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A New $T - X$ Family of Distributions: Structural Properties and Applications

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Abstract

This study introduces a new class of probability distributions within the transformed-transformer (T–X) family framework, using the paralogistic distribution as the baseline generator. The proposed Paralogistic–X family provides a flexible model capable of capturing various distributional shapes such as skewness and heavy tails. General expressions for its structural properties including the density, distribution, quantile function, moments, and entropy measures are developed in terms of the baseline distribution. A special submodel, the Paralogistic–Burr III (PBIII) distribution, is examined in detail. Maximum likelihood estimation is used for parameter inference, and a simulation study is conducted to evaluate the estimators' performance under varying sample sizes. The results confirm the consistency and efficiency of the estimators across different settings. To assess its practical utility, the PBIII distribution is applied to real-world datasets and compared with four established competing distributions. The comparison, based on both graphical tools and model selection criteria such as AIC, BIC, and log-likelihood, shows that the proposed model offers superior fitting capabilities in the two cases. The findings highlight the versatility and robustness of the Paralogistic–X family for statistical modeling.

1 Introduction

In the last decade, generalized $T - X$ families of distributions have been developed after the work of [5] using several upper bounds (the transformers) of the probability integrals to produce different flexible distributions and families of distributions to fit complex lifetime datasets. [5] expressed the cumulative distribution function (cdf) and probability density function (pdf) of any $T - X$ family as

$$G_{T-X}(y;\beta,\eta) = \int_{c}^{W[F(y;\eta)]} r(t;\beta)dt = R(W[F(y;\eta)];\beta) \tag{1}$$

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and

$$g_{T-X}(y;\beta,\eta) = \left[\frac{d}{dy}W[F(y;\eta)]\right] r(W[F(y;\eta)];\beta), \tag{2}$$

where $\beta > 0$ and $\eta > 0$ is the parameter vector of the baseline distribution defined by $\eta = \eta_k, k = 1, 2, ..., m$. $W[F(y;\eta)]$ is the upper bound of (1).

Let $r(t)$ and $R(t)$ define the pdf and cdf of a random variable $T \in (c, d)$ for $-\infty \leq c < d < \infty$ and let $W[F(y;\eta)]$ be a function of $F(y;\eta)$ of random variable X satisfying the conditions:

1. $W[F(y;\eta)] \in [c, d]$,
2. $W[F(y;\eta)]$ is differentiable and monotonically increasing, and
3. $W[F(y;\eta)] \rightarrow c$ as $y \rightarrow -\infty$ and $W[F(y;\eta)] \rightarrow d$ as $y \rightarrow \infty$.

The conditions for the supports for $(c, W[F(y)])$ are presented in [5]. Probability density functions of some well-known baseline distributions as $r(t;\beta)$ have been transformed $G(y;\beta)$ in (1) to new families of distributions.

Several $T - X$ families of distributions have been developed in recent years by employing (1). [1] introduced the generalized Burr X-G family, [20] developed the new generalized Logarithmic-X family of distributions, [10] proposed the continuous and discrete new $T - X^{\theta}$ families of distributions and [11] developed a new five-parameter Weibull-Lomax distribution by extending the Weibull-Lomax distribution [23] using the Weibull-G family.

This paper focuses on the derivation of the general structural properties of a new Paralogistic–X family of distributions. The pdf of a less-studied baseline distribution, the *paralogistic* distribution, is employed to construct a new T–X family of distributions with flexible submodels for modeling complex datasets. The motivations for this study are as follows:

1. To give greater credibility to the paralogistic distribution as a viable baseline distribution;
2. To construct and explore a new T–X family of distributions based on the paralogistic baseline;
3. To develop the structural properties of a particular submodel, the Paralogistic–Burr III (PBIII) distribution; and
4. To apply the PBIII distribution to real lifetime datasets and compare its performance with that of existing distributions in the literature.

The structure of the paper is as follows. Section 2 introduces the construction of the Paralogistic–X family as a new generalized class of distributions. Section 3 discusses the structural properties of the class. In Section 4, the Paralogistic–Burr III (PBIII) distribution is examined, and some of its statistical properties are derived. Section 5 applies the PBIII distribution to two real lifetime datasets, comparing its performance with that of several competing models to evaluate its flexibility. Finally, Section 6 concludes the paper.

2 Construction of the Paralogistic-X Family and its Submodels

The paralogistic distribution is developed as a submodel of the modified generalized gamma distribution (MGGD) [15]. The cdf of the MGGD is given as

$$R(y;\phi,k,\beta,c,\zeta)=\frac{\gamma_\zeta(\phi,k,\beta,c)}{\Gamma_\zeta(\phi,k,\beta)}, y>0,\phi>0,k>0,\beta>0,c>0. \tag{3}$$

where $c=(\frac{x}{\delta})^\eta$ and $\Gamma_\zeta(\phi,k,\beta)=\int_0^\infty w^{(\phi-1)}(w+k)^{-\zeta}e^{-\beta w}dw$ is the modified generalized gamma function. The function $\gamma_\zeta(\phi,k,\beta,c)$ is defined as

$$\gamma_\zeta(\phi,k,\beta,c)=\int_0^c y^{(\phi-1)}(y+k)^{-\zeta}e^{-\beta y}dy=\Gamma_\zeta(\phi,k,\beta)-\Gamma_\zeta(\phi,k,\beta,c), \tag{4}$$

where $\Gamma_\zeta(\phi,k,\beta,c)$ is the upper incomplete modified generalized gamma function.

If $\beta=0$, $k=1$, $\zeta=\beta+1$, $\delta=1$ and $\phi=1$, the cdf of the one-parameter paralogistic distribution is defined as

$$R(y;\beta)=1-(1+y^\beta)^{-\beta}, y>0,\beta>0. \tag{5}$$

The MGGD is a generalization of the works of [22], [28], [12], [9], [3] and [2] for which well known distributions such as Dagum, Singh Maddala, Weibull, exponential, Lomax, paralogistic, Burr III, inverse Lomax, Burr XII, inverse paralogistic and Rayleigh distributions ([17], [14], [24], [13], [7], [25] and [18]) can be derived.

2.1 Paralogistic-X family

Let $r(t)=\beta^2t^{\beta-1}(1+t^\beta)^{-\beta-1}$ be the pdf of the one-parameter paralogistic distribution, $W[F(y;\eta)]=\frac{F(y;\eta)}{1-F(y;\eta)}$ and $c=0$ in (1), the cdf of the new generalized $T-X$ family is expressed as

$$G(y;\beta,\eta)=1-\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^\beta\right)^{-\beta}, y>0,\beta>0. \tag{6}$$

where $F(y;\eta)$ defines the cdf of any baseline distribution. The new generalized $T-X$ family in (6) will be known as the Paralogistic-X family of distributions.

The corresponding pdf, survival, hazard and reversed hazard functions to (6) are expressed as

$$g(y;\beta,\eta)=\frac{\beta^2 f(y;\eta)[F(y;\eta)]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta-1}}{1-F(y;\eta)}, \tag{7}$$

$$\bar{G}(y;\beta,\eta)=\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta} \tag{8}$$

and

$$h(y;\beta,\eta)=\frac{\beta^2 f(y;\eta)[F(y;\eta)]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-1}}{(1-F(y;\eta))}.$$

2.2 Series expansion of the pdf of the Paralogistic-X family

The expansion of the pdf of the Paralogistic-X family plays an important role in the derivations of the general forms of the properties of the family. The useful series expansions that will be useful for simplifying the pdf are given as

$$(1-x)^n=\sum_{i=0}^{\infty}\binom{n}{i}(-1)^i x^i,$$

and

$$(1+x)^{-n}=\sum_{i=0}^{\infty}\binom{n+i-1}{i}(-1)^i x^i,$$

where $|x|>0$ and $n>0$ is a positive interger.

