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A Subclass of Harmonic Multivalent Functions Associated with Differential Operator

Kirti Pal[1,*] and A. L. Pathak[2]

[1] Department of Mathematics, Brahmananad P.G. College (affiliated with Chhatrapati Shahu Ji Maharaj University, Kanpur, Uttar Pradesh, India), The Mall, Kanpur 208011, Uttar Pradesh, India
e-mail: kirtipal.hk@gmail.com

[2] Department of Mathematics, Brahmananad P.G. College (affiliated with Chhatrapati Shahu Ji Maharaj University, Kanpur, Uttar Pradesh, India), The Mall, Kanpur 208011, Uttar Pradesh, India
e-mail: alpathak.bnd@gmail.com

Abstract

In the present paper, we propose and study a new subclass of harmonic multivalent functions in the open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$, which is characterized by its association with a special differential operator. This investigation focuses on establishing several fundamental properties of the introduced subclass including coefficient bounds, convex combination criteria, convolution conditions and the characterization of its extreme points.

1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the family of all functions that are analytic and univalent in open unit disc $U = \{z : z \in \mathbb{C}, |z| < 1\}$ and can be expressed in the form

$$f(z) = z + \sum_{s=2}^{\infty} a_s z^s. \tag{1.1}$$

The subclass $\mathcal{S} \subset \mathcal{A}$ consists of functions that are univalent (*i.e.,* injective) and normalized such that $f(0) = 0$ and $f'(0) = 1$.

In complex analysis, a complex-valued function $f = u + iv$ is said to be harmonic if both the real part u and the imaginary part v are real-valued harmonic functions, that is, they satisfy the Laplace equation and are twice continuously differentiable. Harmonic functions have been extensively studied and appear

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*Corresponding author
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in various fields such as aerodynamics, engineering, electronics, physics, operations research and different branches of pure and applied mathematics.

In 1984, Clunie and Sheil-Small [4] introduced a class of harmonic functions denoted by $\mathcal{SH}$. Any function f in this class can be represented as

$$f(z) = h(z) + \overline{g(z)}, \tag{1.2}$$

where h and g are analytic functions in any simply connected domain $\Omega \subset \mathbb{C}$. Here, h is called analytic part, while g is referred to as co-analytical part of $f(z)$. According to the theory developed by Clunie and Sheil-Small [4], a harmonic function f is locally univalent and sense-preserving in Ω if and only if the condition $|h'(z)| > |g'(z)|$ holds throughout the domain Ω.

The class $\mathcal{SH}$ consists of harmonic functions $f = h + \overline{g}$ that are sense-preserving in an open unit disc U and satisfy the normalization condition $f(0) = h(0) = f_z(0) - 1 = 0$. If the co-analytic part $\overline{g(z)}$ is identically zero, then the function f reduces to an functions of the class $\mathcal{S}$ of analytic univalent functions.

Numerous authors have studied this approach in the context of multivalent function theory, which emerged as a natural extension of univalent function theory after extensive research on harmonic univalent functions. These investigations have revealed several new directions in this field. In 2001, Ahuja and Jahangiri [1] introduced the class $\mathcal{SH}(p)\,(p \in \mathbb{N} = 1, 2, 3, \dots)$ consisting of harmonic multivalent $(p-valent)$ functions of the form $f(z) = h(z) + \overline{g(z)}$, which are sense-preserving in open unit disc U, where

$$h(z) = z^p + \sum_{s=2}^{\infty} a_{s+p-1} z^{s+p-1} \text{ and } g(z) = \sum_{s=1}^{\infty} b_{s+p-1} z^{s+p-1}, \ |b_p| < 1. \tag{1.3}$$

The class $\mathcal{SH}(p)$ converges to the class $A(p)$ of analytic multivalent functions if $\overline{g(z)} = 0$, in which case $f(z) = h(z) = z^p + \sum_{s=2}^{\infty} a_{s+p-1} z^{s+p-1}$. Since then, many mathematicians and researchers have successfully completed the study of various remarkable subclasses of harmonic multivalent functions, like El-Ashwah and Aouf [5], Ezhilarasi et al. [6], Seoudy [10] and Yasar and Yalçın [12].

