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On Two Convex Weighted Integral Inequalities

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Abstract

Thanks to their wide range of applications in mathematics, convex integral inequalities continue to inspire new research. In this paper, we present weighted generalizations of two such modern results, thereby extending their scope and applicability. Several examples involving polynomial and trigonometric (sine) weight functions are provided.

1 Introduction

The concept of a convex function is central to mathematics. A formal definition is given below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that $a < b$. A function $\varphi : [a, b] \to \mathbb{R}$ is said to be convex if and only if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality holds:

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda\varphi(x) + (1 - \lambda)\varphi(y).$$

Convex functions exhibit remarkable analytical properties, such as continuity on open intervals and differentiability almost everywhere. They also provide a natural framework for deriving powerful results, including the Jensen and Hermite-Hadamard integral inequalities, and their many generalized extensions. They have extensive applications in pure and applied mathematics, economics, and optimization theory. Further background on convex functions and related inequalities can be found in [1–14].

Two fundamental results on convex integral inequalities were established by [13], which are [13, Theorems 2.1 and 2.2]. For completeness, they are stated below.

Theorem 1.1. *[13, Theorem 2.1] Let $\varphi : [0, 1] \to \mathbb{R}$ be a convex function. Then we have*

$$\int_0^1 (1 - x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(2x(1 - x))dx \geq \frac{1}{2}\varphi\left(\frac{1}{3}\right).$$

Theorem 1.2. *[13, Theorem 2.2] Let $\varphi : [0, 1] \to \mathbb{R}$ be a convex function. Then we have*

$$\int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(x^2 + (1 - x)^2)dx \geq \frac{1}{2}\varphi\left(\frac{2}{3}\right).$$

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Note that the original statements of these theorems consider $\varphi : [0,1] \to [0,+\infty)$ but this can be extended to $\varphi : [0,1] \to \mathbb{R}$ without mathematical effort. More superficially, note also that, with the change of variables $x = \sqrt{y}$, the first inequality in [13, Theorem 2.2] can be elegantly expressed as follows:

$$\int_0^1 \varphi(\sqrt{x})dx \geq \int_0^1 \varphi(x^2 + (1-x)^2)dx.$$

In summary, the above inequalities provide lower bounds for certain integral means of convex functions, offering elegant alternatives to the Jensen integral inequality on the unit interval.

In this paper, we build on these results by introducing weighted versions of convex integral inequalities. This approach yields new families of convex integral inequalities, with the classical results being special cases. We also provide illustrative examples involving polynomial and sine weight functions. A numerical work is also performed to support the theory.

The remainder of the paper is organized as follows: Section 2 is devoted to presenting and proving our first convex weighted integral inequality, along with examples. Section 3 is analogous to Section 2, but concerns the second convex weighted integral inequality. Section 4 contains concluding remarks and possible directions for further research.

2 First Result with Examples

Our first convex weighted integral inequality is given in the theorem below.

Theorem 2.1. *Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function and let $w : [0,1] \to [0,+\infty)$ be an integrable symmetric weight function, i.e., $w(x) = w(1-x)$ for any $x \in [0,1]$. Define*

$$W := \int_0^1 w(x)dx, \qquad M := \int_0^1 w(x)2x(1-x)dx.$$

Then the following chain of inequalities holds:

$$\int_0^1 w(x)(1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(2x(1-x)\right)dx \geq \frac{W}{2}\varphi\left(\frac{M}{W}\right).$$

In particular, for $w := 1$, this reduces to

$$\int_0^1 (1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(2x(1-x))dx \geq \frac{1}{2}\varphi\left(\frac{1}{3}\right),$$

which coincides with [13, Theorem 2.1].