In (7), the pdf of the Paralogistic-X family is defined as

$$g(y;\beta;\eta)=\frac{\beta^2 f(y;\eta)[F(y;\eta)]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta-1}}{1-F(y;\eta)}.$$

By series expansion,

$$\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta-1}=\sum_{i=0}^{\infty}\binom{\beta+i}{i}(-1)^i\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{i\beta}.$$

The pdf in (7) becomes

$$\begin{aligned} g(y;\beta,\eta) &= \beta^2 \sum_{i=0}^{\infty} \binom{\beta+i}{i} (-1)^i \left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta(i+1)} f(y;\eta)([F(y;\eta)][1-F(y;\eta)])^{-1} \\ &= \beta^2 \sum_{i=0}^{\infty} \binom{\beta+i}{i} (-1)^i [F(y;\eta)]^{\beta(i+1)-1} [1-F(y;\eta)]^{-(\beta(i+1)+1)} f(y;\eta). \end{aligned} \tag{9}$$

If $F(y;\eta) = 1-\bar{F}(y;\eta)$, then (9) becomes

$$g(y;\beta,\eta) = \beta^2 \sum_{i=0}^{\infty} \binom{\beta+i}{i} (-1)^i [1-\bar{F}(y;\eta)]^{\beta(i+1)-1} [\bar{F}(y;\eta)]^{-(\beta(i+1)+1)} f(y;\eta).$$

Substituting $[1-\bar{F}(y;\eta)]^{\beta(i+1)-1} = \sum_{j=0}^{\infty} \binom{\beta(i+1)-1}{j} (-1)^j [\bar{F}(y;\eta)]^j$ into the above expression, the pdf of the Paralogistic-X family is expressed in series expansion as

$$g(y;\beta,\eta) = \beta^2 \sum_{i,j=0}^{\infty} \binom{\beta+i}{i} \binom{\beta(i+1)-1}{j} (-1)^{i+j} [\bar{F}(y;\eta)]^{j-(\beta(i+1)+1)} f(y;\eta). \tag{10}$$

3 Structural Properties of the Paralogistic-X Family

This section considers the derivations of the general forms of the properties of the Paralogistic-X family in terms of the pdf and cdf of any baseline distribution. The properties include quantile function, noncentral (raw) moments, incomplete moments, moment generating function, probability weighted moments, order statistic, entropies and mean deviations.

3.1 Quantile function

Quantile function is used to obtain random numbers which are used in simulation study. For $0 < q < 1$, the quantile function for a random variable X following the Paralogistic-X family is obtained by inverting (6) as

$$y_q = Q(u) = G^{-1}(q).$$

Hence,

$$y_q = F^{-1}\left(\frac{[(1-q)^{-\frac{1}{\beta}}-1]^{\frac{1}{\beta}}}{1+[(1-q)^{-\frac{1}{\beta}}-1]^{\frac{1}{\beta}}}\right) = F^{-1}((((1-q)^{-\frac{1}{\beta}}-1)^{-\frac{1}{\beta}}+1)^{-1}), \tag{11}$$

where $F^{-1}(q)$ defines the quantile function of the baseline distribution with cdf, $F(x)$.

If $q = 0.5$ in (11), then the median of the Paralogistic-X family is defined as

$$y_{0.5} = F^{-1}\left((((0.5)^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\beta}} + 1)^{-1} \right). \tag{12}$$

3.2 Noncentral (raw), incomplete and probability weighted moments

3.2.1 Noncentral (raw) moment

Let Y be a random variable following the pdf in (7), the r^{th} ($r = 1(1)m$, m is a positive integer) raw moments of the Paralogistic-X family is obtained as

$$\mu_r = E(X^r) = \int_0^\infty y^r g(y;\beta,\eta)dx. \tag{13}$$

Substituting the series expansion in (10) into (13), the r^{th} raw moments of the proposed family is expressed as

$$\mu_r = \beta^2 \sum_{i,j=0}^{\infty} \binom{\beta+i}{i} \binom{\beta(i+1)-1}{j} (-1)^{i+j} \int_0^\infty y^r [\bar{F}(y;\eta)]^{j-(\beta(i+1)+1)} f(y;\eta)dx. \tag{14}$$

3.2.2 Incomplete moment

The r^{th} incomplete moments for a random variable Y following the pdf in (7) is obtained as

$$\mu_r^t = E(Y^r/Y > t) = \frac{1}{\bar{G}(t;\beta,\eta)} \int_t^\infty y^r g(y;\beta,\eta)dx. \tag{15}$$

Substituting (10) into (15), the r^{th} incomplete moments for the Paralogistic-X family is given as

$$\begin{aligned} \mu_r^t = & \frac{\beta^2}{1 - \left(1 + \left[\frac{F(t;\eta)}{1-F(t;\eta)}\right]^\beta\right)^{-\beta}} \sum_{i,j=0}^{\infty} \binom{\beta+i}{i} \binom{\beta(i+1)-1}{j} (-1)^{i+j} \\ & \int_t^\infty y^r [\bar{F}(y;\eta)]^{j-(\beta(i+1)+1)} f(y;\eta)dx. \end{aligned} \tag{16}$$

3.2.3 Probability weighted moments

The probability weighted moments, $PWM_{r,l,m}$, for a random variable Y following the pdf in (7) is obtained as

$$PWM_{r,l,m}(Y)=E[Y^r G^l(Y)\bar{G}^m(Y)]=\int_0^\infty y^r G^l(y;\beta,\eta)\bar{G}^m(y;\beta,\eta)g(y;\beta,\eta)dx. \tag{17}$$

Substituting (6), (7) and (8) into (17), the $PWM_{r,l,m}$ for the Paralogistic-X family is given as

$$\begin{aligned}PWM_{r,l,m}(Y)=\beta^2\int_0^\infty y^r f(y;\eta)[F(y;\eta)(1-F(y;\eta))]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\\ \left[1-\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta}\right]^{l}\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta(m+1)-1}dx.\end{aligned} \tag{18}$$

After some algebraic evaluations, (18) becomes

$$\begin{aligned}PWM_{r,l,m}(Y)=\beta^2\sum_{k=0}^{l}\sum_{i=0}^{\infty}\binom{l}{k}\binom{\beta(m+k+1)+i}{i}(-1)^{i+j}\int_0^\infty y^r f(y;\eta)\\ [F(y;\eta)(1-F(y;\eta))]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta(j+1)}dx.\end{aligned}$$

Further algebraic evaluations gives probability weighted moments, $PWM_{r,l,m}$, of the Paralogistic-X family as

$$\begin{aligned}PWM_{r,l,m}(Y)=\beta^2\sum_{k=0}^{l}\sum_{i,j=0}^{\infty}\binom{l}{k}\binom{\beta(m+k+1)+i}{i}\binom{\beta(i+1)-1}{j}\\ (-1)^{i+j+k}\int_0^\infty y^r f(y;\eta)[\bar{F}(y;\eta)]^{-\beta(i+1)-1+j}dy.\end{aligned} \tag{19}$$

If $l=m=0$, then

$$\begin{aligned}PWM_{r,0,0}(Y)=E[Y^r]=\beta^2\sum_{k=0}^{l}\sum_{i,j=0}^{\infty}\binom{l}{k}\binom{\beta(m+k+1)+i}{i}\binom{\beta(i+1)-1}{j}\\ (-1)^{k+i+j}\int_0^\infty y^r f(y;\eta)[\bar{F}(y;\eta)]^{-\beta(i+1)-1+j}dx\end{aligned}$$

gives the r^{th} raw moments of the proposed family. If $m=0$, then

$$\begin{aligned}PWM_{r,l,0}(Y)=E[Y^r G^l(Y)]=\beta^2\sum_{k=0}^{l}\sum_{i,j=0}^{\infty}\binom{l}{k}\binom{\beta(k+1)+i}{i}\binom{\beta(i+1)-1}{j}\\ (-1)^{i+j+k}\int_0^\infty y^r f(y;\eta)[\bar{F}(y;\eta)]^{-\beta(i+1)-1+j}dx\end{aligned}$$

gives the r^{th} raw moments of the proposed family.

3.3 Moment generating function

The moment generating function ($M_Y(t)$) for a random variable Y with pdf in (7) is obtained as

$$M_Y(t) = E(e^{tY}) = \int_0^\infty e^{ty} g(y;\beta,\eta)dx, t > 0. \tag{20}$$

Inputting (10) in (20) gives

$$M_Y(t) = \beta^2 \sum_{i,j=0}^{\infty} \binom{\beta+i}{i}\binom{\beta(i+1)-1}{j}(-1)^{i+j}\int_0^\infty e^{tx}[\bar{F}(y;\eta)]^{j-(\beta(i+1)+1)} f(y;\eta)dx \tag{21}$$

which defines the generalized moment generating function for a random variable Y.

