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Fourier Series and Recurrence Relations for Zeta Functions

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Abstract

This article explores the connection between Fourier series and various zeta functions, including the Riemann zeta function and its generalizations. Specifically, we derive recurrence formulas for even and odd values of zeta functions using Fourier expansions, extending these results to the Hurwitz zeta function.

1 Introduction

The treatment of Fourier series is a fundamental tool for analyzing certain fixed values of functions. Formally, for a function f of period T, we have a description in the basis $\{\sin(\frac{2\pi n}{T}), \cos(\frac{2\pi n}{T})\}$ with $n \in \mathbb{N}$ and in the basis $\{e^{i\frac{2\pi n}{T}}\}$ where $n \in \mathbb{Z}$. These expansions turn out to be [1]

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right),$$

where

$$a_n = \frac{2}{T}\int_{-T}^{T} f(x)\cos\left(\frac{2\pi n}{T}x\right)dx$$

and

$$b_n = \frac{2}{T}\int_{-T}^{T} f(x)\sin\left(\frac{2\pi n}{T}x\right)dx.$$

For the case of the complex expansion, it holds that

$$f(x) \sim \sum_{n\in\mathbb{Z}} c_n e^{i\frac{2\pi n}{T}x},$$

where

$$c_n = \frac{1}{T}\int_{-T}^{T} f(x)e^{-i\frac{2\pi n}{T}x}dx.$$

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In this article, we first employ the classical approach to establish new and existing connections with certain values of the Riemann zeta function, which is defined as follows:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Section 2 is devoted to an analysis of this function. Section 3 then proposes an extension using the Hurwitz zeta function. The article concludes in Section 4.

2 Riemann Zeta Analysis

2.1 First approach

In the article [2], an initial recurrence formula is presented for the even values of the zeta function. A formal statement of this result is given below, along with a detailed proof.

Proposition 2.1 (Recurrence for Even Zeta Values)**.** *By considering the Fourier series of the function $f(x) = x^{2n}$ for $x \in (-\pi, \pi)$, a recurrence formula for the even values of the Riemann zeta function can be obtained as follows:*

$$\zeta(2n) = n\frac{\pi^{2n}(-1)^{n+1}}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}\pi^{2(n-k)}}{(2n-2k+1)!}\zeta(2k), \quad \zeta(2) = \frac{\pi^2}{6}.$$

Proof. To carry out this deduction, note that when performing the expansion in the Fourier basis $\{\sin(nx), \cos(nx)\}$, the odd terms will vanish. On the other hand, we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^{2n}dx = 2\frac{\pi^{2n}}{2n+1}.$$

For the a_m coefficients, we use the parity of the function f and obtain

$$a_m = \frac{2}{\pi}\int_0^{\pi} x^{2n}\cos(mx)dx.$$

Performing $2n$ times integrations by parts and taking into account that $\cos(\pi m) = (-1)^m$ and $\sin(\pi m) = 0$, we get

$$a_m = 2\sum_{k=1}^{n}(-1)^{m+k+1}\frac{\pi^{2n-2k}}{m^{2k}}\frac{(2n)!}{(2n-2k+1)!}.$$

Thus, by evaluating the function at $x = \pi$ and using the definition of the zeta function, we obtain

$$\pi^{2n} = \frac{\pi^{2n}}{2n+1} + 2\sum_{k=1}^{n}(-1)^{k+1}\pi^{2n-2k}\frac{(2n)!}{(2n-2k+1)!}\zeta(2k).$$

Rearranging the equality and extracting the value of $\zeta(2n)$, we obtain the first expression, where the initial value refers to the Basel problem. ■

With this result, it is possible to generate all even values of $\zeta(x)$ analytically. This obtained form has a certain relationship with the known form for even values of the zeta function, given by Bernoulli numbers, as follows [4]:

$$\zeta(2n) = |B_{2n}|\frac{(2\pi)^{2n}}{2(2n)!}.$$

2.2 Second approach

It is possible to generate small variations around the definition of the zeta function for arguments greater than 1, in the following ways [7,9]:

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x},$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^x} = \frac{\zeta(x)}{2^x},$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^x} = \left(\frac{2^x - 1}{2^x}\right)\zeta(x),$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^x} = \left(\frac{2^{x-1} - 1}{2^{x-1}}\right)\zeta(x)$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)^x} = \left(\frac{2^{x-1} - 1}{2^{2x-1}}\right)\zeta(x).$$

However, for the form

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^x},$$

there is no direct relationship with the zeta function. Therefore, we can proceed to use the Hurwitz zeta function, or define a new function, as follows:

$$\hat{\zeta}(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^x}.$$

The next proposition follows Proposition 1 of [3]. The objective is to make a generalization of these results in the next section.

