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Refinements and Extensions of Classical Integral Inequalities: Hölder, Hardy, Minkowski, Clarkson, and Schweitzer

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Abstract

Integral inequalities are a fundamental part of modern mathematical analysis and the theory of function spaces. In this paper, we present several refinements and extensions to classical integral inequalities, with a particular focus on those of Hölder, Hardy, Minkowski, Clarkson, and Schweitzer. First, we apply Hölder's inequality to find new refined bounds. Then, we establish Hölder-type inequalities using extended Young's inequalities. Consequently, we derive Hardy-type derivative inequalities with an optimal weight factor. After that, we introduce the Minkowski-Clarkson relation and variation for two functions. Lastly, we formulate a weighted generalisation of Schweitzer's inequality incorporating parametric functions. Concrete examples involving the beta and gamma functions demonstrate the sharpness and applicability of the proposed bounds, showing measurable improvements upon their classical counterparts.

1 Introduction

Since the invention of calculus by Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, the concept of integration has undergone profound development [2,16]. In the 19th century, Bernhard Riemann formulated a rigorous definition of the integral, establishing a foundation for classical analysis [19]. Later, in 1901, Henri Lebesgue introduced the revolutionary notion of measure theory, extending integration to a far more general and flexible framework [13]. Following these milestones, numerous integral inequalities were developed, amongst which Jensen's, Minkowski's, Cauchy-Schwarz's, and Hölder's inequalities have become fundamental to modern analysis [11].

Building on this historical foundation, integral inequalities have emerged as fundamental tools in modern analysis, providing essential instruments for understanding the behaviour of functions, estimating solutions to differential and integral equations, and establishing quantitative bounds across diverse branches of mathematics. These inequalities forge a crucial link between the local properties of functions—such as convexity, monotonicity, and smoothness, and their global behaviour, typically

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characterised through integrals. Beyond their theoretical significance, integral inequalities also underpin applications in functional analysis, probability theory, and partial differential equations, where they often form the basis for stability and convergence results. See, for instance, [5, 14, 18].

Given their fundamental importance, many mathematicians have sought to investigate new integral inequalities and refine existing ones. For instance, in 2021, Eric Anders Carlen and his team established a remarkable result by showing that, for any $p \in (0,1] \cup [2,\infty)$, one has

$$\int_{\Omega} |f+g|^p d\mu \leq \left(1 + \frac{2^{2/p}\|fg\|_{p/2}}{(\|f\|_p^p + \|g\|_p^p)^{2/p}}\right)^{p-1} \int_{\Omega} (|f|^p + |g|^p) d\mu,$$

where

$$\|f\|_p = \left(\int_{\Omega} |f|^p d\mu\right)^{1/p}$$

denotes the usual L^p-norm. This inequality represents a notable refinement of the standard norm inequality [6]. More recently, in 2023, Jorge Paz Moyado and collaborators established reverse Hölder-type inequalities along with their applications [17]. In 2024, Thabet Abdeljawad and his research group extended Schweitzer's inequality to a Riemann-Liouville fractional calculus setting [1]. In a similar fashion, Noureddine Azzouz and Bouharket Benaissa introduced a reverse Minkowski-type inequality based on the k-weighted fractional integral operator in 2025 [3]. These recent advances stimulate further exploration into new inequalities that can be formulated from these pivotal findings.

Motivated by these developments, in this work, we establish several new refinements and extensions of classical integral inequalities. Our contributions are five-fold, as described below.

- Firstly, we systematically refine fundamental inequalities by exploring their interplay, deriving refined bounds for $\|f_k\|_{p_k}^{p_k}$ through strategic applications of Hölder's inequality (Section 2).
- Secondly, we construct new Hölder-type inequalities using the generalised Young's inequalities by Choi (Section 3).
- Thirdly, we establish Hardy-type derivative inequalities, one with explicit optimal constants, via Cauchy-Schwarz's inequality (Section 4).
- Fourthly, we unify Minkowski's and Clarkson's inequalities to obtain refined and variant two-function bounds (Section 5).
- Finally, we extend Schweitzer's classical inequality to the L^p space with weight functions, establishing new bounds for pairs of functions (Section 6).

Throughout the paper, concrete examples involving the beta and gamma functions illustrate both the sharpness and applicability of these results.

2 Refined Bounds by Hölder's Inequality

Notation. Throughout the paper, n stands for a positive integer.

First of all, we recall the statement of the celebrated Hölder integral inequality, established earlier by Otto Ludwig Hölder in 1889.

Lemma 1. *(Hölder's inequality, [24]) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p_1, \dots, p_n \geq 1$ satisfy $\sum_{j=1}^{n} 1/p_j = 1$. Suppose $f_1, \dots, f_n : \Omega \to \mathbb{R}$ are μ-measurable functions such that $f_j \in L^{p_j}(\Omega)$ for each $j \in \{1, \dots, n\}$. Then, we have*

$$\left\| \prod_{j=1}^{n} f_j \right\|_1 \leq \prod_{j=1}^{n} \|f_j\|_{p_j}.$$

Our first main result provides a lower bound for $\|f_k\|_{p_k}^{p_k}$ by applying Hölder's inequality in multiple configurations. The key insight is to consider different groupings of the exponents $\{p_j\}$.

Theorem 2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p_1, \dots, p_n > 1$ satisfy $\sum_{j=1}^{n} 1/p_j = 1$. Suppose $f_1, \dots, f_n : \Omega \to \mathbb{R}$ are μ-measurable functions such that $0 < \|f_i\|_{p_j} < \infty$ for $i, j \in \{1, \dots, n\}$. Then, for any $k \in \{1, \dots, n\}$, we have*

$$\|f_k\|_{p_k}^{p_k} \geq \prod_{i=1}^{n} \left(\int_\Omega |f_k|^{p_k/p_i} \prod_{j \neq k} |f_j| d\mu \right) \left(\prod_{j \neq k} \left(\|f_j\|_{p_j}^{n-1} \|f_j\|_{p_k} \right) \right)^{-1}.$$

Proof. Fix k. First, applying Hölder's inequality with exponents $\{p_j\}$ gives

$$\int_\Omega \prod_{j=1}^{n} |f_j| d\mu \leq \left(\int_\Omega |f_k|^{p_k} d\mu \right)^{1/p_k} \prod_{j \neq k} \left(\int_\Omega |f_j|^{p_j} d\mu \right)^{1/p_j},$$

which implies

$$\left(\int_\Omega |f_k|^{p_k} d\mu \right)^{1/p_k} \geq \frac{\int_\Omega \prod_{j=1}^{n} |f_j| d\mu}{\prod_{j \neq k} \left(\int_\Omega |f_j|^{p_j} d\mu \right)^{1/p_j}}. \tag{1}$$