3.4 Order statistic

Let $Y_n, n = 1(1)m$ be the n^{th} order statistic where $Y_1, Y_2, ..., Y_m$ define a random sample of m variables, then pdf of n^{th} order statistic for Paralogistic-X family can be obtained as

$$g_{r,n}(y;\beta,\eta) = \frac{n!}{(n-r)!(r-1)!}[G(y;\beta,\eta)]^{r-1}[1-G(y;\beta,\eta)]^{n-r} g(y;\beta,\eta). \tag{22}$$

Replacing $G(y;\beta,\eta)$ with $1-\bar{G}(y;\beta,\eta)$ in (22) gives

$$\begin{aligned} g_{r,n}(y;\beta,\eta) &= \frac{n!}{(n-r)!(r-1)!}\left[1-\bar{G}(y;\beta,\eta)\right]^{r-1}\left[\bar{G}(y;\beta,\eta)\right]^{n-r} g(y;\beta,\eta) \\ &= \frac{n!}{(n-r)!(r-1)!}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i\left[\bar{G}(y;\beta,\eta)\right]^{n+i-r} g(y;\beta,\eta). \end{aligned}$$

Substituting (8) and (10) into the above expression give

$$g_{r,n}(y;\beta,\eta) = \frac{n!\beta^2}{(n-r)!(r-1)!}\sum_{i=0}^{r-1}\binom{r-1}{i}(-1)^i f(y;\eta)\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}$$
$$\left[\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)\right]^{-\beta(n+i+1-r)-1} [F(y;\eta)(1-F(y;\eta))]^{-1}.$$

After some algebraic simplifications, the n^{th} order statistics of the Paralogistic-X family is given as

$$g_{r,n}(y;\beta,\eta) = \frac{n!\beta^2}{(n-r)!(r-1)!}\sum_{i=0}^{r-1}\sum_{j,k=0}^{\infty}\binom{r-1}{i}\binom{\beta(n+i+1-r)+j}{j}\binom{\beta(j+1)-1}{k} (-1)^{i+j+k} f(y;\eta)\left[\bar{F}(y;\eta)\right]^{k-\beta(j+1)-1}. \tag{23}$$

3.5 Rényi and Shannon entropies

The entropies have been used in lifetime modelling to obtain information about the variation of randomness associated with random variables defining failure times of phenomena. These measures have made important contributions in many scientific fields where lifetime modelling is significant.

The Rényi entropy [19] for a random variable Y following the pdf in (7) is obtained as

$$I_R(\rho) = \frac{1}{(1-\rho)} log\left(\int_0^\infty g^\rho(y;\beta,\eta)dy\right), \rho > 0, \rho \neq 1. \tag{24}$$

Substituting (7) into (24) gives

$$I_R(\rho) = \frac{1}{(1-\rho)}\log\left(\int_0^\infty \left[\frac{\beta^2 f(y;\eta)[F(y;\eta)]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^\beta\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^\beta\right)^{-\beta-1}}{1-F(y;\eta)}\right]^\rho dy\right). \tag{25}$$

By expansion of

$$\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^\beta\right)^{-(\beta+1)\rho} = \sum_{i=0}^{\infty}\binom{(\beta+1)\rho+i-1}{i}(-1)^i\left(\frac{F(y;\eta)}{1-F(y;\eta)}\right)^{\beta i},$$

it is seen that

$$I_R(\rho) = \frac{1}{(1-\rho)}\log\left(\beta^{2\rho}\sum_{i=0}^{\infty}\binom{(\beta+1)\rho+i-1}{i}(-1)^i\int_0^\infty f^\rho(y;\eta)[F(y;\eta)]^{-\rho} \left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta(\rho+i)}(1-F(y;\eta))^{-\rho}dy\right).$$

Letting $\bar{F}(y;\eta) = 1 - F(y;\eta)$,

$$I_R(\rho) = \frac{1}{(1-\rho)} log\Bigg(\beta^{2\rho} \sum_{i=0}^{\infty} \binom{(\beta+1)\rho+i-1}{i} (-1)^i \int_0^{\infty} f^{\rho}(y;\eta)[1-\bar{F}(y;\eta)]^{-\rho}$$

$$\left[\frac{1-\bar{F}(y;\eta)}{\bar{F}(y;\eta)}\right]^{\beta(\rho+i)} (\bar{F}(y;\eta))^{-\rho} dy\Bigg)$$

$$= \frac{1}{(1-\rho)} log\Bigg(\beta^{2\rho} \sum_{i=0}^{\infty} \binom{(\beta+1)\rho+i-1}{i} (-1)^i \int_0^{\infty} f^{\rho}(y;\eta)[1-\bar{F}(y;\eta)]^{\beta(\rho+i)-\rho}$$

$$(\bar{F}(y;\eta))^{-\beta(\rho+i)-\rho} dy\Bigg).$$

With some algebraic manipulations, the Rényi entropy for the Paralogistic-X family can be defined as

$$\begin{aligned} I_R(\rho) = \frac{1}{(1-\rho)} \log \Bigg(\beta^{2\rho} \sum_{i,j=0}^{\infty} \binom{(\beta+1)\rho+i-1}{i} \binom{\beta(\rho+i)-\rho}{j} (-1)^{i+j} \\ \int_0^{\infty} f^{\rho}(y;\eta)[\bar{F}(y;\eta)]^{-(\beta(\rho+i)+\rho)+j} dy. \end{aligned} \tag{26}$$

The Shannon entropy [21] for the Paralogistic-X family can be obtained as

$$H_S[g(y;\beta,\eta)] = E[-\log(g(Y))] = -\int_0^{\infty} g(y;\beta,\eta) \log\left(g(y;\beta,\eta)\right) dx. \tag{27}$$

Substituting (7) into (26) gives,

$$H_S[g(y;\beta,\eta)] = E\left[-\log\left(\frac{\beta^2 f(y;\eta)[F(y;\eta)]^{-1}\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\left(1+\left[\frac{F(y;\eta)}{1-F(y;\eta)}\right]^{\beta}\right)^{-\beta-1}}{1-F(y;\eta)}\right)\right].$$

Hence, the Shannon entropy for the Paralogistic-X family is given as

$$\begin{aligned} H_S[g(y;\beta,\eta)] &= -2\log\beta - E[\log f(y;\eta)] - (\beta-1)E[\log F(y;\eta)] \\ &+ (\beta-1)E[\log\left(1-F(y;\eta)\right)] - E\left[\log\left(1+\left(\frac{F(y;\eta)}{1-F(y;\eta)}\right)^{\beta}\right)\right]. \end{aligned} \tag{28}$$

3.6 Mean deviations

Mean deviations (MD) about the mean and median [27] are measures of the variations in a value in a data set from the mean and median. The mean deviations about the mean (μ) and median (M) for any random variable Y following the pdf defined in (7) are defined as

$$MD_\mu = E(|Y-\mu|) = \int_0^\mu |y-\mu| g(y)dy = 2\mu G(\mu) - 2\int_0^\mu yg(y)dy,$$
$$MD_M = E(|Y-M|) = \int_0^M |y-M| g(y)dy = \mu - 2\int_0^M yg(y)dy.$$

Substituting (6) and (10) into the above equations, the mean deviations of the Paralogistic-X family become

$$MD_\mu = 2\mu\left[1-\left(1+\left[\frac{F(\mu;\eta)}{1-F(\mu;\eta)}\right]^\beta\right)^{-\beta}\right] - 2\beta^2\sum_{i,j=0}^{\infty}\binom{\beta+i}{i}\binom{\beta(i+1)-1}{j} (-1)^{i+j}\int_0^\mu xf(y;\eta)[\bar{F}(y;\eta)]^{-\beta(i+1)-1+j}dx \tag{29}$$

and

$$MD_M = \mu - 2\beta^2\sum_{i,j=0}^{\infty}\binom{\beta+i}{i}\binom{\beta(i+1)-1}{j}(-1)^{i+j}\int_0^M xf(y;\eta)[\bar{F}(y;\eta)]^{-\beta(i+1)-1+j}dx. \tag{30}$$

