.970537e-017	0.0601
0.015625	0.00020129476445849724	0.00020129476445858864	9.139824e-017	0.0603
0.018750	0.00024137918953123110	0.00024137918953148554	2.544351e-016	0.0606
0.021875	0.00028140494243011042	0.00028140494243067443	5.640020e-016	0.0609
0.025000	0.00032137181162595881	0.00032137181162705060	1.091792e-015	0.0610
0.028125	0.00036127958616463267	0.00036127958616654927	1.916598e-015	0.0612
0.031250	0.00040112805566908761	0.00040112805567221581	3.128194e-015	0.0615

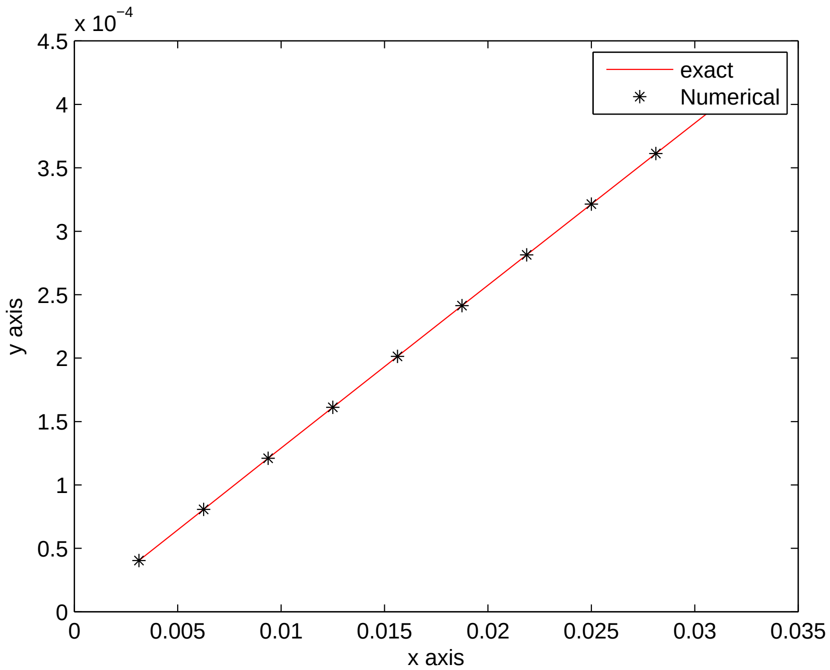

Figure 3: Graph of Exact value and Numerical value for problem 2

It was also compared with exiting methods namely; Atabo and Adee [1] who proposed hybrid block method is of order $p = 16$. The comparison is shown in Table 6.

Table 6: Comparison of the newly developed method with Atabo and Adee [1] for Problem 2.

x-value	Error in New Results, $k=3, h=\frac{1}{320}, p=8$	Error in Atabo and Adee [1], $k=15$, $h=\frac{1}{320}, p=16$
0.003125	9.486769e-020	5.4210 e-20
0.006250	2.710505e-020	5.4210 e-20
0.009375	8.673617e-019	2.7105 e-19
0.001250	1.970537e-017	1.0842 e-19
0.015625	9.139824e-017	3.2526 e-19
0.018750	2.544351e-016	3.2526 e-19
0.021875	5.640020e-016	3.2526 e-19
0.025000	1.091792e-015	4.3368 e-19
0.028125	1.916598e-015	2.1684 e-19
0.031250	3.128194e-015	6.5052 e-19

Problem 3

Consider the fourth order below

$$y^{iv} = y^2 - yy'' - 4x^2 + \exp(x)(1 - 4x + x^2)$$

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3, \quad y'''(0) = 1, \quad h = \frac{1}{320}$$

Exact solution:

$$y(x) = x^2 + \exp(x)$$

Source: Kuboye *et al.* [15].