In geometric function theory, operators are essential tools for generating and analyzing new subclasses of analytic and harmonic functions. Among these, Integral, differential, and convolution operators are particularly significant because of their wide-ranging applications in establishing the properties of various functions.

In 2016, Makinde [8] introduced a Differential Operator, denoted by F^l, where l is a non-negative integer. Using this operator, harmonic multivalent functions can be defined as

$$F^l f(z) = F^l h(z) + (-1)^l \overline{F^l g(z)}, \text{ where } l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.4}$$

The analytic and co-analytical parts of the operator are given by

$$F^l h(z) = z^p + \sum_{s=2}^{\infty} C_{(s+p-1)l} a_{s+p-1} z^{s+p-1}, \quad F^l g(z) = \sum_{s=1}^{\infty} C_{(s+p-1)l} b_{s+p-1} z^{s+p-1},$$

where the coefficient $C_{(s+p-1)l}$ is defined as:

$$C_{(s+p-1)l} = \frac{(s+p-1)!}{|(s+p-l-1)|!}.$$

Based on this differential operator, we define a class of harmonic multivalent functions, denoted by $E_p^l(\alpha)$, which contains functions of the form given in (1.3) that satisfy the inequality:

$$Re\left[\frac{F^{l+1}f(z)}{F^l f(z)}\right] > p\alpha \text{ and } 0 \le \alpha < 1, \ z \in U, \tag{1.5}$$

where $F^l f(z)$ is defined by (1.4).

We also define a related subclass $\overline{E_p^l}(\alpha)$, consisting of functions of the form

$$f_l(z) = h(z) + \overline{g_l}(z), \tag{1.6}$$

where

$$h(z) = z^p - \sum_{s=2}^{\infty} |a_{s+p-1}|\, z^{s+p-1}, \quad g_l(z) = (-1)^l \sum_{s=1}^{\infty} |b_{s+p-1}|\, z^{s+p-1}, \text{ with } \quad |b_p| < 1.$$

The objective of this paper is to derive a sufficient condition for the function $f(z) \in E_p^l(\alpha)$ from (1.3) and also derive the necessary and sufficient condition for the function $f_l(z) \in \overline{E_p^l}(\alpha)$ from (1.6). In addition, we aim to obtain convolution results, convex combination, and extreme points for functions $f_l(z) \in \overline{E_p^l}(\alpha)$.

Remark 1.1. The class $E_p^l(\alpha)$ comprises the following well-known classes.

1. The class $E_p^l(\alpha)$ reduces to the class ${S_H}^*(\alpha)$ when $l = 0$ and $p = 1$. This class ${S_H}^*(\alpha)$ was introduced by Jahangiri [7].

2. The class $E_p^l(\alpha)$ reduces to the class $B_H(l, \alpha)$ when $l \neq 0$ and $p = 1$. This class $B_H(l, \alpha)$ was introduced by Sharma [11].

3. Setting $\rho = 0$ in the class $\mathcal{G}_{\mathcal{H}}(l, \rho, \alpha)$ [3] yields the class $E_p^l(\alpha)$ for $p \neq 0$.

2 Coefficient Bound

We begin by deriving a sufficient condition and estimating coefficient bounds for harmonic functions $f(z) \in E_p^l(\alpha)$.

Theorem 2.1. *Let the harmonic multivalent function $f(z) = h(z) + \overline{g(z)}$ be defined as in equation (1.3). If the following inequality holds:*

$$\sum_{s=2}^{\infty} (|s+p-l-1| - p\alpha)\, C_{(s+p-1)l}\, |a_{s+p-1}| + \sum_{s=1}^{\infty} (|s+p-l-1| + p\alpha)\, C_{(s+p-1)l}\, |b_{s+p-1}| \leq (1-\alpha)p, \tag{2.1}$$

then function $f(z)$ is sense-preserving in the open unit disc U and belongs to the class $E_p^l(\alpha)$, where $0 \leq \alpha < 1$, $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $C_{(s+p-1)l} = \frac{(s+p-1)!}{|(s+p-l-1)|!}$.

Proof. To prove that harmonic multivalent function $f(z)$ is sense-preserving, it suffices to verify that $|h'(z)| > |g'(z)|$.