Proof. Using the symmetry $w(x) = w(1-x)$ and making the change of variables $y = 1-x$, we have

$$\int_0^1 w(x)(1-x)\varphi(x)dx = \int_0^1 w(1-x)(1-x)\varphi(x)dx = \int_0^1 w(x)x\varphi(1-x)dx.$$

Averaging the two integral expressions gives

$$\int_0^1 w(x)(1-x)\varphi(x)dx = \frac{1}{2}\int_0^1 w(x)\left((1-x)\varphi(x) + x\varphi(1-x)\right)dx.$$

For any $x \in [0,1]$, by the basic convex inequality of φ with the parameter $\lambda := x$, we get

$$(1-x)\varphi(x) + x\varphi(1-x) \geq \varphi\left((1-x)x + x(1-x)\right) = \varphi\left(2x(1-x)\right).$$

Multiplying by $(1/2)w(x) \geq 0$ and integrating, we obtain

$$\frac{1}{2}\int_0^1 w(x)\left((1-x)\varphi(x) + x\varphi(1-x)\right)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(2x(1-x)\right)dx.$$

Combining with the previous identity yields the first inequality, i.e.,

$$\int_0^1 w(x)(1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(2x(1-x)\right)dx.$$

For the second inequality, let us define the probability measure

$$d\mu(x) = \frac{w(x)}{W}dx$$

(assuming $W > 0$). By the Jensen integral inequality for the convex function φ, we obtain

$$\int_0^1 \varphi\left(2x(1-x)\right)d\mu(x) \geq \varphi\left(\int_0^1 2x(1-x)d\mu(x)\right).$$

Multiplying by $W/2$ yields

$$\frac{1}{2}\int_0^1 w(x)\varphi\left(2x(1-x)\right)dx \geq \frac{W}{2}\varphi\left(\frac{1}{W}\int_0^1 w(x)2x(1-x)dx\right) = \frac{W}{2}\varphi\left(\frac{M}{W}\right).$$

Setting $w := 1$ gives exactly [13, Theorem 2.1], i.e., we have $W = 1$ and $M = \int_0^1 2x(1-x)dx = 1/3$, so that

$$\int_0^1 (1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(2x(1-x))dx \geq \frac{1}{2}\varphi\left(\frac{1}{3}\right).$$

This completes the proof. □

Two examples of applications of Theorem 2.1 are given below.

Example 1. Taking $w(x) := 1 + cx(1-x)$ with $c \geq 0$, which is clearly an integrable symmetric weight function, we have

$$W = \int_0^1 (1 + cx(1-x))dx, \qquad M = \int_0^1 (1 + cx(1-x))\, 2x(1-x)dx.$$

After some integral developments, we get

$$W = 1 + \frac{c}{6}, \qquad M = \frac{1}{3} + \frac{c}{15}.$$

It follows from Theorem 2.1 that

$$\begin{aligned}\int_0^1 (1 + cx(1-x))(1-x)\varphi(x)dx &\geq \frac{1}{2}\int_0^1 (1 + cx(1-x))\varphi(2x(1-x))\,dx \\ &\geq \frac{1+c/6}{2}\varphi\left(\frac{1}{1+c/6}\left(\frac{1}{3}+\frac{c}{15}\right)\right).\end{aligned}$$

In particular, by considering the convex function $\varphi(x) := x^p$ with $p > 1$, we get

$$\begin{aligned}\int_0^1 (1 + cx(1-x))(1-x)x^p dx &\geq \frac{1}{2}\int_0^1 (1 + cx(1-x))(2x(1-x))^p\,dx \\ &\geq \frac{1+c/6}{2}\left(\frac{1}{1+c/6}\left(\frac{1}{3}+\frac{c}{15}\right)\right)^p,\end{aligned}$$

so that

$$\begin{aligned}\int_0^1 (1 + cx(1-x))(1-x)x^p dx &\geq 2^{p-1}\int_0^1 (1 + cx(1-x))\,x^p(1-x)^p dx \\ &\geq \frac{1}{2}\frac{1}{(1+c/6)^{p-1}}\left(\frac{1}{3}+\frac{c}{15}\right)^p.\end{aligned}$$

Eventually, we can express the two first integrals as follows:

$$\int_0^1 (1 + cx(1-x))(1-x)x^p dx = \frac{1}{p^2+3p+2} + cB(p+2,3)$$

and

$$\int_0^1 (1 + cx(1-x))\,x^p(1-x)^p dx = B(p+1,p+1) + cB(p+2,p+2),$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ denotes the beta function at a and b, with $a, b > 0$.