Proposition 2.2 (Recurrence for Alternating Odd Zeta Values)**.** *A recurrence formula can be derived for the alternating odd zeta function involving odd numbers. If we consider the function $f(x) = x^{2n-1}$ over the interval $(-\pi, \pi)$, we derive the following recurrence formula:*

$$\hat{\zeta}(2n-1) = \frac{\pi^{2n-1}(-1)^{n+1}}{2^{2n}(2n-1)!} + \sum_{k=1}^{n-1} \frac{(-1)^{k+n+1}\pi^{2(n-k)}}{(2n-2k+1)!}\hat{\zeta}(2k-1), \quad \hat{\zeta}(1) = \frac{\pi}{4}.$$

To the best of our knowledge, this is a new addition to the existing literature on this topic.

Proof. For the deduction of this formula, the idea is analogous to the first formulation. Initially, we observe that the function to be treated is odd, so $a_n = 0$. Then we get

$$b_m = \frac{2}{\pi}\int_0^{\pi} x^{2n-1}\sin(mx)dx.$$

Integrating by parts, we obtain

$$b_m = 2\sum_{k=1}^{n}(-1)^{m+k+1}\frac{\pi^{2n-2k}}{m^{2k-1}}\frac{(2n-1)!}{(2n-2k+1)!}.$$

Thus, evaluating at the point $x = \pi/2$, we have

$$\frac{\pi^{2n-1}}{2^{2n-1}} = 2\sum_{k=1}^{n}(-1)^{k+1}\pi^{2n-2k}\frac{(2n-1)!}{(2n-2k+1)!}\hat{\zeta}(2k-1).$$

Finally, we only need to rearrange the expression, extracting the term $\hat{\zeta}(2n-1)$, to obtain the recurrence formula mentioned at the beginning. ■

Again, due to the form given by the recurrence formula, there is an association with Euler numbers, as follows [5]:

$$\hat{\zeta}(2n+1) = (-1)^n\frac{E_{2n}}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}.$$

3 Extension using Hurwitz Zeta Function

Looking at the different formulas obtained, it is possible to make a generalization using the Hurwitz zeta function, which is defined as [6]

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}.$$

In this case, we should consider the even and odd parts of the previously defined function, as follows:

$$G(s,a) = \zeta(s,a) + \zeta(s,-a)$$

and

$$\overline{G}(s,a) = \zeta(s,a) - \zeta(s,-a).$$

It should be noted that these functions must converge for $s > 1$. Furthermore, there is a special case for $\overline{G}(1,a)$, which converges considering the series of the sum.

Proposition 3.1 (Recurrence Relations for Hurwitz Zeta Extensions)**.** *For the Hurwitz zeta function extensions, the following recurrence relations hold:*

Part 1: *For $f(x) = x^{2n}e^{iax}$ with $|x| \leq \pi$, evaluating at $x = \pi$ and taking the average yields*

$$\begin{aligned}
\frac{\sin(\pi a)}{\pi}(-1)^n\overline{G}(2n+1,a) + \sum_{k=0}^{n-1}\left(\frac{(-1)^k\pi^{2(n-k)}}{(2n-2k-1)!}\left(\frac{\sin(\pi a)\overline{G}(2k+1,a)}{(2n-2k)\pi} + \frac{\cos(\pi a)G(2k+2,a)}{\pi^2}\right)\right) \\
= \frac{\pi^{2n}}{(2n)!}\cos(\pi a) + \frac{\sin(\pi a)}{\pi a}\frac{(-1)^{n+1}}{a^{2n}} \\
+ \sum_{k=0}^{n-1}\left(\frac{(-1)^{k+1}\pi^{2(n-k)}}{(2n-2k-1)!a^{2k}}\left(\frac{\sin(\pi a)}{(2n-2k)\pi a} + \frac{\cos(\pi a)}{(\pi a)^2}\right)\right).
\end{aligned}$$