Now,

$$\begin{aligned}
\left|h'(z)\right| &\geqslant p\,|z|^{p-1} - \sum_{s=2}^{\infty}(s+p-1)\left|a_{(s+p-1)}\right| |z|^{s+p-2} \\
&\geqslant |z|^{p-1}\left[p - \sum_{s=2}^{\infty}(s+p-1)\left|a_{(s+p-1)}\right| |z|^{s-1}\right] \\
&> p - \sum_{s=2}^{\infty} \frac{(|s+p-l-1| - p\alpha)\, C_{(s+p-1)l}\, |a_{s+p-1}|}{(1-\alpha)p} \\
&\geqslant \sum_{s=1}^{\infty} \frac{(|s+p-l-1| - p\alpha)\, C_{(s+p-1)l}\, |b_{s+p-1}|}{(1-\alpha)p} \\
&> \sum_{s=1}^{\infty} \frac{(|s+p-l-1| - p\alpha)\, C_{(s+p-1)l}\, |b_{s+p-1}|\, |z|^{s-1}}{(1-\alpha)p} \\
&> \sum_{s=1}^{\infty}(s+p-1)\left|b_{(s+p-1)}\right| |z|^{s+p-2} = \left|g'(z)\right| \\
&> \left|g'(z)\right|.
\end{aligned}$$

Thus, function $f(z)$ is indeed sense-preserving within the open unit disc U. For showing $f(z) \in E_p^l(\alpha)$, we have to prove that (1.5) is true.

If

$$\omega = \frac{F^{l+1} f(z)}{F^l f(z)} = \frac{A_1(z)}{A_2(z)}$$

and using

$$Re\,[\omega] > p\alpha \Leftrightarrow |\omega - (1+\alpha)p| < |\omega + (1-\alpha)p|\,,$$

then it will be enough to prove that

$$|A_1(z) - (1+\alpha)pA_2(z)| - |A_1(z) + (1-\alpha)pA_2(z)| \leq 0.$$

Now, solve the part $|A_1(z) - (1+\alpha)pA_2(z)|$

$$\Rightarrow \left| \begin{array}{l} \left(z^p + \sum_{s=2}^{\infty} C_{(s+p-1)(l+1)} a_{(s+p-1)} z^{s+p-1} + (-1)^{l+1} \sum_{s=1}^{\infty} C_{(s+p-1)(l+1)} b_{(s+p-1)} \bar{z}^{s+p-1} \right) \\ - (1+\alpha)p \left(z^p + \sum_{s=2}^{\infty} C_{(s+p-1)l} a_{(s+p-1)} z^{s+p-1} + (-1)^{l+1} \sum_{s=1}^{\infty} C_{(s+p-1)l} b_{(s+p-1)} \bar{z}^{s+p-1} \right) \end{array} \right|$$

$$\begin{aligned} &\leq \alpha p\,|z^p| + \sum_{s=2}^{\infty} \Big((1+\alpha)p - |s+p-l-1| \Big) C_{(s+p-1)l} \left| a_{(s+p-1)} \right| |z|^{s+p-1} \\ &+ \sum_{s=1}^{\infty} \Big((1+\alpha)p - |s+p-l-1| \Big) C_{(s+p-1)l} \left| b_{(s+p-1)} \right| |\bar{z}|^{s+p-1}. \end{aligned}$$

Now, solve the part $|A_1(z) + (1-\alpha)pA_2(z)|$

$$\Rightarrow \left| \begin{array}{l} \left(z^p + \sum_{s=2}^{\infty} C_{(s+p-1)(l+1)} a_{(s+p-1)} z^{s+p-1} + (-1)^{l+1} \sum_{s=1}^{\infty} C_{(s+p-1)(l+1)} b_{(s+p-1)} \bar{z}^{s+p-1} \right) \\ + (1-\alpha)p \left(z^p + \sum_{s=2}^{\infty} C_{(s+p-1)l} a_{(s+p-1)} z^{s+p-1} + (-1)^{l+1} \sum_{s=1}^{\infty} C_{(s+p-1)l} b_{(s+p-1)} \bar{z}^{s+p-1} \right) \end{array} \right|$$

$$\begin{aligned} &\geq (2-\alpha)p\,|z^p| - \sum_{s=2}^{\infty} \Big| (\alpha-1)p - |s+p-l-1| \Big| C_{(s+p-1)l} \left| a_{(s+p-1)} \right| |z|^{s+p-1} \\ &- \sum_{s=1}^{\infty} \Big| |s+p-l-1| - (1-\alpha)p \Big| C_{(s+p-1)l} \left| b_{(s+p-1)} \right| |\bar{z}|^{s+p-1}. \end{aligned}$$