Table 7: Numerical Results for Problem 3 using $h = \frac{1}{320}$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.003125	1.00313965352773900000	1.00104329445926270000	1.776357e-015	0.0568
0.006250	1.00630863450376200000	1.00630863450377390000	1.199041e-014	0.0609
0.009375	1.00950697358907090000	1.00950697358904470000	2.620126e-014	0.0700
0.012500	1.01273470154063450000	1.01057961595501580000	2.464695e-014	0.0707
0.015625	1.01599184921168570000	1.01599184921167200000	1.376677e-014	0.0708
0.018750	1.01927844755202620000	1.01927844755197250000	5.373479e-014	0.0711
0.021875	1.02259452760832640000	1.02038053028855620000	5.195844e-014	0.0713
0.025000	1.02594012052442900000	1.02594012052438810000	4.085621e-014	0.0715
0.028125	1.02931525754165380000	1.02931525754157200000	8.171241e-014	0.0717
0.031250	1.03271996999910280000	1.03044687398359770000	7.949197e-014	0.0720

Problem 3 is a nonlinear fourth-order problem which is solved using the newly proposed method of order $p = 8$, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 7.

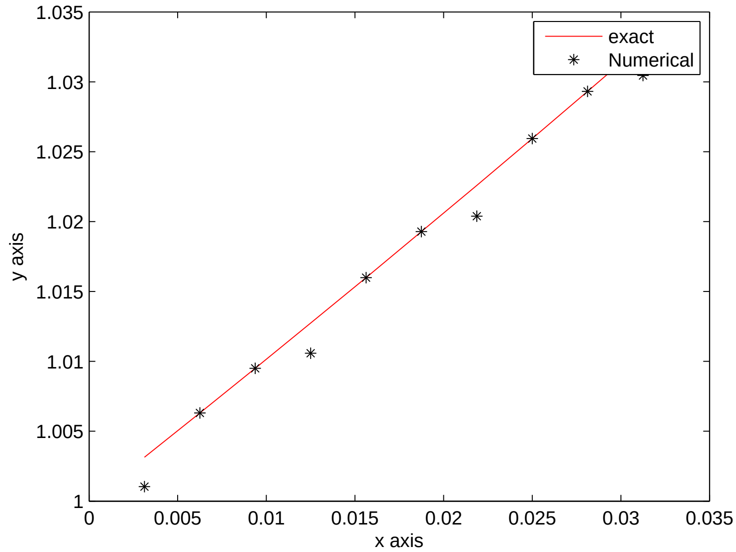

Figure 4: Graph of exact value and Numerical value for problem 3.

It was also compared with exiting methods namely; Kayode *et al.* [14] proposed a four step Predicted-corrector method with order of accuracy of $p = 10$.

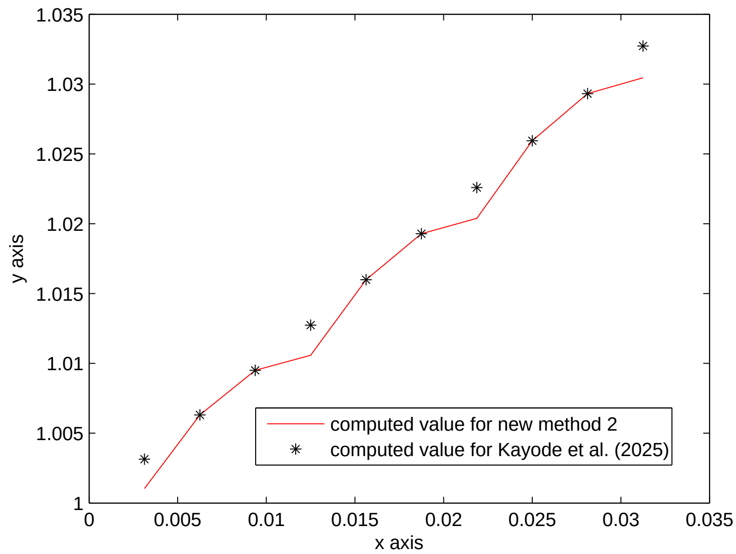

Figure 5: Comparison of computed value in the new method and Kayode *et al.* [14].