Next, for each $i \neq k$, we apply Hölder's inequality to the factors $|f_k|^{p_k/p_i}$, $|f_i|$, and $|f_j|$ (for $j \neq k, i$) with exponents p_i, p_k, and p_j, respectively. Note that $1/p_i + 1/p_k + \sum_{j \neq k,i} 1/p_j = 1$. This yields

$$\int_\Omega |f_k|^{p_k/p_i} \prod_{j \neq k} |f_j| d\mu \leq \left(\int_\Omega |f_k|^{p_k} d\mu \right)^{1/p_i} \left(\int_\Omega |f_i|^{p_k} d\mu \right)^{1/p_k} \prod_{j \neq k,i} \left(\int_\Omega |f_j|^{p_j} d\mu \right)^{1/p_j},$$

which can be rearranged to

$$\left(\int_\Omega |f_k|^{p_k} d\mu \right)^{1/p_i} \geq \frac{\int_\Omega |f_k|^{p_k/p_i} \prod_{j \neq k} |f_j| d\mu}{\left(\int_\Omega |f_i|^{p_k} d\mu \right)^{1/p_k} \prod_{j \neq k,i} \left(\int_\Omega |f_j|^{p_j} d\mu \right)^{1/p_j}}. \tag{2}$$

Multiplying Equation (1) and all the Equation (2) over $i \neq k$, the left-hand side becomes $\int_\Omega |f_k|^{p_k} d\mu$ since $\sum_{j=1}^n 1/p_j = 1$. The right-hand side becomes

$$\frac{\left(\int_\Omega \prod_{j=1}^n |f_j| d\mu\right) \prod_{i\neq k} \left(\int_\Omega |f_k|^{p_k/p_i} \prod_{j\neq k} |f_j| d\mu\right)}{\left[\prod_{j\neq k} \left(\int_\Omega |f_j|^{p_j} d\mu\right)^{1/p_j}\right] \prod_{i\neq k} \left[\left(\int_\Omega |f_i|^{p_k} d\mu\right)^{1/p_k} \prod_{j\neq k,i} \left(\int_\Omega |f_j|^{p_j} d\mu\right)^{1/p_j}\right]}.$$

Performing some algebraic manipulation, we obtain the desired result. □

For the special case that $n = 2$, we have

$$\int_\Omega |f|^p d\mu \geq \left(\int_\Omega |fg| d\mu\right) \left(\int_\Omega |f|^{p/q} |g| d\mu\right) \left(\int_\Omega |g|^q d\mu\right)^{-1/q} \left(\int_\Omega |g|^p d\mu\right)^{-1/p},$$

where $1/p + 1/q = 1$.

As a demonstration, we now proceed to a sequence of corollaries illustrating specific parameter choices of Theorem 2.

Corollary 3. *Let $p_1, \ldots, p_n > 1$ satisfy $\sum_{j=1}^n 1/p_j = 1$, and let $\alpha_1, \ldots, \alpha_n > 0$. Suppose $f : (0, 1) \to \mathbb{R}$ is a Lebesgue integrable function and $0 < \int_0^1 |f(x)|^{p_j} dx < \infty$ for each j. Then, for any $k \in \{1, \ldots, n\}$, we have*

$$\int_0^1 |f(x)|^{p_k} dx \geq \prod_{i=1}^n \left(\int_0^1 |f(x)|^{p_k/p_i} \prod_{j\neq k} x^{\alpha_j} dx\right) \prod_{j\neq k} (\alpha_j p_j + 1)^{(n-1)/p_j} (\alpha_j p_k + 1)^{1/p_k}.$$

Proof. Fix k. The result is immediate by applying Theorem 2, taking $f_k = f$ and $f_j(x) = x^{\alpha_j}$ for all $j \neq k$ together with some further simplification. □

For the special case that $n = 2$, we have

$$\int_0^1 |f(x)|^p dx \geq \left(\int_0^1 |f(x)| x^\alpha dx\right) \left(\int_0^1 |f(x)|^{p/q} x^\alpha dx\right) (\alpha q + 1)^{1/q} (\alpha p + 1)^{1/p},$$

where $1/p + 1/q = 1$.

Example 4. By taking $f(x) = 1 - x$ and $n = 2$ in Corollary 3, we get

$$\int_0^1 (1-x)^p dx \geq (\alpha q + 1)^{1/q} (\alpha p + 1)^{1/p} \left(\int_0^1 x^\alpha (1-x) dx\right) \left(\int_0^1 x^\alpha (1-x)^{p/q} dx\right),$$

equivalently,

$$\frac{1}{p+1} \geq (\alpha q + 1)^{1/q} (\alpha p + 1)^{1/p} B(\alpha + 1, 2) B\left(\alpha + 1, 1 + \frac{p}{q}\right),$$

where $B(\cdot, \cdot)$ is the beta function [4].

Corollary 5. *Let $p_1,\dots,p_n > 1$ satisfy $\sum_{j=1}^{n} 1/p_j = 1$, and let $\alpha_1,\dots,\alpha_n > 0$. Suppose $f:(0,\infty)\to\mathbb{R}$ is a Lebesgue integrable function and $0 < \int_0^\infty |f(x)|^{p_j}dx < \infty$ for each j. Then, for any $k \in \{1,\dots,n\}$, we have*

$$\int_0^\infty |f(x)|^{p_k}dx \ge \prod_{i=1}^{n}\left(\int_0^\infty |f(x)|^{p_k/p_i}\prod_{j\neq k} e^{-\alpha_j x}dx\right)\prod_{j\neq k}(\alpha_j p_j)^{(n-1)/p_j}(\alpha_j p_k)^{1/p_k}.$$

Proof. Fix k. The result is immediate by applying Theorem 2, taking $f_k = f$ and $f_j(x) = e^{-\alpha_j x}$ for all $j \neq k$ together with some further simplification. □

For the special case that $n = 2$, we have

$$\int_0^\infty |f(x)|^p dx \ge \left(\int_0^\infty |f(x)|e^{-\alpha x}dx\right)\left(\int_0^\infty |f(x)|^{p/q}e^{-\alpha x}dx\right)(\alpha q)^{1/q}(\alpha p)^{1/p},$$

where $1/p + 1/q = 1$.

One may naturally inquire whether it is possible to obtain an upper bound for $\|f_k\|_{p_k}^{p_k}$ as well. In this regard, Panagiotis Krasopoulos and Lazhar Bougoffa have established a reverse-type Hölder inequality in 2022, stated below.

Lemma 6. *[12] Let $(\Omega,\mathcal{A},\mu)$ be a measure space, and let $p_1,\dots,p_n > 1$ satisfy $\sum_{j=1}^{n} 1/p_j = 1$. Suppose $f_1,\dots,f_n : \Omega\to\mathbb{R}$ are μ-measurable functions such that $0 < \|f_j\|_{p_j} < \infty$ and $0 < m_j \le f_j \le M_j$ μ-a.e. for $j \in \{1,\dots,n\}$. Then, we have*

$$\prod_{j=1}^{n}\|f_j\|_{p_j}^{p_j} \le M\left\|\prod_{j=1}^{n} f_j\right\|_1^n,$$

where $M = \prod_{j=1}^{n} M_j^{p_j-1}/\prod_{j=1}^{n} m_j^{n-1}$.

Using this reverse Hölder inequality, we easily obtain an upper bound complementing Theorem 2:

$$\|f_k\|_{p_k}^{p_k} \le \frac{M\left\|\prod_{j=1}^{n} f_j\right\|_1^n}{\prod_{j\neq k}\|f_j\|_{p_j}^{p_j}}. \tag{3}$$

However, we note that this result is limited by the restriction that $m_j > 0$ μ-a.e., as the essential infimum of many special functions over interesting domains equals zero.