Now, implies that

$$\begin{aligned}
&|A_1(z)-(1+\alpha)pA_2(z)|-|A_1(z)+(1-\alpha)pA_2(z)|\\
&\le \alpha p\,|z^p|+\sum_{s=2}^{\infty}\Big((1+\alpha)p-|s+p-l-1|\Big)C_{(s+p-1)l}\left|a_{(s+p-1)}\right||z|^{s+p-1}\\
&\quad+\sum_{s=1}^{\infty}\Big((1+\alpha)p-|s+p-l-1|\Big)C_{(s+p-1)l}\left|b_{(s+p-1)}\right||\bar{z}|^{s+p-1}\\
&\quad+(\alpha-2)p\,|z^p|+\sum_{s=2}^{\infty}\Big|(\alpha-1)p-|s+p-l-1|\Big|C_{(s+p-1)l}\left|a_{(s+p-1)}\right||z|^{s+p-1}\\
&\quad+\sum_{s=1}^{\infty}\Big||s+p-l-1|-(1-\alpha)p\Big|C_{(s+p-1)l}\left|b_{(s+p-1)}\right||\bar{z}|^{s+p-1}.
\end{aligned}$$

Now,

$$\begin{aligned}
\Longrightarrow 2&\sum_{s=2}^{\infty}\Big||s+p-l-1|-\alpha p\Big|C_{(s+p-1)l}\left|a_{(s+p-1)}\right||z|^{s+p-1}\\
&+2\sum_{s=1}^{\infty}\Big||s+p-l-1|+\alpha p\Big|C_{(s+p-1)l}\left|b_{(s+p-1)}\right||\bar{z}|^{s+p-1}-2(1-\alpha)p\,|z^p|\le 0.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\sum_{s=2}^{\infty}\Big||s+p-l-1|-\alpha p)\Big|C_{(s+p-1)l}\left|a_{(s+p-1)}\right|\\
&\quad+\sum_{s=1}^{\infty}\Big||s+p-l-1|+\alpha p)\Big|C_{(s+p-1)l}\left|b_{(s+p-1)}\right|\le(1-\alpha)p.
\end{aligned}$$

This completes the proof. □

If $p=1$ in Theorem (2.1), then Corollary (2.2) is obtained.

Corollary 2.2. *[11] Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.2). If*

$$\sum_{s=2}^{\infty}(|s-l|-\alpha)\,C_{sl}\,|a_s|+\sum_{s=1}^{\infty}(|s-l|+\alpha)\,C_{sl}\,|b_s|\le(1-\alpha),$$

where $0\le\alpha<1$, $l\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $C_{sl}=\frac{s!}{|s-l|!}$, then the function $f(z)$ is sense-preserving in U and $f(z)\in B_H(l,\alpha)$.

The harmonic multivalent function

$$\begin{aligned}
f(z)=z&+\sum_{s=2}^{\infty}\frac{(1-\alpha)p}{\Big(|s+p-l-1|-\alpha p\Big)C_{(s+p-1)l}}X_s z^s\\
&+\sum_{s=1}^{\infty}\frac{(1-\alpha)p}{\Big(|s+p-l-1|+\alpha p\Big)C_{(s+p-1)l}}\overline{Y_s z^s},
\end{aligned}\tag{2.2}$$

where $0 \leq \alpha < 1$, $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $C_{(s+p-1)l} = \frac{(s+p-1)!}{|(s+p-l-1)|!}$ and $\sum_{s=2}^{\infty} |X_s| + \sum_{s=1}^{\infty} |Y_s| = 1$, shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.2) belongs to the class $E_p^l(\alpha)$ because

$$\begin{aligned}
&\sum_{s=2}^{\infty} \frac{\Big(|s+p-l-1| - \alpha p\Big) C_{(s+p-1)l}}{(1-\alpha)p} |a_{s+p-1}| \\
&\qquad + \sum_{s=1}^{\infty} \frac{\Big(|s+p-l-1| + \alpha p\Big) C_{(s+p-1)l}}{(1-\alpha)p} |b_{s+p-1}| \\
&= 1 + \sum_{s=2}^{\infty} |X_s| + \sum_{s=1}^{\infty} |Y_s| = 2.
\end{aligned} \tag{2.3}$$

The following theorem demonstrates that condition (2.1) is not only sufficient but also necessary for the functions $f_l = h + \overline{g_l}$, where h and g_l are of the form(1.6).