3.7 Parameter Estimation

The maximum likelihood estimates (MLEs) of the parameters (β,η) of the Paralogistic-X family will be determined for aan n uncensored dataset. Let $x_1, x_1, ..., x_n$ be n observed values from the Paralogistic-X family and $\Omega = (\beta,\eta)^T$ be the $(v\times 1)$ parameter vector. The total log-likelihood function for $\Omega = (\beta,\eta)^T$ is

$$l(\Omega) = 2n\log\beta + \sum_{i=1}^{n}\log[f(y;\eta)] + (\beta-1)\sum_{i=1}^{n}\log[F(y;\eta)] - (\beta+1)\sum_{i=1}^{n}\log[1-F(y;\eta)] - (\beta+1)\sum_{i=1}^{n}\log\left[1-\left(\frac{F(y;\eta)}{1-F(y;\eta)}\right)^\beta\right]. \tag{31}$$

The partial derivatives to (31) with respect to $\Omega = (\beta, \eta)^T$ are given by

$$\begin{aligned}\frac{\partial l(\Omega)}{\partial \beta} &= \frac{2n}{\beta} + \sum_{i=1}^{n} \log F(y_i;\eta) - \sum_{i=1}^{n} \log\left[1 - F(y_i;\eta)\right] \\ &- \sum_{i=1}^{n} \log\left(1 + \left(\frac{F(y_i;\eta)}{1 - F(y_i;\eta)}\right)^{\beta}\right)\end{aligned} \tag{32}$$

and

$$\begin{aligned}\frac{\partial l(\Omega)}{\partial \eta_k} &= \sum_{i=1}^{n} \frac{\partial f(y_i;\eta)/\partial \eta_k}{f(y_i;\eta)} + (\beta - 1)\sum_{i=1}^{n} \frac{\partial F(y_i;\eta)/\partial \eta_k}{F(y_i;\eta)} + (\beta + 1)\sum_{i=1}^{n} \frac{\partial F(y_i;\eta)/\partial \eta_k}{(1 - F(y_i;\eta))} \\ &- (\beta + 1)\sum_{i=1}^{n} \frac{(F(y_i;\eta))^{\beta-1}\partial F(y_i;\eta)/\partial \eta_k}{(1 - F(y_i;\eta))^{\beta+1}\left(1 + \left(\frac{F(y_i;\eta)}{1-F(y_i;\eta)}\right)^{\beta}\right)}.\end{aligned} \tag{33}$$

The nonlinear expressions in (32) and (33) are equated to zero to obtain the estimators for the parameters in $\Omega = (\beta, \eta)^T$. These equations are solved by iterative methods such as Newton–Raphson type algorithms.

4 Paralogistic-Burr III Distribution as a Special Model

In the subsequent sections, we derive several new models from the proposed Paralogistic–X family. Particular attention is given to a specific submodel, the Paralogistic–Burr III (PBIII) distribution, which is discussed in detail. Some of its statistical properties are explored comprehensively. Parameter estimation is performed using the maximum likelihood estimation (mle) technique, and a simulation study is conducted to assess the performance of the estimators under different sample sizes. Finally, the practical applicability of the PBIII distribution is demonstrated through its application to real-world datasets, where its performance is compared with that of existing competing models.

4.1 Some new submodels

Several new submodels can be generated from the general formulation in (6) by selecting appropriate baseline distributions. Below are illustrative examples of such submodels:

1. When the cdf of the exponential distribution is given by $F(y) = 1 - e^{-\delta y}$, the resulting distribution

is referred to as the Paralogistic-Exponential (PExp) distribution, with cdf as

$$G(y;\beta,\delta)=1-\left(1+\left(e^{\delta y}-1\right)^{\beta}\right)^{-\beta},\quad y>0,\ \beta,\delta>0.$$

2. If the Weibull distribution is used with cdf as $F(y)=1-e^{-\delta y^{\phi}}$, the resulting distribution is termed the Paralogistic-Weibull (PWeib) distribution:

$$G(y;\beta,\delta,\phi)=1-\left(1+\left(e^{\delta y^{\phi}}-1\right)^{\beta}\right)^{-\beta},\quad y>0,\ \beta,\delta,\phi>0.$$

3. Substituting the Burr XII cdf given as $F(y)=1-(1+y^{\delta})^{-\phi}$ into (6) yields the Paralogistic-Burr XII (PBXII) distribution:

$$G(y;\beta,\delta,\phi)=1-\left(1+\left((1+y^{\delta})^{\phi}-1\right)^{\beta}\right)^{-\beta},\quad y>0,\ \beta,\delta,\phi>0.$$

4. If the Lomax distribution, expressed as $F(y)=1-(1+\delta y)^{-\phi}$, is used, the resulting model is the Paralogistic-Lomax (PLom) distribution:

$$G(y;\beta,\delta,\phi)=1-\left(1+\left((1+\delta y)^{\phi}-1\right)^{\beta}\right)^{-\beta},\quad y>0,\ \beta,\delta,\phi>0.$$

These derivations highlight the flexibility of the proposed Paralogistic–X family in generating a wide array of submodels by choosing different baseline distributions. A key novelty of this work lies in the use of the paralogistic distribution, a baseline that has received relatively limited attention in the literature. Its incorporation into the T–X framework introduces new distributional forms with enhanced flexibility and modeling power, particularly for skewed and heavy-tailed data. In the following section, we focus on a particularly important submodel, the Paralogistic–Burr III (PBIII) distribution. This model is studied in detail, including its structural properties, parameter estimation, and applications to real-life datasets.

4.2 Paralogistic-Burr III distribution

The cdf and pdf of the Burr III distribution [7] are expressed, respectively, by $F(y)=(1+y^{-\delta})^{-\phi}$ and $f(y)=\delta\phi y^{-\delta-1}(1+y^{-\delta})^{-\phi-1}$, where $y>0$, $\delta>0$, $\phi>0$. Inserting the cdf and pdf of the Burr III distribution into (6) and (7), the cdf of the Paralogistic-Burr III (PBIII) distribution is given as

$$G(y;\beta,\delta,\phi)=1-\left(1+\left(\frac{(1+y^{-\delta})^{-\phi}}{1-(1+y^{-\delta})^{-\phi}}\right)^{\beta}\right)^{-\beta},y>0,\beta,\delta,\phi>0. \tag{34}$$

Alternatively, equation (36) can be written as

$$G(y;\beta,\delta,\phi) = 1 - \left(1 + \left((1+y^{-\delta})^{\phi} - 1\right)^{-\beta}\right)^{-\beta}. \tag{35}$$

The corresponding pdf of the PBIII distribution is given as

$$g(y;\beta,\delta,\phi) = \beta^2\delta\phi y^{-\delta-1}(1+y^{-\delta})^{\phi-1}((1+y^{-\delta})^{\phi}-1)^{-\beta-1}(1+((1+y^{-\delta})^{\phi}-1)^{-\beta})^{-\beta-1}.$$

The hazard function of the PBIII distribution is expressed as

$$h(y;\beta,\delta,\phi) = \beta^2\delta\phi y^{-\delta-1}(1+y^{-\delta})^{\phi-1}((1+y^{-\delta})^{\phi}-1)^{-\beta-1}(1+((1+y^{-\delta})^{\phi}-1)^{-\beta})^{-1}.$$