Part 2: *For $f(x) = x^{2n+1}e^{iax}$ in the same domain, evaluating at $x = \pi$ yields*

$$\begin{aligned}
\sum_{k=0}^{n}\left(\frac{(-1)^k\pi^{2(n-k)}}{(2n-2k)!}\left(\frac{\sin(\pi a)G(2k+2,a)}{\pi} - \frac{\cos(\pi a)\overline{G}(2k+1,a)}{2n-2k+1}\right)\right) \\
= \frac{\pi^{2n+1}}{(2n+1)!}\sin(\pi a)\sum_{k=0}^{n}\left(\frac{(-1)^k\pi^{2(n-k)}}{(2n-2k)!a^{2k+1}}\left(\frac{\cos(\pi a)}{(2n-2k+1)} - \frac{\sin(\pi a)}{\pi a}\right)\right).
\end{aligned}$$

Proof. **Part 1:** Initially, it is possible to find a Fourier series expansion for the function $f(x) = x^{2n}e^{iax}$ with $|x| \leq \pi$. In this case, we proceed analogously to the real cases and obtain, for the complex series,

$$\begin{aligned}
c_m = (-1)^{m+n}\frac{\pi^{2(n-m)}}{(a-m)^{2n-1}}\frac{\sin(\pi a)}{\pi}(2n)! \\
+ \sum_{k=0}^{n-1}\left((-1)^{k+m}\frac{\pi^{2(n-k)}}{(a-m)^{2k+1}}\frac{(2n)!}{(2n-2k-1)!}\left(\frac{\sin(\pi a)}{(2n-2k)\pi} + \frac{\cos(\pi a)}{(a-n)\pi^2}\right)\right).
\end{aligned}$$

Evaluating again at the endpoint $x = \pi$ and taking the average yields the recurrence form.

Note that, in the case $n = 0$, we obtain the first initial condition

$$\overline{G}(1,a) = \frac{\pi}{\tan(\pi a)} - \frac{1}{a}.$$

Part 2: Performing the same process for the function $f(x) = x^{2n+1}e^{iax}$ in the same domain, we get

$$c_m = i\sum_{k=0}^{n}\left((-1)^{k+m+1}\frac{\pi^{2(n-k)}}{(a-m)^{2k+1}}\frac{(2n+1)!}{(2n-2k)!}\left(\frac{\cos(\pi a)}{(2n-2k+1)} - \frac{\sin(\pi a)}{(a-n)\pi}\right)\right).$$

Again, evaluating at $x = \pi$, we obtain the second necessary recurrence formula.

Taking $n = 0$ and using the previous result for $\overline{G}(1,a)$, we obtain the second initial term

$$G(2,a) = \pi^2 + \left(\frac{\pi}{\tan(\pi a)}\right)^2 - \frac{1}{a^2}.$$

∎

In this way, all values for $G(2k, a)$ and $\overline{G}(2k-1, a)$ for $k = 1, 2, \ldots$ can be generated.

Note that these results provide a generalization of the values obtained in the first approach of [2]. The drawback is the emergence of a 'conjugate' function, which requires a system of equations to solve for the even values of the desired function. It is worth noting that, for the second approach, it would remain to use our Fourier series and perform the approximation for $x = \pi/2$, which could be an interesting result for future work.

4 Conclusions

In this article, we gave recursive formulas that allow reducing problems of numerical bounding of the given series to solving linear systems. The result can be obtained analytically, through symbolic programming, which allows obtaining functions $G(2k+2, a)$ and $\overline{G}(2k+1, a)$, with $k \in \mathbb{N}$ and $a \in \mathbb{C}$. The remaining question is whether there exists a closed formula for $G(2k+2, s)$ and $\overline{G}(2k+1, a)$ that involves special numbers, such as Euler numbers or Bernoulli numbers. A good generalization in an associated way is the result obtained at [8].

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