Theorem 2.3. *Let the harmonic multivalent function $f_l(z) = h(z) + \overline{g_l(z)}$ be defined as in equation (1.6). Then $f_l(z) \in \overline{E_p^l}(\alpha)$ if and only if the following inequality holds:*

$$\begin{aligned}
&\sum_{s=2}^{\infty} \left(|s+p-l-1| - \alpha p\right) C_{(s+p-1)l} \left|a_{s+p-1}\right| \\
&\qquad + \sum_{s=1}^{\infty} \left(|s+p-l-1| + \alpha p\right) C_{(s+p-1)l} \left|b_{s+p-1}\right| \leq (1-\alpha)p,
\end{aligned} \tag{2.4}$$

where $0 \leq \alpha < 1$, $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $C_{(s+p-1)l} = \frac{(s+p-1)!}{|(s+p-l-1)|!}$.

Proof. For this theorem, we just need to show "only if" part. Since $\overline{E_p^l}(\alpha) \subseteq E_p^l(\alpha)$, we observe that the condition (1.5) is equivalent for the functions $f_l(z)$ of the form (1.6),

$$Re\left[\begin{array}{l}
\dfrac{z^p - \sum_{s=2}^{\infty} C_{(s+p-1)(l+1)} \left|a_{(s+p-1)}\right| z^{s+p-1} + (-1)^{2l+1} \sum_{s=1}^{\infty} C_{(s+p-1)(l+1)} \left|b_{(s+p-1)}\right| \bar{z}^{s+p-1}}{z^p - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s+p-1} + (-1)^{2l} \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s+p-1}} \\[3ex]
- \dfrac{\alpha p\Big(z^p - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s+p-1} + (-1)^{2l} \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s+p-1}\Big)}{z^p - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s+p-1} + (-1)^{2l} \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s+p-1}}
\end{array}\right] \geq 0,$$

$$Re\left[\begin{array}{l}
\dfrac{(1-\alpha p) - \sum_{s=2}^{\infty} \left(|s+p-l-1| - \alpha p\right) C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s-1}}{1 - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s-1} + \frac{\bar{z}}{z} \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s-1}} \\[3ex]
- \dfrac{\frac{\bar{z}}{z} \left\{\sum_{s=1}^{\infty} \left(|s+p-l-1| + \alpha p\right) C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s-1}\right\}}{1 - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| z^{s-1} + \frac{\bar{z}}{z} \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{z}^{s-1}}
\end{array}\right] \geq 0. \tag{2.5}$$

The above equation (2.5) must be true for all values of z on the positive real axis, where $0 \leq |z| < r < 1$. Now, we get

$$\left[\begin{array}{c} \dfrac{(1-\alpha p) - \sum_{s=2}^{\infty} \left(|s+p-l-1| - \alpha p\right) C_{(s+p-1)l} \left|a_{(s+p-1)}\right| r^{s-1}}{1 - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| r^{s-1} + \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{r}^{s-1}} \\ \\ -\dfrac{\sum_{s=1}^{\infty} \left(|s+p-l-1| + \alpha p\right) C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{r}^{s-1}}{1 - \sum_{s=2}^{\infty} C_{(s+p-1)l} \left|a_{(s+p-1)}\right| r^{s-1} + \sum_{s=1}^{\infty} C_{(s+p-1)l} \left|b_{(s+p-1)}\right| \bar{r}^{s-1}} \end{array}\right] \geq 0. \tag{2.6}$$

We observe that, if condition (2.1) is not satisfied for the numerator in (2.6) when $r \longrightarrow 1$, it results in negative value. This is contradicting the condition for $f_l(z) \in \overline{E_p^l}(\alpha)$ and hence the proof is completed. □