Table 8: : Comparison of error in the newly developed method and Kayode *et al.* [14] for Problem 3.

x-value	Error in New Results, $p = 8$ $h = \frac{1}{320}, k = 3$	Error in Kayode *et al.* [14], $p = 10$ $h = \frac{1}{320}, k = 4$
0.003125	1.776357e-015	2.220446e-15
0.006250	1.199041e-014	5.628831e-13
0.009375	2.620126e-014	5.704326e-13
0.001250	2.464695e-014	1.895373e-13
0.015625	1.376677e-014	4.039213e-12
0.018750	5.373479e-014	8.015588e-12
0.021875	5.195844e-014	1.202927e-11
0.025000	4.085621e-014	1.891598e-11
0.028125	8.171241e-014	2.800626e-11
0.031250	7.949197e-014	4.028355e-11

Problem 4

Special fourth order problem

$$y^{iv} = x$$

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0, \quad h = 0.1$$

Exact solution:

$$y(x) = \frac{x^5}{120} + x$$

Source: Kayode *et al.* [14].

Table 9: Numerical Results for Problem 4 using $h = 0.1$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.100000	0.10000008333333334000	0.10000008333333334000	0.000000e+000	0.0573
0.200000	0.20000266666666661000	0.19791919730797874000	2.775558e-017	0.0650
0.300000	0.30002025000000004000	0.30002024999999999000	5.551115e-017	0.0715
0.400000	0.40008533333333357000	0.40008533333333335000	2.220446e-016	0.0801
0.500000	0.50026041666666721000	0.49817170300934993000	5.551115e-016	0.0873
0.600000	0.60064800000000074000	0.60064800000000007000	6.661338e-016	0.0954
0.700000	0.70140058333333433000	0.70140058333333333000	9.992007e-016	0.1030
0.800000	0.80273066666666792000	0.80061196248133726000	1.443290e-015	0.1112
0.900000	0.90492075000000149000	0.90492074999999994000	1.554312e-015	0.1174
1.000000	1.00833333333333510000	1.00833333333333330000	1.776357e-015	0.1239

Problem 4 is a special fourth-order problem that has previously been addressed by scholars such as and Kayode *et al.* [14]. In this study, the problem is solved using the newly proposed method of order $p = 8$. The numerical results, which include the exact solution y, the computed solution, the absolute error, and the execution time, are presented in Table 9.

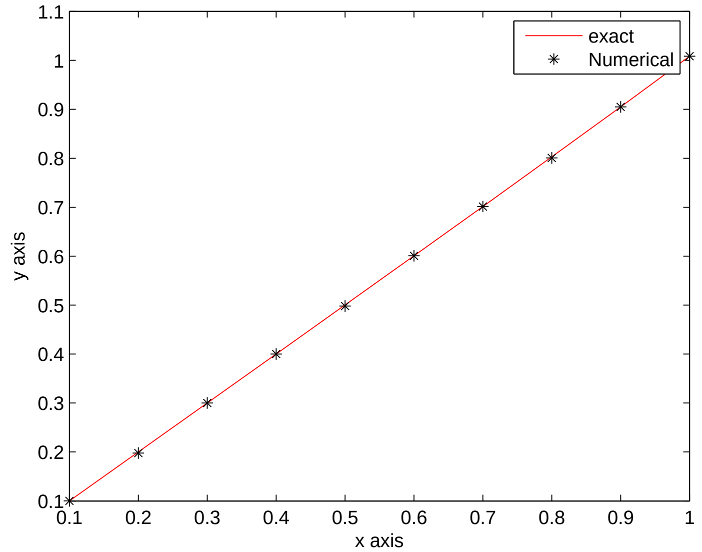

Figure 6: Graph of Exact value and Numerical value for problem 4

Furthermore, a comparison is made with existing methods, notably the Four-step predictor-corrector method with five hybrid points of order $p = 10$ proposed by Kayode *et al.* [14].