Example 2. Taking $w(x) := \sin(\pi x)$, which is clearly an integrable symmetric weight function, we have

$$W = \int_0^1 \sin(\pi x)dx, \qquad M = \int_0^1 \sin(\pi x)2x(1-x)dx.$$

After some integral developments, we get

$$W = \frac{2}{\pi}, \qquad M = \frac{8}{\pi^3}.$$

It follows from Theorem 2.1 that

$$\int_0^1 \sin(\pi x)(1-x)\varphi(x)dx \geq \frac{1}{2}\int_0^1 \sin(\pi x)\varphi(2x(1-x))\,dx \geq \frac{1}{\pi}\varphi\left(\frac{4}{\pi^2}\right).$$

In order to perform a numerical study, let us consider the convex function $\varphi(x) := e^{-x}$. Then we compute

$$\int_0^1 \sin(\pi x)(1-x)\varphi(x)dx \approx 0.21628,$$

$$\frac{1}{2}\int_0^1 \sin(\pi x)\varphi\left(2x(1-x)\right)dx \approx 0.21343$$

and

$$\frac{1}{\pi}\varphi\left(\frac{4}{\pi^2}\right) \approx 0.21224.$$

It is clear that $0.21628 > 0.21343 > 0.21224$, which illustrates our theoretical inequality.

3 Second Result with Examples

Our second convex weighted integral inequality is inspired by [13, Theorem 2.2]. It is given in the theorem below.

Theorem 3.1. *Let $\varphi : [0,1] \to \mathbb{R}$ be a convex function and let $w : [0,1] \to [0,+\infty)$ be an integrable symmetric weight function, i.e., $w(x) = w(1-x)$ for any $x \in [0,1]$. Define*

$$W := \int_0^1 w(x)dx, \qquad N := \int_0^1 w(x)\left(x^2 + (1-x)^2\right)dx.$$

Then the following chain of inequalities holds:

$$\int_0^1 w(x)x\varphi(x)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(x^2+(1-x)^2\right)dx \geq \frac{W}{2}\varphi\left(\frac{N}{W}\right).$$

In particular, for $w := 1$, this reduces to

$$\int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi(x^2+(1-x)^2)dx \geq \frac{1}{2}\varphi\left(\frac{2}{3}\right),$$

which coincides with [13, Theorem 2.2].

Proof. Using the symmetry $w(x) = w(1-x)$ and making the change of variables $y = 1-x$, we have

$$\int_0^1 w(x)x\varphi(x)dx = \int_0^1 w(1-x)(1-x)\varphi(1-x)dx = \int_0^1 w(x)(1-x)\varphi(1-x)dx.$$

Averaging the two integral expressions gives

$$\int_0^1 w(x)x\varphi(x)dx = \frac{1}{2}\int_0^1 w(x)\left(x\varphi(x) + (1-x)\varphi(1-x)\right)dx.$$

For any $x \in [0,1]$, by the basic convex inequality of φ with the parameter $\lambda := 1-x$, we get

$$x\varphi(x) + (1-x)\varphi(1-x) \geq \varphi\left(x \times x + (1-x) \times (1-x)\right) = \varphi\left(x^2 + (1-x)^2\right).$$

Multiplying by $(1/2)w(x) \geq 0$ and integrating, we find that

$$\frac{1}{2}\int_0^1 w(x)\left(x\varphi(x) + (1-x)\varphi(1-x)\right)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(x^2 + (1-x)^2\right)dx.$$

Combining with the previous identity yields the first inequality, i.e.,

$$\int_0^1 w(x)x\varphi(x)dx \geq \frac{1}{2}\int_0^1 w(x)\varphi\left(x^2 + (1-x)^2\right)dx.$$

For the second inequality, let us define the probability measure

$$d\mu(x) = \frac{w(x)}{W}dx$$

(assuming $W > 0$). By the Jensen integral inequality for the convex function φ, we obtain

$$\int_0^1 \varphi\left(x^2 + (1-x)^2\right)d\mu(x) \geq \varphi\left(\int_0^1 \left(x^2 + (1-x)^2\right)d\mu(x)\right) = \varphi\left(\frac{N}{W}\right).$$

Multiplying by $W/2$ yields

$$\frac{1}{2}\int_0^1 w(x)\varphi\left(x^2 + (1-x)^2\right)dx \geq \frac{W}{2}\varphi\left(\frac{N}{W}\right).$$

Setting $w := 1$ gives exactly [13, Theorem 2.2], i.e., we have $W = 1$ and $N = \int_0^1 \left(x^2 + (1-x)^2\right)dx = 2/3$, so that

$$\int_0^1 x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \varphi\left(x^2 + (1-x)^2\right)dx \geq \frac{1}{2}\varphi\left(\frac{2}{3}\right).$$

This completes the proof. □

Two examples of applications of Theorem 3.1 are given below. In fact, we have adapted the examples introduced for the first convex weighted integral inequality for use in this new context.