3 Hölder-type Inequalities via Choi-Young's Inequalities

Having established refined bounds through strategic applications of Hölder's inequality, we now turn to constructing new Hölder-type inequalities using a different approach based on the generalised Young inequality founded by Daeshik Choi in 2018. First, we present this inequality, the Jensen inequality, and the alternative reverse Hölder inequality (which is a different version from the one in Section 2 for a specific purpose).

Lemma 7. *[7] Let $a, b > 0$, and let $p, q > 1$ such that $1/p + 1/q = 1$. Then, for any positive integer n, we have*

$$\left(\frac{a}{p} + \frac{b}{q}\right)^n \geq (a^{1/p}b^{1/q})^n + (2r)^n \left(\left(\frac{a+b}{2}\right)^n - (ab)^{n/2}\right), \tag{4}$$

$$\left(\frac{a}{p} + \frac{b}{q}\right)^n \leq (a^{1/p}b^{1/q})^n + (2R)^n \left(\left(\frac{a+b}{2}\right)^n - (ab)^{n/2}\right), \tag{5}$$

where $r = \min\{1/p, 1/q\}$ and $R = \max\{1/p, 1/q\}$.

Lemma 8. *(Jensen's inequality, [22]) Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. Let $f : \Omega \to \mathbb{R}$ be a μ-measurable function, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, we have*

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \varphi \circ f d\mu.$$

Lemma 9. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space, and let $f : \Omega \to \mathbb{R}$ be a μ-measurable function. If $n \geq 1$, then we have*

$$\int_\Omega f^n d\mu \geq \left(\int_\Omega f d\mu\right)^n.$$

Proof. This follows from Jensen's inequality, taking $\varphi(x) = x^n$ which is convex by [20]. □

Lemma 10. *[15] Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $p, q > 1$ with $1/p+1/q = 1$. Suppose $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions such that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, and*

$$0 < m \leq \frac{|f|^p}{|g|^q} \leq M \quad \mu\text{-a.e.}$$

Then, we have

$$\|f\|_p^p \|g\|_q^q \leq \left(\frac{M}{m}\right)^{1/(pq)} \|fg\|_1.$$

Now, we can state the main result.

Theorem 11. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. Let $p, q > 1$ with $1/p + 1/q = 1$. Suppose $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions such that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, and*

$$0 < m \leq \frac{|f|^p}{|g|^q} \leq M \quad \mu\text{-a.e.}$$

Let $r = \min\{1/p, 1/q\}$. Then, for any positive integer n, we have

$$\left(\frac{1}{(p^{1/p}q^{1/q})^n} \cdot \left(\frac{m}{M}\right)^{1/(pq)} + r^n\right) (\|f\|_p^p \|g\|_q^q)^n$$

$$\leq \frac{(2r)^n}{(pq)^{n/2}} \left(\|f\|_p^p \|g\|_q^q \int_\Omega |f|^p |g|^q d\mu\right)^{n/2} + \frac{1}{(pq)^{2n}} \int_\Omega \left(q^2 |f|^p \|g\|_q^q + p^2 |g|^q \|f\|_p^p\right)^n d\mu.$$

Proof. Note that $\|f\|_p, \|g\|_q > 0$ since $\mu(\Omega) = 1 > 0$ and $|f|, |g|$ are positive μ-a.e. We start by substituting

$$a = \frac{|f|^p}{p \int_\Omega |f|^p d\mu} \quad \text{and} \quad b = \frac{|g|^q}{q \int_\Omega |g|^q d\mu}$$

in Lemma 7, Equation (4). This gives

$$\begin{aligned}
&\left(\frac{|f|^p}{p^2 \int_\Omega |f|^p d\mu} + \frac{|g|^q}{q^2 \int_\Omega |g|^q d\mu}\right)^n \\
&\geq \left(\frac{|f||g|}{\left(p \int_\Omega |f|^p d\mu\right)^{1/p} \left(q \int_\Omega |g|^q d\mu\right)^{1/q}}\right)^n + r^n \left(\frac{|f|^p}{p \int_\Omega |f|^p d\mu} + \frac{|g|^q}{q \int_\Omega |g|^q d\mu}\right)^n \\
&\quad - (2r)^n \left(\frac{|f|^p |g|^q}{pq \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu}\right)^{n/2} \\
&= \frac{1}{p^{n/p} q^{n/q}} \cdot \frac{(|f||g|)^n}{\left(\left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}\right)^n} + \left(\frac{r}{pq}\right)^n \cdot \frac{\left(q|f|^p \int_\Omega |g|^q d\mu + p|g|^q \int_\Omega |f|^p d\mu\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\quad - \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\left(|f|^p |g|^q \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n}.
\end{aligned} \tag{6}$$

Next, we integrate Equation (6) over Ω with respect to μ. On the right-hand side, we have

$$\begin{aligned}
&\int_\Omega \frac{1}{(p^{1/p} q^{1/q})^n} \cdot \frac{(|f||g|)^n}{\left(\left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}\right)^n} d\mu + \int_\Omega \left(\frac{r}{pq}\right)^n \cdot \frac{\left(q|f|^p \int_\Omega |g|^q d\mu + p|g|^q \int_\Omega |f|^p d\mu\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} d\mu \\
&\quad - \int_\Omega \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\left(|f|^p |g|^q \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} d\mu \\
&= \frac{1}{(p^{1/p} q^{1/q})^n} \cdot \frac{\int_\Omega |fg|^n d\mu}{\left(\left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}\right)^n} + \left(\frac{r}{pq}\right)^n \cdot \frac{\int_\Omega \left(q|f|^p \int_\Omega |g|^q d\mu + p|g|^q \int_\Omega |f|^p d\mu\right)^n d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\quad - \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\int_\Omega \left(|f|^p |g|^q \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2} d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\geq \frac{1}{(p^{1/p} q^{1/q})^n} \left(\frac{\int_\Omega |fg| d\mu}{\left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}}\right)^n + \left(\frac{r}{pq}\right)^n \cdot \frac{\left(\int_\Omega q|f|^p d\mu \int_\Omega |g|^q d\mu + \int_\Omega p|g|^q d\mu \int_\Omega |f|^p d\mu\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\quad - \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\int_\Omega \left(|f|^p |g|^q \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2} d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\geq \frac{1}{(p^{1/p} q^{1/q})^n} \left(\frac{\left(\frac{m}{M}\right)^{1/(pq)} \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}}{\left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}}\right)^n + \left(\frac{r(p+q)}{pq}\right)^n \\
&\quad - \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\int_\Omega \left(|f|^p |g|^q \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2} d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&= \frac{1}{(p^{1/p} q^{1/q})^n} \left(\frac{m}{M}\right)^{1/(pq)} + r^n - \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\left(\int_\Omega |f|^p |g|^q d\mu \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n}.
\end{aligned}$$

Above, we apply Lemma 9 in the second inequality and Lemma 10 in the third inequality. Moreover, the last equality is due to $(p+q)/(pq) = 1/p + 1/q = 1$. Next, on the left-hand side of Equation (6), we have

$$\int_\Omega \left(\frac{|f|^p}{p^2 \int_\Omega |f|^p d\mu} + \frac{|g|^q}{q^2 \int_\Omega |g|^q d\mu} \right)^n d\mu = \frac{1}{(pq)^{2n}} \int_\Omega \left(\frac{q^2 |f|^p \int_\Omega |g|^q d\mu + p^2 |g|^q \int_\Omega |f|^p d\mu}{\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu} \right)^n d\mu$$
$$= \frac{\int_\Omega \left(q^2 |f|^p \int_\Omega |g|^q d\mu + p^2 |g|^q \int_\Omega |f|^p d\mu \right)^n d\mu}{(pq)^{2n} \left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^n}.$$