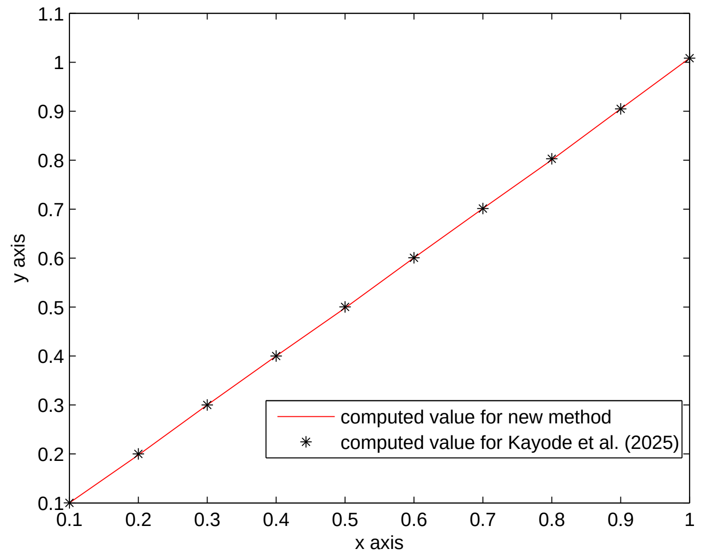

Figure 7: Comparison of computed value in the new method and Kayode *et al.* [14] for problem 4.

Table 10: Comparison of error in the newly developed method and Kayode *et al.* [14] for Problem 4.

x-value	Error in New Results, $p = 8, k = 3, h = 0.1$	Error in Kayode *et al.* [14] $p = 10, k = 4, h = 0.1$
0.1	0.000000e+000	0.000000e+00
0.2	2.775558e-017	9.242607e-015
0.3	5.551115e-017	2.498002e-015
0.4	2.220446e-016	6.494805e-015
0.5	5.551115e-016	3.885781e-014
0.6	6.661338e-016	1.629807e-013
0.7	9.992007e-016	3.756995e-013
0.8	1.443290e-015	7.696066e-013
0.9	1.554312e-015	1.487144e-012
1.0	1.776357e-015	2.586598e-012

Problem 5: Application Problems

To confirm the application of the newly developed method, we solve a physical problem from ship dynamics. As stated by Atabo and Adee [1], when a sinusoidal wave of frequency passes along a ship or offshore structure, the resultant fluid actions vary with time t. Therefore, consider the fourth-order problem as:

$$y^{iv} + 3y'' + y\,(2 + \varepsilon\cos(\Omega t)) = 0, \qquad t \geq 0,$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0, \quad h = \frac{1}{320},$$

where $\varepsilon = 0$ for the existence of the theoretical solution:

$$y(t) = 2\cos t - \cos\left(t\sqrt{2}\right).$$

Source: Atabo and Adee *et al.* [1] and Kayode *et al.* [14].

Table 11: Numerical Results for Problem 5 using $h = \frac{1}{320}$

x-value	y-exact-solution	y-computed solution	Error in New Results	Time
0.003125	0.99999999999205269000	0.99999999999205269000	0.000000e+000	0.0005
0.006250	0.99999999987284383000	0.99999999987284394000	1.110223e-016	0.0006
0.009375	0.99999999935627548000	0.99999999935627548000	0.000000e+000	0.0008
0.012500	0.99999999796552663000	0.99999999901886427000	1.110223e-016	0.0011
0.015625	0.99999999503306747000	0.99999999503306747000	0.000000e+000	0.0012
0.018750	0.99999998970067949000	0.99999998970067949000	0.000000e+000	0.0014
0.021875	0.99999998091947950000	0.99999998721410266000	0.000000e+000	0.0017
0.025000	0.99999996744995123000	0.99999996744995112000	1.110223e-016	0.0018
0.028125	0.99999994786198110000	0.99999994786198121000	1.110223e-016	0.0020
0.031250	0.99999992053490105000	0.99999993969837697000	0.000000e+000	0.0023

Finally, Problem 5 which is an application problem earlier solved by Atabo and Adee [1] and Kayode *et al.* [14] is resolved using the newly developed method of order $p = 8$, the numerical results which contains the y-exact results, y-computed results, absolute error and the time of execution is presented in Table 11.