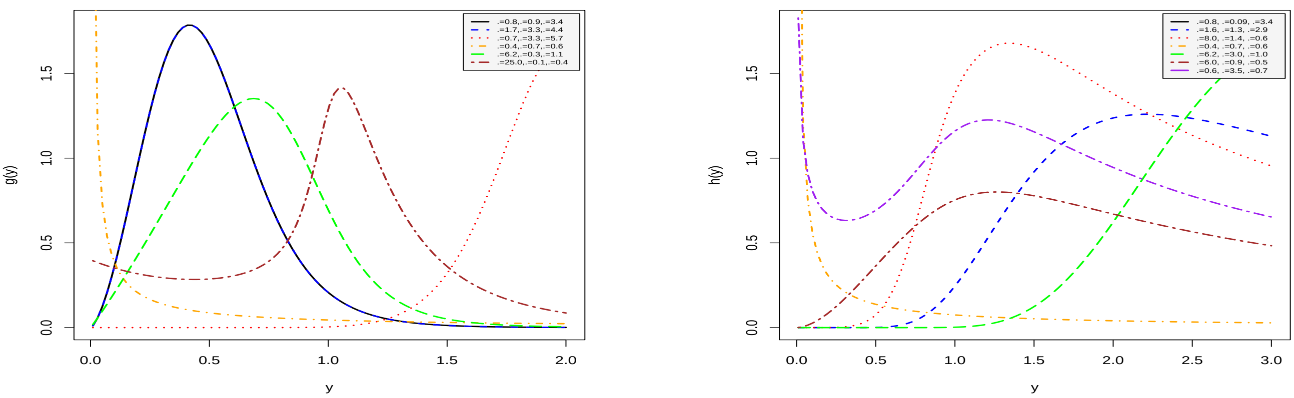

Figure 1: Plots of the PBIII PDF $g(y)$ and hazard function $h(y)$.

Figure 1 presents the plots for the pdf and hazard function of the PBIII distribution. The important statistical properties of the PBIII distribution are presented as follows.

(i) **Quantile function**

The quantile function of the Burr III distribution be defined as

$$y_u = F^{-1}(u) = (u^{-\frac{1}{\phi}} - 1)^{-\frac{1}{\delta}}. \tag{36}$$

Substuting $u = (((1-q)^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\beta}} + 1)^{-1}$ in (11) into (38) gives the quantile function of the PBIII distribution which is expressed as

$$y_q = F^{-1}(q) = ((((1-q)^{-\frac{1}{\beta}} - 1)^{-\frac{1}{\beta}} + 1)^{\frac{1}{\phi}} - 1)^{-\frac{1}{\delta}}. \tag{37}$$

Table 1 presents the quantiles of the PBIII distribution at combinations of parameter values.

Table 1: Quantile values for PBIII distribution

	(0.7,3.0,0.5)	(5.0,0.8,14.0)	(6.0,0.2,0.4)	(3.0,1.5,8.0)	(0.9,0.8,0.9)
0.1	0.02032186	1.381978	0.4389873	1.548551	0.04692977
0.2	0.03618052	1.518528	1.5570301	1.701445	0.16190295
0.3	0.05164806	1.644823	3.2545591	1.821672	0.36784451
0.4	0.06779923	1.778321	6.7413920	1.932325	0.72127466
0.5	0.08557811	1.930564	15.0133670	2.043796	1.34203660
0.6	0.10620795	2.116137	38.7288806	2.164782	2.51092215
0.7	0.13182411	2.361218	129.2388349	2.307279	5.01475087
0.8	0.16711515	2.727856	379.8201492	2.495974	11.84475270
0.9	0.22795051	3.440451	379.8202471	2.810713	44.62486650

Table 1 presents selected quantile values of the PBIII distribution under five different combinations of parameter values. It highlights how the distribution's shape and spread are highly sensitive to parameter changes. For some parameter sets, the quantiles increase gradually, indicating a relatively light-tailed behavior. In contrast, parameter values (6.0, 0.2, 0.4) exhibits extremely increase for higher quantiles, showing heavy-tailed characteristics and strong skewness. Other combinations produce moderate but steadily increasing pattern of quantiles, reflecting a wider spread. In general, the table demonstrates the flexibility of the PBIII distribution in modeling diverse data behaviors, ranging from light-tailed to heavy-tailed cases, depending on the parameter values.

(ii) **Moments**

The pdf and survival function of the Burr III distribution are defined as

$$f(y;\delta,\phi) = \delta\phi y^{-\delta-1}(1+y^{-\delta})^{-\phi-1}, y > 0$$

and

$$\bar{F}(y;\beta,\delta) = 1 - (1+y^{-\delta})^{-\phi}.$$

Substituting the $f(y;\beta,\delta)$ and $\bar{F}(y;\beta,\delta)$ into (14) gives

$$\mu_r = \beta^2\delta\phi \sum_{i,j=0}^{\infty} \binom{\beta+i}{i}\binom{\beta(i+1)-1}{j}(-1)^{i+j}$$
$$\int_0^{\infty} y^{r-\delta-1}(1-(1+y^{-\delta})^{-\phi})^{j-(\beta(i+1)+1)}(1+y^{-\delta})^{-\phi-1}dy. \tag{38}$$

Using series expansion and further evaluations of the integrand in (40) gives

$$\mu_r = \beta^2\phi \sum_{i,j,k=0}^{\infty} \binom{\beta+i}{i}\binom{\beta(i+1)-1}{j}\binom{j-(\beta(i+1)+1)}{k}(-1)^{i+j+k}$$
$$\mathcal{B}\left(\phi(k+1)+\frac{r}{\delta}, 1-\frac{r}{\delta}\right). \tag{39}$$

Equation (41) gives the r^{th} raw moments of the PBIII distribution.

The raw moments for $r = 1, 2, 3, 4, 5$ and related statistics are computed for some sets of parameter values of the PBIII distribution and presented in Table 2.

Table 2: First four raw moments and related measures of PBIII distribution

	(0.5,0.8,0.6)	(0.8,4.0,1.3)	(0.7,1.9,0.3)	(5,6.4,0.9)	(3,2.7,1)
E(X)	0.05303267	0.02612847	0.027837802	0.01553630	0.12786139
$E(X^2)$	0.02774262	0.01996596	0.014474696	0.01467527	0.10876932
$E(X^3)$	0.01873979	0.01609442	0.009727933	0.01389816	0.09428802
$E(X^4)$	0.01414059	0.01345512	0.007318086	0.01319390	0.08299503
$E(X^5)$	0.01135185	0.01154770	0.005863368	0.01255322	0.07398179
SD	0.15789288	0.13886418	0.117045943	0.12014113	0.30400787
CV	2.97727593	5.31467021	4.204568405	7.73293123	2.37763613
CS	3.71524534	5.43927888	5.339722403	7.62449202	2.01969015
CK	17.07080901	31.87736886	33.569167881	59.28504417	5.22610848

Table 2 summarizes the first four raw moments and related measures of the PBIII distribution under five sets of parameter values. The computed means are small signifying concentration near the lower tail, while higher-order moments increase steadily, reflecting sensitivity to parameter variation. Standard deviations and large coefficients of variation show substantial relative dispersion. From the table, it is evident that the distribution is consistently right-skewed and leptokurtic, suggesting heavy-tailed behavior which indicates the fit of the distribution for extreme values. These characteristics demonstrate the flexibility of the PBIII distribution for modeling data exhibiting high variability, skewness, and heavy tails, particularly in applications such as finance, reliability, and environmental studies.

4.2.1 Simulation study

In this section, the Monte-Carlo simulation study for the PBIII distribution is to determine the asymptotic properties of maximum likelihood estimators of unknown parameters given as $\Omega = (\delta, \phi, \beta)$. Some sets of parameter values for parameter vector, Ω, at $n = 25, 50, 100, 200, 400$ samples generated from a large $N = 2000$ samples. This enables the true sampling distribution of data be randomly generated using the PBIII quantile function.

Four sets of true parameter values are presented for the simulation study, which are; Set I: $\delta = 0.6, \phi = 1.7, \beta = 0.8$, Set II: $\delta = 1.3, \phi = 1.0, \beta = 0.5$, Set III: $\delta = 1.5, \phi = 0.3, \beta = 4.2$ and Set IV: $\delta = 2.5, \phi = 1.7, \beta = 1.0$. Three quantities are computed in the simulation study to determine the behaviour of the maximum likelihood estimates as sample size (n) increases. These quantities are

(i) $\text{Average}(\Omega) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\hat{\Omega}_i.$

(ii) $Bias(\Omega) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}(\hat{\Omega}_i - \Omega).$

(iii) $RMSE(\Omega) = \sqrt{\dfrac{1}{N}\displaystyle\sum_{i=1}^{N}(\hat{\Omega}_i - \Omega)^2}.$

Table 3 gives simulation results for the PBIII distribution across four sets of parameter values for varying sample sizes. For all parameters values of δ, ϕ, β, the average estimates approach the true values as n increases, with corresponding decreases in bias and RMSE, indicating the consistency of the estimators. At small samples (such as $n = 25, 50$), estimates exhibit noticeable bias, particularly for β, but this diminishes rapidly with larger n. For all parameter value sets, δ and ϕ demonstrate stable convergence with modest error, while β shows relatively higher initial variability before stabilizing at $n \geq 200$. Generally, the simulation validates the reliability of the proposed estimation method for the PBIII distribution, with efficiency improving markedly as sample size increases.