Combining these developments, we obtain

$$\frac{\int_\Omega \left(q^2 |f|^p \int_\Omega |g|^q d\mu + p^2 |g|^q \int_\Omega |f|^p d\mu \right)^n d\mu}{(pq)^{2n} \left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^n} + \frac{(2r)^n}{(pq)^{n/2}} \cdot \frac{\left(\int_\Omega |f|^p |g|^q d\mu \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^n}$$
$$\geq \frac{1}{(p^{1/p} q^{1/q})^n} \left(\frac{m}{M} \right)^{1/(pq)} + r^n,$$

or

$$\left(\frac{1}{(p^{1/p} q^{1/q})^n} \cdot \left(\frac{m}{M} \right)^{1/(pq)} + r^n \right) \left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^n$$
$$\leq \frac{(2r)^n}{(pq)^{n/2}} \left(\int_\Omega |f|^p |g|^q d\mu \int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \right)^{n/2} + \frac{1}{(pq)^{2n}} \int_\Omega \left(q^2 |f|^p \int_\Omega |g|^q d\mu + p^2 |g|^q \int_\Omega |f|^p d\mu \right)^n d\mu.$$

This completes the proof. □

For the special case that $p = q = 2$ and $n = 1$, Theorem 11 reduces to

$$\left(\frac{1}{2} \sqrt[4]{\frac{m}{M}} + \frac{1}{2} \right) \|f\|_2^2 \|g\|_2^2 \leq \frac{1}{2} \sqrt{\|f\|_2^2 \|g\|_2^2 + \int_\Omega |fg|^2 d\mu} + \frac{1}{16} \left(4 \|g\|_2^2 \int_\Omega |f|^2 d\mu + 4 \|f\|_2^2 \int_\Omega |g|^2 d\mu \right).$$

With some algebraic manipulation, this is simply equivalent to

$$\sqrt{\frac{m}{M}} \|f\|_2^4 \|g\|_2^4 \leq \|f\|_2^2 \|g\|_2^2 + \|fg\|_2^2,$$

where $0 < \sqrt{m} \leq |f/g| \leq \sqrt{M}$ for μ-a.e.

By repeating a similar argument, we deduce a complementary identity as shown below.

Theorem 12. *Let $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. Let $p, q > 1$ with $1/p + 1/q = 1$. Suppose $f, g : \Omega \to \mathbb{R}$ are μ-measurable function such that $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, and $\int_\Omega |f|^p |g|^q d\mu < \infty$. Let $R = \max\{1/p, 1/q\}$. Then, for any positive integer $n \geq 2$, we have*

$$(\|f\|_p^p \|g\|_q^q)^n + (2R)^n \left(\|f\|_p^p \|g\|_q^q \int_\Omega |f|^p |g|^q d\mu \right)^{n/2}$$
$$\leq \left(\|f^{p/q}\|_q \|g^{q/p}\|_p \right)^n \int_\Omega |fg|^n d\mu + R^n \int_\Omega \left(|f|^p \|g\|_q^q + |g|^q \|f\|_p^p \right)^n d\mu.$$

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then the inequality trivially holds. So, let us suppose that both of them are nonzero. Then, we substitute

$$a = \frac{|f|^p}{\int_\Omega |f|^p d\mu} \quad \text{and} \quad b = \frac{|g|^q}{\int_\Omega |g|^q d\mu}$$

in Lemma 7, Equation (5). Therefore, we have

$$\begin{aligned}
&\left(\frac{|f|^p}{p\int_\Omega |f|^p d\mu} + \frac{|g|^q}{q\int_\Omega |g|^q d\mu}\right)^n \\
&\leq \left(\frac{|f||g|}{\left(\int_\Omega |f|^p d\mu\right)^{1/p}\left(\int_\Omega |g|^q d\mu\right)^{1/q}}\right)^n + R^n\left(\frac{|f|^p}{\int_\Omega |f|^p d\mu} + \frac{|g|^q}{\int_\Omega |g|^q d\mu}\right)^n \\
&\quad - (2R)^n\left(\frac{|f|^p|g|^q}{\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu}\right)^{n/2} \\
&= \frac{\left(|fg|\left(\int_\Omega |f|^p d\mu\right)^{1/q}\left(\int_\Omega |g|^q d\mu\right)^{1/p}\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} + R^n \cdot \frac{\left(|f|^p\int_\Omega |g|^q d\mu + |g|^q \int_\Omega |f|^p d\mu\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\quad - (2R)^n \cdot \frac{\left(|f|^p|g|^q\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n}.
\end{aligned} \tag{7}$$

In the second equality, we apply the assumption $1/p + 1/q = 1$. Next, we integrate Equation (7) over Ω with respect to μ. On the left-hand side, we have

$$\int_\Omega \left(\frac{|f|^p}{p\int_\Omega |f|^p d\mu} + \frac{|g|^q}{q\int_\Omega |g|^q d\mu}\right)^n d\mu \geq \left(\frac{\int_\Omega |f|^p d\mu}{p\int_\Omega |f|^p d\mu} + \frac{\int_\Omega |g|^q d\mu}{q\int_\Omega |g|^q d\mu}\right)^n = \left(\frac{1}{p} + \frac{1}{q}\right)^n = 1,$$

where we apply Lemma 9 in the first inequality. Then on the right-hand side of Equation (7), we have

$$\begin{aligned}
&\int_\Omega \frac{\left(|fg|\left(\int_\Omega |f|^p d\mu\right)^{1/q}\left(\int_\Omega |g|^q d\mu\right)^{1/p}\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} d\mu + R^n \int_\Omega \frac{\left(|f|^p\int_\Omega |g|^q d\mu + |g|^q \int_\Omega |f|^p d\mu\right)^n}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} d\mu \\
&\quad - (2R)^n \int_\Omega \frac{\left(|f|^p|g|^q\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} d\mu \\
&\leq \frac{\left(\left(\int_\Omega |f|^p d\mu\right)^{1/q}\left(\int_\Omega |g|^q d\mu\right)^{1/p}\right)^n \int_\Omega |fg|^n d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} + R^n \cdot \frac{\int_\Omega \left(|f|^p\int_\Omega |g|^q d\mu + |g|^q \int_\Omega |f|^p d\mu\right)^n d\mu}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n} \\
&\quad - (2R)^n \cdot \frac{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \int_\Omega |f|^p|g|^q d\mu\right)^{n/2}}{\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n}.
\end{aligned}$$

Note that the second inequality is due to Lemma 9. Combining these facts, we obtain

$$\begin{aligned}
&\left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu\right)^n \leq \left(\left(\int_\Omega |f|^p d\mu\right)^{1/q}\left(\int_\Omega |g|^q d\mu\right)^{1/p}\right)^n \int_\Omega |fg|^n d\mu \\
&+ R^n \int_\Omega \left(|f|^p \int_\Omega |g|^q d\mu + |g|^q \int_\Omega |f|^p d\mu\right)^n d\mu - (2R)^n \left(\int_\Omega |f|^p d\mu \int_\Omega |g|^q d\mu \int_\Omega |f|^p |g|^q d\mu\right)^{n/2}.
\end{aligned}$$