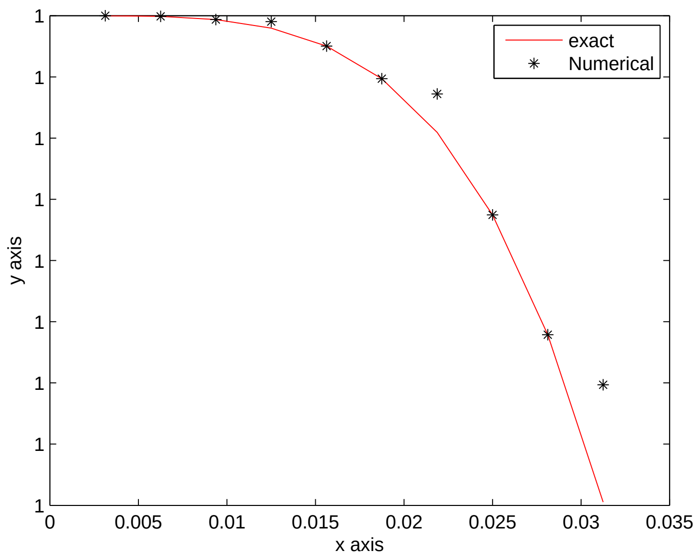

Figure 8: Graph of exact value and numerical value for problem 5.

It was also compared with exiting methods namely; Atabo and Adee [1] who proposed a special 15-step block method of order sixteen, while Kayode *et al.* [14] Proposed predictor-corrector method of order of accuracy of $p = 10$.

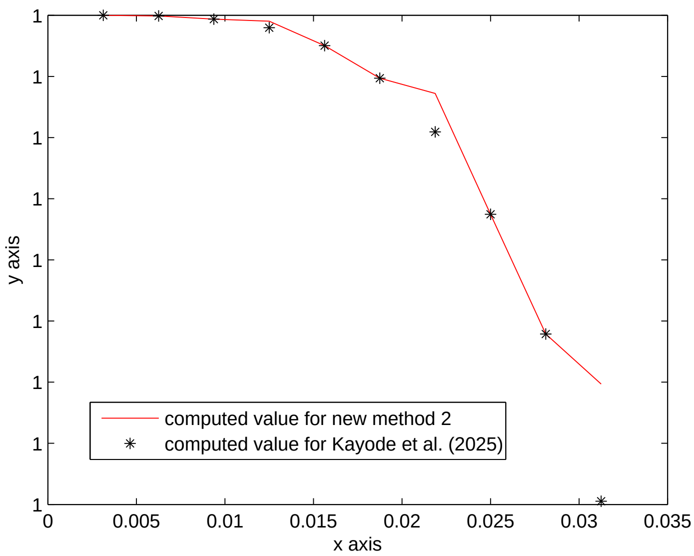

Figure 9: Comparison of computed value in the new method and Kayode *et al.* [14] for problem 5.

Table 12: Comparison of error in the newly developed method, Atabo and Adee [1] and Kayode *et al.* [14] for Problem 5.

x	Error in New Results, $p = 8$ $h = \frac{1}{320}$, $k = 3$	Error in Kayode *et al.* [14], $p = 10$ $h = \frac{1}{320}, k = 4$	Atabo and Adee [1] $p = 16$ $h = \frac{1}{320}, p = 15$
0.003125	0.000000e+000	0.000000e-00	0.000000e-00
0.006250	1.110223e-016	1.304512e-15	1.110223e-16
0.009375	0.000000e+000	5.884182e-15	0.000000e-00
0.012500	1.110223e-016	1.920686e-14	2.220446e-16
0.015625	0.000000e+000	4.196643e-14	7.771561e-16
0.018750	0.000000e+000	9.592327e-14	2.886580e-15
0.021875	0.000000e+000	1.596501e-13	8.548717e-15
0.025000	1.110223e-016	3.291811e-13	2.187139e-14
0.028125	1.110223e-016	5.845324 e-13	4.951595e-14
0.031250	0.000000e+000	1.105338 e-12	1.035838e-13