4.3 Parameter estimation

Let $X\sim$ PBIII distribution with a random sample of complete observations given as $x_1, x_2, ..., x_n$. The corresponding total log-likelihood function is

Table 3: Simulation study results for the PBIII distribution

Ω	n	Set I			Set II		
		$Av(\Omega)$	$Bias(\Omega)$	$RMSE(\Omega)$	$Av(\Omega)$	$Bias(\Omega)$	$RMSE(\Omega)$
δ	25	0.581713	-0.018287	0.375565	0.818575	-0.481425	0.718718
	50	0.584726	-0.015274	0.316005	0.822321	-0.477679	0.652822
	100	0.603362	0.003362	0.267844	0.842094	-0.457906	0.601910
	200	0.617512	0.017512	0.224793	0.878003	-0.421997	0.546001
	400	0.618226	0.018226	0.190038	0.932975	-0.367025	0.487422
	800	0.615808	0.015808	0.150183	1.027880	-0.272120	0.400010
ϕ	25	1.780376	0.080376	0.512471	1.493276	0.493276	0.718198
	50	1.744973	0.044973	0.342526	1.449948	0.449948	0.617729
	100	1.726460	0.026460	0.291496	1.401711	0.401711	0.543495
	200	1.705457	0.005457	0.244160	1.353173	0.353173	0.478623
	400	1.693653	-0.006347	0.200818	1.302163	0.302163	0.421260
	800	1.688815	-0.011185	0.159820	1.213402	0.213402	0.331742
β	25	1.323371	0.523371	2.120285	0.953922	0.453922	0.960600
	50	1.029981	0.229981	0.662590	0.806030	0.306030	0.518458
	100	0.908036	0.108036	0.312063	0.734161	0.234161	0.342063
	200	0.848523	0.048523	0.199660	0.687029	0.187029	0.265462
	400	0.825720	0.025720	0.148974	0.648411	0.148411	0.213618
	800	0.811362	0.011362	0.110691	0.601604	0.101604	0.162979
Ω	n	Set III			Set IV		
		$Av(\Omega)$	$Bias(\Omega)$	$RMSE(\Omega)$	$Av(\Omega)$	$Bias(\Omega)$	$RMSE(\Omega)$
δ	25	0.914689	0.014689	0.589527	2.886853	0.386853	2.076241
	50	0.878732	-0.021268	0.436084	2.908595	0.308595	1.735216
	100	0.892739	-0.007261	0.407306	2.775802	0.275802	1.446074
	200	0.899281	-0.000719	0.208620	2.746958	0.246958	1.203202
	400	0.900802	0.000802	0.144522	2.663545	0.163545	0.883404
	800	0.901138	0.001138	0.101818	2.593582	0.093582	0.603627
ϕ	25	0.566102	0.166102	0.395337	1.667401	-0.032599	0.390254
	50	0.515792	0.115792	0.314933	1.650831	-0.049169	0.308279
	100	0.457750	0.057750	0.210530	1.652180	-0.047820	0.252849
	200	0.424467	0.024467	0.122971	1.657997	-0.042003	0.217038
	400	0.410178	0.010178	0.069959	1.670248	-0.029752	0.162890
	800	0.405189	0.005189	0.045340	1.684267	-0.015733	0.114847
β	25	1.040561	0.340561	1.089626	1.600392	0.600392	2.215453
	50	0.905457	0.205457	0.717811	1.242741	0.242741	0.803303
	100	0.782590	0.082590	0.308885	1.103239	0.103239	0.452424
	200	0.727273	0.027273	0.177066	1.035056	0.035056	0.253369
	400	0.708810	0.008810	0.059218	1.012863	0.012863	0.180747
	800	0.704186	0.004186	0.037396	1.004198	0.004198	0.127941

$$\begin{aligned}\log[g(y_i)] &= 2n\log(\beta) + n\log(\delta) + n\log(\phi) - (\delta+1)\sum_{i=1}^{n}\log(y_i)\\ &+ (\phi-1)\sum_{i=1}^{n}\log(1+y_i^{-\delta}) - (\beta+1)\left[\sum_{i=1}^{n}\log((1+y_i^{-\delta})^{\phi}-1)\right.\\ &\left.+\sum_{i=1}^{n}\log(1+((1+y_i^{-\delta})^{\phi}-1))^{-\beta})\right].\end{aligned} \tag{40}$$

The first partial derivatives of $\mathcal{L}_n$ with respect to δ, ϕ, and β are obtained as

$$\frac{\partial\log[g(y_i)]}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\log y_i - (\phi-1)\sum_{i=1}^{n}\frac{y_i^{-\delta}\log y_i}{1+y_i^{-\delta}} - \phi(\beta+1)$$
$$\left[\sum_{i=1}^{n}\frac{\beta y_i^{\delta}(1+y_i^{-\delta})^{\phi-1}((1+y_i^{-\delta})^{\phi}-1)^{-\beta-1}\log y_i}{1+((1+y_i^{-\delta})^{\phi}-1)^{-\beta}} - \sum_{i=1}^{n}\frac{y_i^{-\delta}(1+y_i^{-\delta})^{\phi-1}\log y_i}{((1+y_i^{-\delta})^{\phi}-1)}\right] = 0.$$

$$\begin{aligned}\frac{\partial\log[g(y_i)]}{\partial\phi} &= \frac{n}{\phi} + \sum_{i=1}^{n}\log(1+y_i^{-\delta}) - (\beta+1)\left[\sum_{i=1}^{n}\frac{(1+y_i^{-\delta})^{\phi}\log(1+y_i^{-\delta})}{((1+y_i^{-\delta})^{\phi}-1)}\right.\\ &\left.-\sum_{i=1}^{n}\frac{\beta(1+y_i^{-\delta})^{\phi}((1+y_i^{-\delta})^{\phi}-1)^{-\beta-1}\log(1+y_i^{-\delta})}{1+((1+y_i^{-\delta})^{\phi}-1)^{-\beta}}\right] = 0.\\ \frac{\partial\log[g(y_i)]}{\partial\beta} &= \frac{2n}{\beta} + \sum_{i=1}^{n}\log((1+y_i^{-\delta})^{\phi}-1) - \sum_{i=1}^{n}\log(1+((1+y_i^{-\delta})^{\phi}-1)^{-\beta})\\ &+ (\beta+1)\sum_{i=1}^{n}\frac{((1+y_i^{-\delta})^{\phi}-1)^{-\beta}\log((1+y_i^{-\delta})^{\phi}-1)}{1+((1+y_i^{-\delta})^{\phi}-1)^{-\beta}} = 0.\end{aligned} \tag{41}$$

The system of nonlinear equations in (41) is solved numerically by using the *AdequacyModel* package in *R programming language*. The solution vector, $\hat{\Omega} = (\hat{\delta}, \hat{\phi}, \hat{\beta})$ gives the maximum likelihood estimates (mles) of the parameters of the PBIII distribution.

5 Applications and Results

This section considers empirical illustration of the proposed family by the application of the Paralogistic-BIII (PBIII) distribution as a submodel to two uncensored datasets. This is to authenticate the applicability and adaptability of the PBIII distribution over the competing lifetime distributions in the analysis of the datasets. The performance of these models is compared based on some discrepancy criteria such as the Akaike information criterion (AIC), consistent Akaike information criterion ($CAIC$), Bayesian information criterion (BIC), Hannan-Quinn information criterion ($HQIC$), Cramer von Mises (W^{*}), Anderson-Darling (A^{*}) and Kolmogorov-Smirov (KS) tests. The smaller the values of these discrepancy statistics, the better fit of the data is achieved.