This completes the proof. □

Remark 13. *Both Theorems 11 and 12 can be generalised to any finite positive measure space (not only probability space) by introducing the factor $\mu(\Omega)$ appropriately in Lemma 9. We leave this straightforward extension to interested readers.*

Remark 14. *There are two significant differences between the assumptions in Theorems 11 and 12. First, we do not assume boundedness conditions on the ratio $|f|^p/|g|^q$ in Theorem 12 since we do not use Lemma 10. Second, in Theorem 12, we suppose that $n \geq 2$ to ensure the convexity argument on Lemma 9.*

Example 15. Let $p = q = 2$ and $n = 2$. Consider $\Omega = (0, 1)$ and μ the Lebesgue measure. Choose $f(x) = x^\alpha$ and $g(x) = (1-x)^\beta$ with $\alpha, \beta > -1/2$. Note that $R = 1/2$. We compute the following expressions involved in Theorem 12:

$$\|f\|_2^2 = \int_0^1 x^{2\alpha} dx = \frac{1}{2\alpha+1}, \quad \|g\|_2^2 = \int_0^1 (1-x)^{2\beta} dx = \frac{1}{2\beta+1},$$

$$\int_0^1 x^{2\alpha}(1-x)^{2\beta} dx = B(2\alpha+1, 2\beta+1), \quad \|f^{2/2}\|_2 = \|f\|_2 = \frac{1}{\sqrt{2\alpha+1}},$$

$$\|g^{2/2}\|_2 = \|g\|_2 = \frac{1}{\sqrt{2\beta+1}}, \quad \int_0^1 (x^\alpha(1-x)^\beta)^2 dx = B(2\alpha+1, 2\beta+1),$$

$$\int_0^1 \left(\frac{x^{2\alpha}}{2\beta+1} + \frac{(1-x)^{2\beta}}{2\alpha+1}\right)^2 dx = \frac{1}{(2\beta+1)^2(4\alpha+1)} + \frac{2B(2\alpha+1, 2\beta+1)}{(2\alpha+1)(2\beta+1)} + \frac{1}{(2\alpha+1)^2(4\beta+1)}.$$

Plugging all these into the inequality, we have

$$\left(\frac{1}{(2\alpha+1)(2\beta+1)}\right)^2 + \frac{B(2\alpha+1, 2\beta+1)}{(2\alpha+1)(2\beta+1)}$$
$$\leq \frac{B(2\alpha+1, 2\beta+1)}{(2\alpha+1)(2\beta+1)} + \frac{1}{4}\left(\frac{1}{(2\beta+1)^2(4\alpha+1)} + \frac{2B(2\alpha+1, 2\beta+1)}{(2\alpha+1)(2\beta+1)} + \frac{1}{(2\alpha+1)^2(4\beta+1)}\right),$$

or simply

$$\frac{4}{(2\alpha+1)^2(2\beta+1)^2} \leq \frac{1}{(2\beta+1)^2(4\alpha+1)} + \frac{2B(2\alpha+1, 2\beta+1)}{(2\alpha+1)(2\beta+1)} + \frac{1}{(2\alpha+1)^2(4\beta+1)}. \tag{8}$$

As a numerical example, substituting $\alpha = \beta = 1/4$ into Equation (8) gives

$$\frac{4}{(3/2)^4} \leq \frac{1}{(3/2)^2(2)} + \frac{2(\pi/8)}{(3/2)^2} + \frac{1}{(3/2)^2(2)}.$$

Approximately, the left-hand side is 0.7901 while the right-hand side is 0.7935 up to four significant digits.

4 Hardy-type Derivative Inequalities using Cauchy-Schwarz's Inequality

The techniques developed in the preceding sections rely primarily on Hölder's inequality and its variants. We now demonstrate how the Cauchy-Schwarz inequality, a special case of Hölder's inequality, can yield derivative-type estimates with optimal constants.

Theorem 16. *Let $f : [0, \infty) \to \mathbb{R}$ be an absolutely continuous function with $f(0) = 0$. Let $w : [0, \infty) \to [0, \infty]$ be a Lebesgue measurable function. For $x \geq 0$, we define*

$$W(x) = \int_x^\infty yw(y)dy \in [0, \infty].$$

Then, for every measurable derivative f' (μ-a.e. defined), we have

$$\int_0^\infty w(y)(f(y))^2 dy \leq \int_0^\infty (f'(x))^2 W(x)dx.$$

Proof. Since $f(0) = 0$ and f is absolutely continuous, the fundamental theorem of calculus implies that $f(y) = \int_0^y f'(x)dx$ for each $y \geq 0$. Applying Cauchy-Schwarz's inequality to the integral on the right-hand side gives

$$(f(y))^2 = \left(\int_0^y f'(x)dx\right)^2 \leq \left(\int_0^y 1^2 dx\right)\left(\int_0^y (f'(x))^2 dx\right) = y\int_0^y (f'(x))^2 dx$$

for each fixed $y \geq 0$. Next, we multiply both sides by the nonnegative weight $w(y)$ and integrate in y over $[0, \infty)$. This gives

$$\int_0^\infty w(y)(f(y))^2 dy \leq \int_0^\infty yw(y)\left(\int_0^y (f'(x))^2 dx\right) dy. \tag{9}$$

The right-hand side is nonnegative. So, we can apply Fubini-Tonelli's theorem to change the order of integrations and deduce

$$\int_0^\infty yw(y)\left(\int_0^x (f'(x))^2 dx\right) dy = \int_0^\infty (f'(x))^2 \left(\int_x^\infty yw(y)dy\right) dx = \int_0^\infty (f'(x))^2 W(x)dx. \tag{10}$$

Combining Equations (9) and (10), the proof is complete. □

Although the above result is only stated on the real half-line with Lebesgue measure, it can be regarded as a one-dimensional illustration of how Hölder's inequality yields derivative-type estimates in the general measure-space framework.

Corollary 17. *Let $\alpha > 0$. Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$. Then, we have*

$$\int_0^\infty e^{-\alpha x^2}(f(x))^2 dx \leq \frac{1}{2\alpha}\int_0^\infty e^{-\alpha x^2}(f'(x))^2 dx. \tag{11}$$

Proof. This easily follows from Theorem 16 by taking $w(x) = e^{-\alpha x^2}$. We note that

$$W(x) = \int_x^\infty yw(y)dy = \int_x^\infty ye^{-\alpha y^2}dy = \left[-\frac{1}{2\alpha}e^{-\alpha y^2}\right]_{y=x}^{y=\infty} = \frac{1}{2\alpha}e^{-\alpha x^2}.$$

This completes the proof. □

Remark 18. *In fact, the constant $1/(2\alpha)$ in Corollary 17 is optimal; that is, it cannot be replaced by a smaller value. To illustrate this, consider the test function $f(x) = x$ for $x \geq 0$. Clearly, $f(0) = 0$ and $f'(x) \equiv 1$. We can evaluate the two integrals appearing in Equation (11) explicitly. On one hand, we obtain*

$$\int_0^\infty e^{-\alpha x^2}(f(x))^2 dx = \int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{1}{2}\alpha^{-3/2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\alpha^{-3/2} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{4}\sqrt{\pi}\alpha^{-3/2},$$

where $\Gamma(\cdot)$ is the gamma function [4]. On the other hand, we find

$$\int_0^\infty e^{-\alpha x^2}(f'(x))^2 dx = \int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}.$$

It follows that equality holds in Equation (11).