Example 1. Taking $w(x) := 1 + cx(1-x)$ with $c \geq 0$, we have

$$W = \int_0^1 \left(1 + cx(1-x)\right)dx, \qquad N = \int_0^1 \left(1 + cx(1-x)\right)\left(x^2 + (1-x)^2\right)dx.$$

After some integral developments, we get

$$W = 1 + \frac{c}{6}, \qquad N = \frac{2}{3} + \frac{c}{10}.$$

It follows from Theorem 3.1 that

$$\begin{aligned}\int_0^1 (1+cx(1-x))\, x\varphi(x)dx &\geq \frac{1}{2}\int_0^1 (1+cx(1-x))\, \varphi\left(x^2+(1-x)^2\right) dx \\ &\geq \frac{1+c/6}{2}\varphi\left(\frac{1}{1+c/6}\left(\frac{2}{3}+\frac{c}{10}\right)\right).\end{aligned}$$

In particular, by considering the convex function $\varphi(x) := x^p$ with $p > 1$, we obtain

$$\begin{aligned}\int_0^1 (1+cx(1-x))\, x \times x^p dx &\geq \frac{1}{2}\int_0^1 (1+cx(1-x))\left(x^2+(1-x)^2\right)^p dx \\ &\geq \frac{1+c/6}{2}\left(\frac{1}{1+c/6}\left(\frac{2}{3}+\frac{c}{10}\right)\right)^p,\end{aligned}$$

so that

$$\begin{aligned}\int_0^1 (1+cx(1-x))\, x^{p+1} dx &\geq \frac{1}{2}\int_0^1 (1+cx(1-x))\left(x^2+(1-x)^2\right)^p dx \\ &\geq \frac{1}{2}\frac{1}{(1+c/6)^{p-1}}\left(\frac{2}{3}+\frac{c}{10}\right)^p.\end{aligned}$$

Eventually, we can express the first integral as follows:

$$\int_0^1 (1+cx(1-x))\, x^{p+1} dx = \frac{1}{p+2} + cB(p+3,2).$$

Example 2. Taking $w(x) := \sin(\pi x)$, we have

$$W = \int_0^1 \sin(\pi x)dx, \qquad N = \int_0^1 \sin(\pi x)\left(x^2+(1-x)^2\right) dx.$$

After some integral developments, we get

$$W = \frac{2}{\pi}, \qquad N = \frac{2}{\pi^3}(\pi^2-4).$$

It follows from Theorem 3.1 that

$$\int_0^1 \sin(\pi x) x\varphi(x)dx \geq \frac{1}{2}\int_0^1 \sin(\pi x)\varphi\left(x^2+(1-x)^2\right) dx \geq \frac{1}{\pi}\varphi\left(1-\frac{4}{\pi^2}\right).$$

In order to perform a numerical study, let us consider the convex function $\varphi(x) := e^{-x}$. Then we compute

$$\int_0^1 \sin(\pi x) x\varphi(x)dx \approx 0.17907,$$

$$\frac{1}{2}\int_0^1 \sin(\pi x)\varphi\left(x^2+(1-x)^2\right) dx \approx 0.17651$$

and

$$\frac{1}{\pi}\varphi\left(1-\frac{4}{\pi^2}\right) \approx 0.17561.$$

It is clear that $0.17907 > 0.17651 > 0.17561$, which illustrates our theoretical inequality.

Last but not least, in the statements of Theorems 2.1 and 3.1, if φ is concave instead of convex, then the final inequalities are reversed.

4 Conclusion

In this paper, we have introduced weighted extensions of convex integral inequalities established in [13], thereby broadening their analytical scope and range of applications. Future research could explore multidimensional analogues, connections with fractional integral operators and applications in optimization and information theory.

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