5.1 Applications

The competing distributions that will be applied to two real-life datasets are; Kumaraswamy Burr III (KBIII) [6], modified Burr III (MBIII) [4], New Extended Burr III (NEBIII) [8] and Burr III (BIII) [7] distributions. The datasets and the tables of computations are presented.

Data 1: Monthly Road Accidents Dataset.
The first dataset consists of monthly cases of road accidents on Oyo-Ibadan Express Road, Oyo State, Nigeria (2004-2014). The dataset was previously analyzed by [26]. The road accidents data are: 15, 4, 7, 12, 6, 8, 5, 6, 5, 4, 7, 11, 10, 6, 15, 13, 5, 7, 11, 9, 4, 6, 4, 14, 8, 4, 10, 5, 6, 4, 10, 7, 5, 9, 19, 16, 8, 7, 16, 9, 4, 6, 12, 10, 7, 11, 18, 20, 7, 16, 12, 13, 5, 7, 8, 12, 9, 11, 19, 21, 7, 6, 10, 8, 8, 14, 16, 12, 18, 10, 18, 21,8, 12, 18, 12, 22, 11, 20, 17, 19, 13, 27, 20, 11, 12, 19, 18, 15, 19, 23, 25, 22, 26, 13, 23, 26, 26, 14, 16, 23, 26, 24, 25, 24, 13, 22, 24, 28, 25, 25, 20, 23, 22, 30, 29, 19, 31, 25, 26, 25, 21, 27, 19, 22, 24, 28, 25, 18, 23, 18, 29.

Table 4: Parameter estimates and discrepancy statistics for the distributions

Model	δ (std. error)	ϕ (std. error)	β (std. error)	γ (std. error)	$-2LL$	AIC	CAIC	BIC	HQIC	W^{*}	A^{*}	KS (p-value)
PBIII	0.14079	1.54049	13.72976	-	896.1082	902.1082	902.2957	910.7566	905.6226	0.27017	1.76814	0.09733
	(0.13790)	(0.59428)	(14.44969)	-								(0.1639)
KBIII	0.46347	1.70830	114.60231	11.45261	898.7083	906.7083	907.0233	918.2395	911.3941	0.29448	1.95290	0.12004
	(0.12320)	(1.01351)	(181.84330)	(6.79461)								(0.0446)
MBIII	1.93623	29.81964	100.55039	-	926.0830	932.0830	932.2705	940.7314	935.5973	0.54690	3.52387	0.11996
	(0.16589)	(30.00329)	(47.90910)	-								(0.0448)
NEBIII	0.17259	1.40271	15.27103	-	915.5525	921.5525	921.7400	930.2009	925.0668	0.42302	2.77796	0.11790
	(0.19082)	(0.54740)	(18.28643)	-								(0.0510)
BIII	1.73357	51.19311	-	-	932.0818	936.0818	936.1748	941.8474	938.4247	0.65990	4.21031	0.12833
	(0.10778)	(11.61818)	-	-								(0.0259)

Table 4 presents parameter estimates, their standard errors, and various discrepancy statistics for the fitted distributions. While the other competing distributions show wider variability in estimates, the PBIII distribution achieves stable parameters relative to its complexity. Across all model selection criteria, the PBIII distribution consistently provides the best trade-off between goodness-of-fit and parsimony. Goodness-of-fit tests further confirm this result that the PBIII distribution displays the smallest Anderson-Darling and Cramér-von Mises statistics. The corresponding Kolmogorov-Smirnov statistic with its non-significant p-value illustrates no evidence to reject the PBIII distribution as a plausible model for the data.

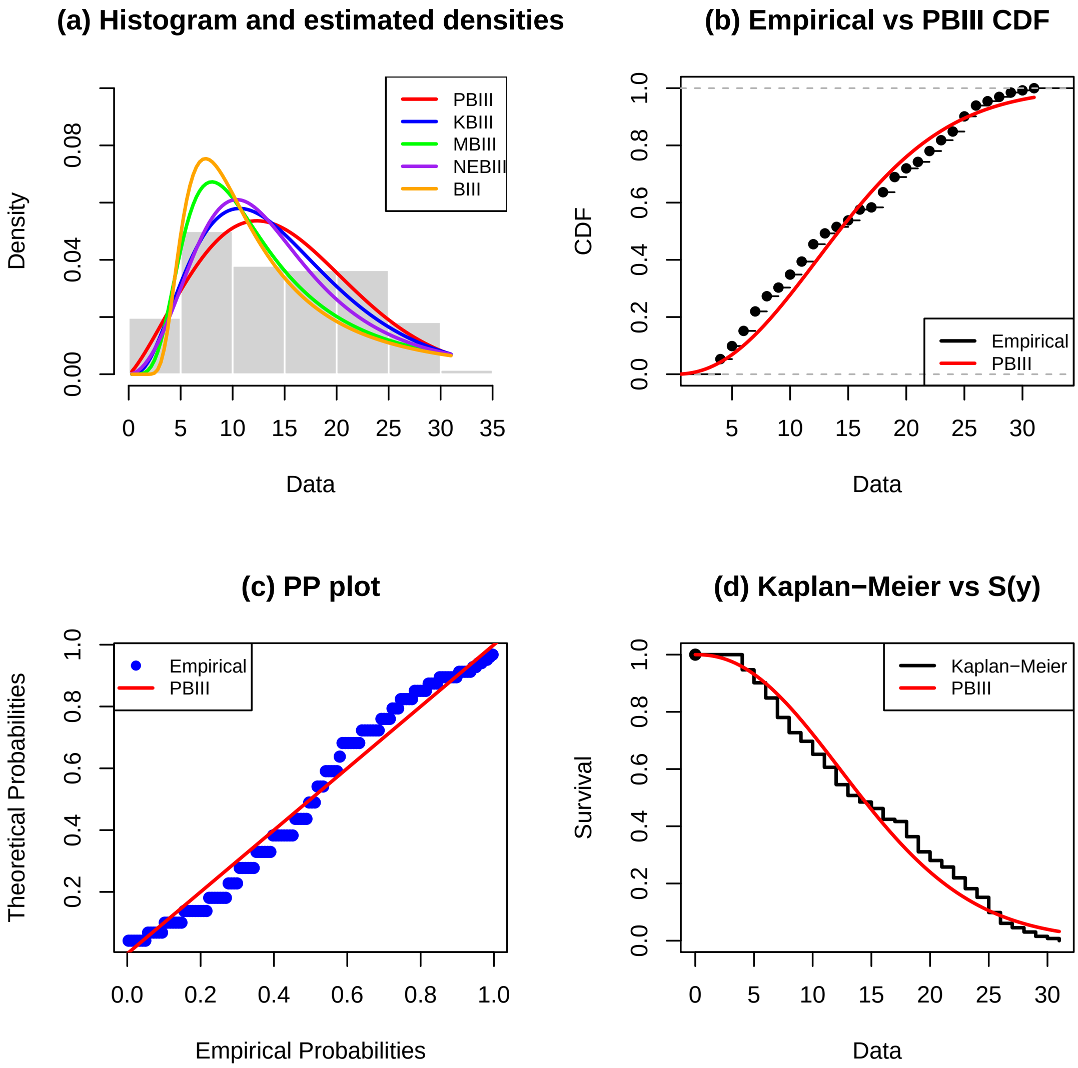

Figure 2: (a) Histogram and fitted densities (b) Empirical vs PBIII cdfs (c) PP plot (d) Kaplan Meier vs S(y) for Data 1.

The graphical assessments provide multiple lines of statistical evidence supporting the adequacy of the PBIII distribution. The histogram and fitted density indicate that the PBIII model effectively captures both the central tendency and variability of the data, outperforming competing distributions. The close alignment between the empirical and theoretical CDFs demonstrates that the PBIII distribution accurately represents the observed distribution across the entire range. Furthermore, the P-P plot shows points lying near the diagonal, indicating that the model fits well across all quantiles without systematic deviations. The Kaplan–Meier curve closely follows the PBIII survival function, confirming that the model also captures the tail behavior appropriately. Collectively, these results establish PBIII distribution as the most suitable model for Data 1, exhibiting superior parameter stability, discrepancy measures, and overall fit compared to alternative distributions.