In light of the preceding developments, it is natural to investigate whether similar operator-type inequalities can be formulated for bilinear expressions involving two functions. By carefully applying the Hölder inequality in this context and leveraging the symmetry of the underlying kernel, we obtain the following new result, which offers a further extension of the inequalities discussed above.

Theorem 19. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions such that $f(a) = 0$ and $g(b) = 0$, where we adopt the conventions $f(-\infty) = \lim_{x\to-\infty} f(x)$ and $g(\infty) = \lim_{x\to\infty} g(x)$. Suppose $f', g' \in L^2([a, b])$ (which exist μ-a.e.). Then, we have*

$$\left|\int_a^b f(x)g(x)dx\right| \leq \frac{1}{\sqrt{3}}\left(\int_a^b (b-x)^3|f'(x)|^2\right)^{1/2}\left(\int_a^b (x-a)|g'(x)|^2 dx\right)^{1/2}. \tag{12}$$

Proof. If either of the two integrals on the right-hand side of Equation (12) diverge, then we are done. So, let us consider the case where both are finite. Because f and g are absolutely continuous with $f(a) = g(b) = 0$, we can write

$$f(x) = \int_a^x f'(t)dt \quad \text{and} \quad g(x) = -\int_x^b g'(s)ds$$

by the fundamental theorem of calculus. Using Fubini's theorem, we have

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= -\int_a^b \int_a^x \int_x^b f'(t)g'(s)dtdsdx \\
&= -\int_a^b \int_t^b \int_t^s f'(t)g'(s)dxdsdt \\
&= -\int_a^b \int_t^b (s-t)f'(t)g'(s)dsdt.
\end{aligned}$$

Therefore we obtain

$$\left|\int_a^b f(x)g(x)dx\right| = \left|\iint_{\{(s,t):a\leq t\leq s\leq b\}} (s-t)f'(t)g'(s)dsdt\right|.$$

Applying Cauchy-Schwarz's inequality gives

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\iint_{\{(s,t):a\leq t\leq s\leq b\}} (s-t)^2|f'(t)|^2 dsdt\right)^{1/2} \left(\iint_{\{(s,t):a\leq t\leq s\leq b\}} |g'(s)|^2 dsdt\right)^{1/2}. \quad (13)$$

We examine each integral above separately. We have

$$\begin{aligned}\iint_{\{(s,t):a\leq t\leq s\leq b\}} (s-t)^2|f'(t)|^2 dsdt &= \int_a^b |f'(t)|^2 \left(\int_t^b (s-t)^2 ds\right) dt \\ &= \frac{1}{3}\int_a^b (b-t)^3|f'(t)|^2 dt\end{aligned} \quad (14)$$

and

$$\begin{aligned}\iint_{\{(s,t):a\leq t\leq s\leq b\}} |g'(s)|^2 dsdt &= \int_a^b |g'(s)|^2 \left(\int_a^s dt\right) ds \\ &= \int_a^b (s-a)|g'(s)|^2 ds.\end{aligned} \quad (15)$$

Substituting Equations (14) and (15) into Equation (13), the proof is complete. □

Let us see a demonstration below involving the beta function.

Example 20. Let $[a,b]=[0,1]$. For $\alpha,\beta > 1/2$, define $f(x)=x^\alpha$ and $g(x)=(1-x)^\beta$. Note that $f(0)=g(1)=0$ as desired. Moreover, $f'(x)=\alpha x^{\alpha-1}$ and $g'(x)=-\beta(1-x)^{\beta-1}$. By Theorem 12, we have

$$\left|\int_0^1 x^\alpha(1-x)^\beta dx\right| \leq \frac{1}{\sqrt{3}}\left(\int_0^1 (1-x)^3|f'(x)|^2 dx\right)^{1/2}\left(\int_0^1 x|g'(x)|^2 dx\right)^{1/2}. \quad (16)$$

The left-hand side of Equation (16) is exactly $B(\alpha+1,\beta+1)$ by definition. Now, we look at each integral involved on the right-hand side of Equation (16). We have

$$\int_0^1 (1-x)^3|f'(x)|^2 dx = \alpha^2\int_0^1 (1-x)^3 x^{2\alpha-2} dx = \alpha^2 B(2\alpha-1,4)$$

and

$$\int_0^1 x|g'(x)|^2 dx = \beta^2\int_0^1 x(1-x)^{2\beta-2} dx = \beta^2 B(2,2\beta-1).$$

Putting these all together, we obtain the inequality

$$B(\alpha+1,\beta+1) \leq \frac{\alpha\beta}{\sqrt{3}}\sqrt{B(2\alpha-1,4)B(2,2\beta-1)}.$$

As a numerical illustration, the left-hand side equals 0.1667 while the right-hand side equals 0.2041, up to 4 significant digits.

5 Minkowski-Clarkson Relations and Variants

While Sections 2, 3, and 4 focused on refinements of Hölder-type inequalities, we now investigate relationships between Minkowski's and Clarkson's inequalities. These results are fundamental in understanding the structure and uniform convexity properties of L^p spaces. We start by recalling the famous Minkowski inequality, introduced by Hermann Minkowski in 1896.

Lemma 21. *(Minkowski's inequality, [9]) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \geq 1$. Suppose $f_1, \ldots, f_n : \Omega \to \mathbb{R}$ are μ-measurable functions such that $f_j \in L^p(\Omega)$ for each $j \in \{1, \ldots, n\}$. Then, we have*

$$\left\| \sum_{j=1}^{n} f_j \right\|_p \leq \sum_{j=1}^{n} \|f_j\|_p.$$

To facilitate our study, we use several classical inequalities stated below.

Lemma 22. *(Jensen's discrete inequality, [11]) Let $\Omega \subseteq \mathbb{R}$ be an interval. Suppose $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{j=1}^{n} \lambda_j = 1$. Let $\varphi : \Omega \to \mathbb{R}$ be a continuous function. Let $x_1, \ldots, x_n \in \Omega$. If φ is concave, then we have*

$$\varphi\left(\sum_{j=1}^{n} \lambda_j x_j\right) \geq \sum_{j=1}^{n} \lambda_j \varphi(x_j).$$

On the other hand, if φ is convex, then the inequality is reversed.

Lemma 23. *Let $x_1, \ldots, x_n \geq 0$. If $k \in (0, 1]$, then*

$$\sum_{j=1}^{n} x_j^k \leq n^{1-k} \left(\sum_{j=1}^{n} x_j\right)^k.$$

Proof. First we observe that the function $x \mapsto x^k$ defined on $[0, \infty)$ is concave if $k \in (0, 1]$ [20]. So, we apply Jensen's discrete inequality with coefficients $\lambda_j = 1/n$. As f is concave, we get

$$\frac{1}{n} \sum_{j=1}^{n} x_j^k \leq \left(\frac{1}{n} \sum_{j=1}^{n} x_j\right)^k \quad \Rightarrow \quad \sum_{j=1}^{n} x_j^k \leq n^{1-k} \left(\sum_{j=1}^{n} x_j\right)^k.$$

This completes the proof. □

Now, we can present our next main result.