5 Discussion of Results

One non-linear ODEs was considered because of a small number of literature that appeared on it. While other four problems ranging from linear to special and non-homogeneous fourth-order ordinary differential equations problems are examined by the new developed block methods.

In Table 3, the results produced when the new method $k = 3$ was applied to Problem 1 are found better than the methods of: Ahamad and Charan [3] and Jena *et al.* [10]. Ahamad and Charan [3] employed Runge-kutta method to solve problem 1 after transforming to system of first order ordinary differential equation. On the other hand, Jena *et al.* [10] presented a block algorithm with nine concurrent step with order $p = 11$. The results displayed in Table 4 for solving Problem 1 implies that the new block method $k = 3$ is high in accuracy than Jena *et al.* [10] even with the same step-length $k = 9$. Problem 1 was also solved by the new block method $k = 3$ and the produced numerical results are better than the existing methods in the literature as shown in Table 4.

Problem 2 is a linear problem which was solving by Atabo and Adee [1] who proposed hybrid block method of order 16 The numerical results is shown in Table 5. It could be seen that when the new method were compared with Atabo and Adee [1] for solving Problem 2 in Table 6. In term of accuracy, the generated numerical results of the new block method $k = 3$ claim superiority over Atabo and Adee [1] for solving Problem 2.

The numerical results of nonlinear problem is presented in Table 7, while In Tables 8, it can be observed that the accuracy of the new block method $k = 3$ for solving Problems 3 is higher than Kayode *et al.* [14], who proposed a four step predictor corrector method with order of accuracy of $p = 10$. This actually shows that the method is computational reliable in handling nonlinear problem more efficient.

Problem 4 is a special fourth-order problem. The application of the new method to Problem 4 in Table 9. Table 10 is also better in terms of error than Kayode *et al.* [14]. It should be noted that the exiting method compared with are also of order $p = 10$. Furthermore, by comparing errors in Tables 10 for solving Problem 4, it can be seen that the accuracy of the new block method $k = 3$ is more advanced than Kayode *et al.* [14].

In Table 11, the numerical results derived from the new block method $k = 3$ when the method was applied to Problem 5 which is physical problem from ship dynamics show the efficiency of the method in terms of error over Kayode *et al.* [14] and Atabo and Adee [1]. The accuracy of the new method in Table 12 is better when comparison was made with existing methods.

Finally, this research has demonstrated that block hybrid method with multiple interpolation points is more efficient and perform better than other block algorithms for solving fourth-order ordinary differential equation. This claim has been demonstrated with several examples presented ranging from linear, nonlinear, special non-homogeneous problems and application problem. It should also be noted that most of the existing methods compared with are of higher order, despite the proposed method give a more accurate and efficient results. Hence it is recommended for general purpose.

5.1 Conclusion and Future Research

The proposed method which is design for the solution of fourth order initial value problems of ordinary differential equations with continuous hybrid block method of orders eight proposed for the direct solutions of fourth order initial value problems. The block methods are found to be zero-stable, consistency, hence they are convergent. five numerical examples were solved using the newly derived methods, comparing their accuracy with existing methods, they performed favorably well. The main method and the additional methods are obtained via interpolation and collocation procedures from the same continuous scheme derived. Numerical results obtained using the proposed block approach shows that it is adequate for the solution of special, linear, non-linear problems of fourth-order ordinary differential equations.

Acknowledgments

We thank the referees for the positive enlightening comments and suggestions, which have greatly helped us in making improvements to this paper.

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