Data 2: Fracture Toughness of Alumina (Al_2O_3) Dataset.
The second dataset consists of 119 observations on fracture toughness of Alumina (Al_2O_3)(in the units of MPa $m^{1/2}$). The dataset was analyzed by [16]. The data are: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

Table 5: Parameter estimates and discrepancy statistics for the distributions

Model	δ (std. error)	ϕ (std. error)	β (std. error)	γ (std. error)	$-2LL$	AIC	CAIC	BIC	HQIC	W^*	A^*	KS (p-value)
PBIII	0.27128	1.54399	16.73081	-	337.7828	343.7828	343.9915	352.1202	347.1684	0.09877	0.61392	0.07295
	(0.27089)	(0.62714)	(17.86739)	-								(0.5509)
KBIII	17.39709	8.34571	0.04833	8.34571	619.7286	627.7286	628.0794	638.8451	632.2426	1.11623	6.32951	0.41075
	(0.01291)	(1.74826)	(0.00443)	(1.74826)								(¡2.2e-16)
MBIII	4.43752	248.11313	492.48411	-	381.8647	387.8647	388.0734	396.2021	391.2502	0.72999	4.32389	0.14560
	(0.25613)	(117.21777)	(182.96399)	-								(0.0129)
NEBIII	0.17680	1.21141	37.63235	-	356.8376	362.8376	363.0463	371.1750	366.2231	0.36010	0.26100	0.08319
	(0.10409)	(0.14035)	(23.38497)	-								(0.3823)
BIII	3.05779	51.86600	-	-	419.5353	423.5353	423.6387	429.0935	425.7923	1.36491	7.65754	0.19633
	(0.11989)	(11.17552)	-	-								(0.0002)

Table 5 presents the estimated parameters and associated discrepancy statistics for the five Burr III-type distributions fitted to Data 2. The PBIII model demonstrates superior performance, with favorable goodness-of-fit and discrepancy statistics over the four competing distributions for Data 2.

The graphical assessments reinforce the findings reported in Table 5. The PBIII density closely follows the observed histogram, the fitted cdf aligns with the empirical cdf, the P–P plot demonstrates points near the line and the PBIII survival function is in close agreement with the Kaplan–Meier estimate. In contrast, alternative models such as KBIII and BIII distributions exhibit poorer consistency with the data, highlighting their relative inadequacy. Collectively, these numerical and graphical evaluations indicate that the PBIII distribution provides the best-fitting model for capturing the distributional and survival behavior of Data 2. The PBIII distribution consistently outperformed the four competing models

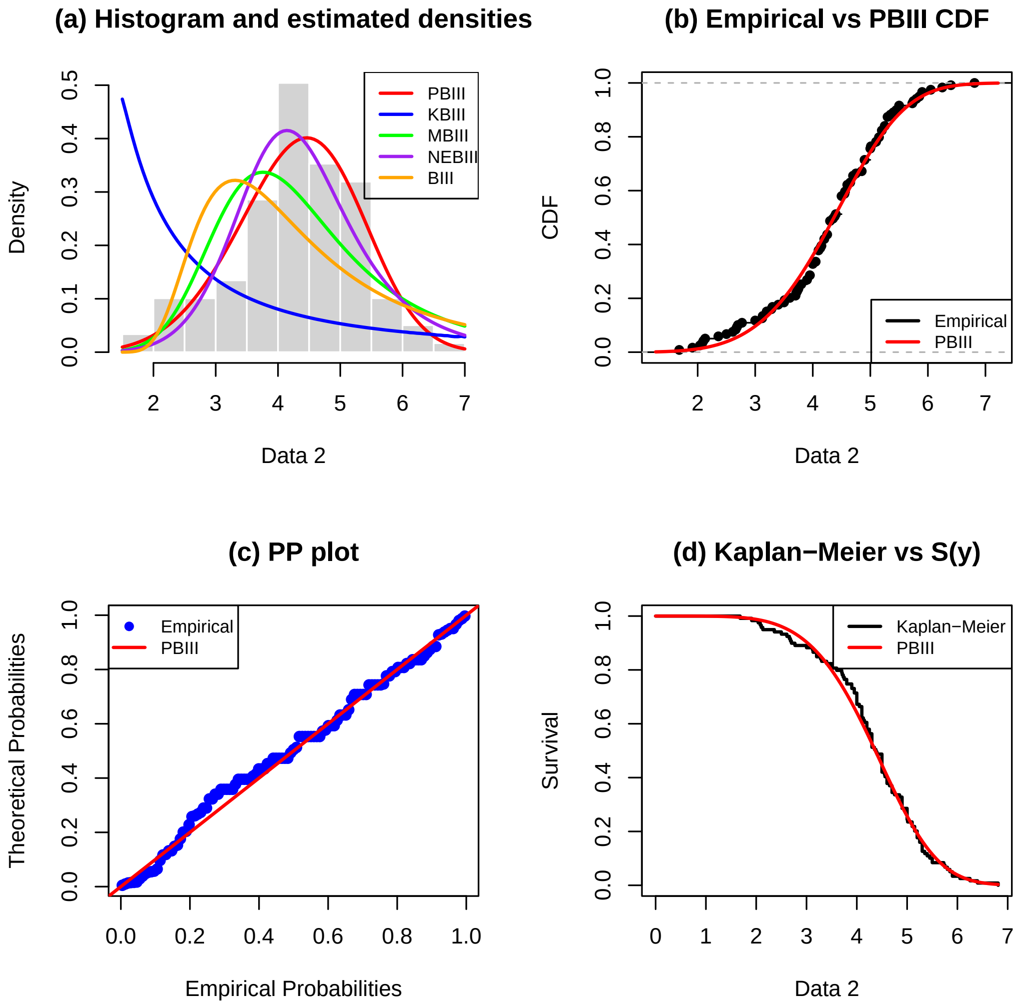

Figure 3: (a) Histogram and fitted densities (b) Empirical vs PBIII cdfs (c) PP plot (d) Kaplan Meier vs S(y) for Data 2.

when fitted to both datasets. Model selection criteria, including AIC, BIC, CAIC, and HQIC, as well as the log-likelihood, highlighting PBIII distribution as the model that achieves both statistical robustness and parsimony. This was further corroborated by the Kolmogorov–Smirnov (KS) statistic, which yielded the smallest value and a high p-value, reflecting strong agreement between the empirical and theoretical distributions. Graphical assessments reinforced these findings: the histogram and fitted density closely followed the observed data patterns, the empirical and fitted cdfs were in close alignment, and the P–P plot exhibited minimal deviation from the diagonal, demonstrating consistent fit across quantiles. Additionally, the PBIII survival function closely tracked the Kaplan–Meier estimate, highlighting the model's suitability for representing both the distributional and survival characteristics of the datasets.

6 Conclusion

In this paper, we introduced a new class of probability distributions referred to as the Paralogistic-X family, within the transformed-transformer (T-X) framework, using the less-studied paralogistic distribution as the baseline generator. The proposed family was shown to be highly flexible and capable of modeling a wide range of data patterns due to its ability to accommodate varying shapes of skewness, kurtosis, and tail behavior. The general forms of key structural properties of the family, including the quantile function, moments, moment generating function, entropy measures, and order statistics were derived. A specific submodel, the Paralogistic-Burr III (PBIII) distribution, was examined in detail, and its parameters were estimated via the method of maximum likelihood. A simulation study confirmed the consistency and efficiency of the maximum likelihood estimators across different sample sizes. The practical applicability of the proposed submodel was demonstrated using two real-world datasets. In both cases, the PBIII distribution provided superior fits compared to four existing Burr III-type models. These findings highlight the versatility, robustness, and statistical efficiency of the Paralogistic-X family, positioning it as a valuable tool for modeling real-life data across various disciplines related to lifetime data analyses.

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Authors' Contributions

The first author developed the model, derived all expressions, and interpreted the results. The second author assisted with result interpretation and manuscript preparation. All authors reviewed and approved the final manuscript.

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