Theorem 24. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \geq 1$. Suppose $f_1, \ldots, f_n \in L^p(\Omega)$. Then, we have*

$$\left\| \sum_{j=1}^{n} f_j \right\|_p \leq \sum_{j=1}^{n} \|f_j\|_p \leq n^{1-1/p} \left(\sum_{j=1}^{n} \|f_j\|_p^p\right)^{1/p}.$$

Proof. Note that $1/p \in (0,1]$ because $p \geq 1$. The former inequality is basically the Minkowski's inequality, while the latter is trivial by applying Lemma 23 with $x_j = \int_\Omega |f_j|^p d\mu$ for each $j \in \{1, \ldots, n\}$. $\square$

While this result is not technically demanding, it possesses certain applicability and offers new insight into its relationship with other inequalities, as we shall see below.

Example 25. Let $\alpha_1, \cdots, \alpha_n > 0$. Consider $\Omega = [0, \infty]$ and μ the Lebesgue measure. For each j, take $f_j(x) = e^{-x} x^{\alpha_j} > 0$ on $[0, \infty]$. Then, we have

$$\int_0^\infty f_j^p(x)dx = \int_0^\infty e^{-px} x^{\alpha_j p} dx = \frac{1}{p^{\alpha_j p+1}} \int_0^\infty e^{-y} y^{\alpha_j p} dy = \frac{1}{p^{\alpha_j p+1}} \Gamma(\alpha_j p + 1).$$

Note that the second equality above is yielded from the change of variable $y = px$. Thus applying Theorem 24, we obtain

$$\left(\int_0^\infty \left(e^{-x} \sum_{j=1}^n x^{\alpha_j} \right)^p dx \right)^{1/p} \leq \sum_{j=1}^n \left(\frac{1}{p^{\alpha_j p+1}} \Gamma(\alpha_j p + 1) \right)^{1/p} \leq n^{1-1/p} \left(\sum_{j=1}^n \frac{1}{p^{\alpha_j p+1}} \Gamma(\alpha_j p + 1) \right)^{1/p}.$$

For further instance, if we take $p = 2$ and $\alpha_j = j$, then we deduce that

$$\sqrt{\int_0^\infty e^{-2x} \left(\frac{x^{n+1} - 1}{x - 1} \right)^2 dx} \leq \sum_{j=1}^n \sqrt{\frac{1}{2^{2j+1}} \Gamma(2j+1)} \leq \sqrt{n \sum_{j=1}^n \frac{1}{2^{2j+1}} \Gamma(2j+1)}.$$

When $n = 2$, Theorem 24 specialises to

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \leq 2^{1-1/p} \left(\|f\|_p^p + \|g\|_p^p \right)^{1/p}.$$

To further refine the two-function case, we make use of Clarkson's inequality, established by John Clarkson in 1936. It is stated below.

Lemma 26. *(Clarkson's inequality, [8]) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \geq 2$. Suppose $f, g \in L^p(\Omega)$. Then, we have*

$$\|f + g\|_p^p + \|f - g\|_p^p \leq 2^{p-1} \left(\|f\|_p^p + \|g\|_p^p \right).$$

Remark 27. *In fact, this inequality can easily be extended to n functions in a cyclic form. Suppose $f_1, f_2, \ldots, f_n \in L^p(\Omega)$. By adopting the convention $f_{n+1} := f_1$, we have*

$$\sum_{j=1}^n \left(\|f_j + f_{j+1}\|_p^p + \|f_j - f_{j+1}\|_p^p \right) \leq 2^p \sum_{j=1}^n \|f_j\|_p^p.$$

From our main finding and this inequality, we deduce the following immediately.

Corollary 28. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \geq 2$. Suppose $f, g \in L^p(\Omega)$. Then, we have*

$$\|f + g\|_p + \|f - g\|_p \leq 2^{2(1-1/p)} \left(\|f\|_p^p + \|g\|_p^p \right)^{1/p}.$$

Proof. We consider Theorem 24 for the case $n = 2$. Substituting f by $f + g$ and g by $f - g$, we have

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} + \left(\int_\Omega |f-g|^p d\mu\right)^{1/p} \leq 2^{1-1/p}\left(\int_\Omega (|f+g|^p + |f-g|^p) d\mu\right)^{1/p}.$$

Applying Clarkson's inequality on the right-hand side gives

$$\left(\int_\Omega |f+g|^p d\mu\right)^{1/p} + \left(\int_\Omega |f-g|^p d\mu\right)^{1/p} \leq 2^{(1-1/p)} 2^{(1-1/p)}\left(\int_\Omega (|f|^p + |g|^p) d\mu\right)^{1/p}$$

as desired. □

We note that the classical Clarkson inequality describes the uniform convexity of L^p spaces in the unweighted setting. We next present a weighted and mixed-exponent variant Clarkson-type inequality, where a measurable weight function modifies the argument in a nontrivial way.

Lemma 29. *[11] Let $a, b \in \mathbb{R}$. If $p \geq 2$, then we have*

$$|a|^p + |b|^p \leq (|a|^2 + |b|^2)^{p/2}.$$

Theorem 30. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $p \geq 2$. Suppose $f, g : \Omega \to \mathbb{R}$ and $w : \Omega \to (0, \infty)$ are μ-measurable functions such that $wf + g, f - wg \in L^p(\Omega)$. Then, we have*

$$\|wf+g\|_p^p + \|f-wg\|_p^p \leq 2^{p/2-1}\|w^2+1\|_p^{p/2}(\|f\|_{2p}^p + \|g\|_{2p}^p).$$

Proof. We consider the substitution $a = (wf + g)/2$ and $b = (f - wg)/2$ in Lemma 29. We first compute the sum of squares which appears on the right-hand side, as follows:

$$|a|^2 + |b|^2 = \frac{1}{4}(|wf+g|^2 + |f-wg|^2) = \frac{1}{4}(w^2+1)(|f|^2+|g|^2).$$

Hence, we have

$$|a|^p + |b|^p \leq \left(\frac{(w^2+1)(|f|^2+|g|^2)}{4}\right)^{p/2}. \tag{17}$$

Applying the Jensen's discrete inequality with coefficients $\{1/2, 1/2\}$ on the right-hand side of Equation (17) (since the function $x \mapsto x^{p/2}$ is convex by [20]), we see that

$$\begin{aligned}\left(\frac{(w^2+1)(|f|^2+|g|^2)}{4}\right)^{p/2} &= \left(\frac{w^2+1}{2}\right)^{p/2}\left(\frac{|f|^2+|g|^2}{2}\right)^{p/2}\\ &\leq \frac{1}{2^{p/2}}(w^2+1)^{p/2}\frac{1}{2}(|f|^p+|g|^p)\\ &= 2^{-p/2-1}(w^2+1)^{p/2}(|f|^p+|g|^p).\end{aligned}$$

Now, merging this and Equation (17), plugging in $a = (wf + g)/2$ and $b = (f - wg)/2$, then multiplying both sides by 2^p, we have

$$|wf+g|^p + |f-wg|^p \leq 2^{p/2-1}(w^2+1)^{p/2}(|f|^p+|g|^p).$$

Integrating this over Ω with respect to μ, we get

$$\int_\Omega |wf+g|^p d\mu + \int_\Omega |f-wg|^p d\mu \leq 2^{p/2-1} \int_\Omega (w^2+1)^{p/2}(|f|^p+|g|^p) d\mu.$$

Finally, applying the Hölder's inequality with exponents $\{2,2\}$ to each term, we obtain

$$\int_\Omega (w^2+1)^{p/2}|f|^p d\mu \leq \left(\int_\Omega (w^2+1)^p d\mu\right)^{1/2} \left(\int_\Omega |f|^{2p} d\mu\right)^{1/2},$$
$$\int_\Omega (w^2+1)^{p/2}|g|^p d\mu \leq \left(\int_\Omega (w^2+1)^p d\mu\right)^{1/2} \left(\int_\Omega |g|^{2p} d\mu\right)^{1/2}.$$

Combining all these developments, we obtain the desired result. □

Let us see an example.

Example 31. Let $\Omega = [0,1]$ and μ the Lebesgue measure. Take $p = 2$. Fix parameters $\alpha, \beta, \gamma \geq 0$ and set $f(x) = x^\alpha(1-x)^\beta$, $g(x) = (1-x)^\beta$, $w(x) = x^\gamma$ for $x \in [0,1]$. Then $f, g \in L^4([0,1])$ for all $\alpha, \beta > -1/4$. Put $m := \gamma + \alpha$. For $p = 2$, the left-hand side of Theorem 30 becomes

$$\begin{aligned} \|wf+g\|_2^2 + \|f-wg\|_2^2 &= \int_0^1 (1-x)^{2\beta}(1+x^m)^2 dx + \int_0^1 (1-x)^{2\beta}(x^\alpha - x^\gamma)^2 dx \\ &= \int_0^1 (1-x)^{2\beta}(1+2x^m+x^{2m}+x^{2\alpha}-2x^{\alpha+\gamma}+x^{2\gamma}) dx \\ &= B(1,2\beta+1) + 2B(m+1,2\beta+1) + B(2m+1,2\beta+1) \\ &\quad + B(2\alpha+1,2\beta+1) - 2B(\alpha+\gamma+1,2\beta+1) + B(2\gamma+1,2\beta+1). \end{aligned} \tag{18}$$

The right-hand side in Theorem 30 for $p = 2$ equals

$$\begin{aligned} &\|w^2+1\|_2(\|f\|_4^2 + \|g\|_4^2) \\ &= \left(\int_0^1 (x^{2\gamma}+1)^2\right)^{1/2} \left(\int_0^1 x^{4\alpha}(1-x)^{4\beta} dx + \int_0^1 (1-x)^{4\beta} dx\right)^{1/2} \\ &= \sqrt{\left(\frac{1}{4\gamma+1} + \frac{2}{2\gamma+1} + 1\right)} \left(\sqrt{B(4\alpha+1,4\beta+1)} + \sqrt{B(1,4\beta+1)}\right). \end{aligned} \tag{19}$$

Hence Equation (18) is bounded above by Equation (19). As a numerical demonstration, take $\alpha = \beta = \gamma = 1/2$. Then the left-hand side is roughly 0.9167 while the right-hand side is 1.160 up to 4 significant digits.

6 Weighted Generalised Schweitzer's Inequality

Our final result extends Schweitzer's classical inequality to a weighted setting in L^p spaces. The original version, proven by P. Schweitzer in 1914, is stated below [23].

Let $a, b \in \mathbb{R}$ with $b > a$, and let $m, M \in (0, \infty)$. Suppose $f : [a, b] \to (0, \infty)$ satisfies $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then, we have

$$\left(\int_a^b f(x)dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right) \leq \frac{(b-a)^2(m+M)^2}{4mM}.$$

Here is the result.

Theorem 32. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $m, M \in (0, \infty)$. Suppose $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions such that $|f| \geq m$ and $|g| \leq M$ for μ-a.e. Let $w : \Omega \to (0, \infty)$ be a μ-measurable function. Then, we have*

$$\left(\int_\Omega w\sqrt{|fg|}d\mu\right)\left(\int_\Omega \frac{w}{\sqrt{|fg|}}d\mu\right) \leq \frac{1}{4mM}\left(M\int_\Omega w\sqrt{\left|\frac{f}{g}\right|}d\mu + m\int_\Omega w\sqrt{\left|\frac{g}{f}\right|}d\mu\right)^2,$$

provided that the four integrals involved converge.

Proof. First, because $|f| \geq m > 0$ and $|g| \leq M$, μ-a.e., $(|f| - m)(M - |g|) \geq 0$ μ-a.e. Expanding this inequality gives

$$|fg| + mM \leq M|f| + m|g| \quad \mu\text{-a.e.}$$

Multiplying both sides by the positive factor $w/\sqrt{|fg|}$ (note that $w \geq 0$ and $|f|, |g| > 0$ μ-a.e.) gives

$$w\sqrt{|fg|} + \frac{mMw}{\sqrt{|fg|}} \leq Mw\sqrt{\left|\frac{f}{g}\right|} + mw\sqrt{\left|\frac{g}{f}\right|}.$$

Next, we integrate this over Ω with respect to μ, which yields

$$\int_\Omega w\sqrt{|fg|}d\mu + mM\int_\Omega \frac{w}{\sqrt{|fg|}}d\mu \leq M\int_\Omega w\sqrt{\left|\frac{f}{g}\right|}d\mu + m\int_\Omega w\sqrt{\left|\frac{g}{f}\right|}d\mu. \tag{20}$$

Applying the classical AM-GM inequality to the left-hand side, we see that

$$2\sqrt{\int_\Omega w\sqrt{|fg|}d\mu}\sqrt{mM\int_\Omega \frac{w}{\sqrt{|fg|}}d\mu} \leq \int_\Omega w\sqrt{|fg|}d\mu + mM\int_\Omega \frac{w}{\sqrt{|fg|}}d\mu. \tag{21}$$

Finally, combining Equations (20) and (21) together with some further simplification, we obtain the desired result. □

Remark 33. *When $|f| \equiv |g|$ and $w \equiv 1$, Theorem 32 reduces to the classical Schweitzer inequality.*

Example 34. Consider $\Omega = [0, 1]$ and μ the Lebesgue measure. Choose $m = 1/2$ and $M = 2$. Define functions (all positive μ-a.e.): $f(x) = 1 + x$, $g(x) = 2 - x$, and $w(x) = 1 + \sin(2\pi x)$. We note that the assumptions in Theorem 32 are met. Thus we have

$$\frac{1}{4}\left(2\int_0^1 (1+\sin(2\pi x))\sqrt{\frac{1+x}{2-x}}dx + \frac{1}{2}\int_0^1 (1+\sin(2\pi x))\sqrt{\frac{2-x}{1+x}}dx\right)^2$$
$$\geq \left(\int_0^1 (1+\sin(2\pi x))\sqrt{2+x-x^2}dx\right)\left(\int_0^1 \frac{1+\sin(2\pi x)}{\sqrt{2+x-x^2}}\right).$$

Numerically, we know that the left-hand side equals 1.291, while the right-hand side equals 0.9997, up to 4 significant digits.

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Conflict of Interest

The authors declare no conflict of interest.

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