3 Convolution

In this part, we establish the closure property of the class $\overline{E_p^l}(\alpha)$ under convolution. In this part, we establish the closure nature of the class $\overline{E_p^l}(\alpha)$ under convolution. For harmonic multivalent functions

$$f_l(z) = z^p - \sum_{s=2}^{\infty} |a_{s+p-1}|\, z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty} |b_{s+p-1}|\, \bar{z}^{s+p-1}$$

and

$$F_l(z) = z^p - \sum_{s=2}^{\infty} |a_{s+p-1}|\, z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty} |b_{s+p-1}|\, \bar{z}^{s+p-1}.$$

The convolution relation for $f_l(z)$ and $F_l(z)$ is defined as

$$\begin{aligned}(f_l * F_l)(z) &= f_l(z) * F_l(z) \\ &= z^p - \sum_{s=2}^{\infty} |a_{s+p-1}A_{s+p-1}|\, z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty} |b_{s+p-1}B_{s+p-1}|\, \bar{z}^{s+p-1}. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1$, $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $f_l(z) \in \overline{E_p^l}(\alpha_2)$ and $F_l(z) \in \bar{E}_p^l(\alpha_1)$, then $f_l(z) * F_l(z)$ belongs to $\overline{E_p^l}(\alpha_2) \subset \overline{E_p^l}(\alpha_1)$.*

Proof. We aim to obtain the coefficient for $f_l * F_l$, that satisfies the necessary condition defined in Theorem

2.3. Now $F_l(z) \in \overline{E_p^l}(\alpha_1)$, we see that $|A_{s+p-1}| \leq 1$ and $|B_{s+p-1}| \leq 1$. Now,

$$\begin{aligned}
&\sum_{s=2}^{\infty} \frac{\Big(|s+p-l-1| - \alpha_1 p\Big) C_{(s+p-1)l}}{(1-\alpha_1)p} |a_{s+p-1}| \, |A_{s+p-1}| \\
&\qquad + \sum_{s=1}^{\infty} \frac{\Big(|s+p-l-1| + \alpha_1 p\Big) C_{(s+p-1)l}}{(1-\alpha_1)p} |b_{s+p-1}| \, |B_{s+p-1}| \\
&\leq \sum_{s=2}^{\infty} \frac{\Big(|s+p-l-1| - \alpha_1 p\Big) C_{(s+p-1)l}}{(1-\alpha_1)p} |a_{s+p-1}| \\
&\qquad + \sum_{s=1}^{\infty} \frac{\Big(|s+p-l-1| + \alpha_1 p\Big) C_{(s+p-1)l}}{(1-\alpha_1)p} |b_{s+p-1}| \\
&\leq \sum_{s=2}^{\infty} \frac{\Big(|s+p-l-1| - \alpha_2 p\Big) C_{(s+p-1)l}}{(1-\alpha_2)p} |a_{s+p-1}| \\
&\qquad + \sum_{s=1}^{\infty} \frac{\Big(|s+p-l-1| + \alpha_2 p\Big) C_{(s+p-1)l}}{(1-\alpha_2)p} |b_{s+p-1}| \\
&\leq 1.
\end{aligned}$$

Because $0 \leq \alpha_1 \leq \alpha_2 < 1$ and $f_l(z) \in \overline{E_p^l}(\alpha_2)$, so $f_l(z) * F_l(z) \in \overline{E_p^l}(\alpha_2) \subset \overline{E_p^l}(\alpha_1)$. Therefore proof of the theorem is complete. □

4 Convex Combination

In this part, we establish the closure property under convex combination for the class $\overline{E_p^l}(\alpha)$. We consider harmonic multivalent functions $f_{l_i}(z)$ defined for $i = 1, 2, 3, \cdots, m$ by

$$f_{l_i}(z) = z^p - \sum_{s=2}^{\infty} |a_{s+p-1,i}| \, z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty} |b_{s+p-1,i}| \, \bar{z}^{s+p-1}. \tag{4.1}$$

Theorem 4.1. *Let the functions $f_{l_i}(z)$, for $i = 1, 2, 3, \ldots, m$, be given by (4.1) and suppose that $f_{l_i}(z) \in \overline{E_p^l}(\alpha)$ for each i. Consider a new function $t_i(z)$ defined as a linear convex combination of these functions:*

$$t_i(z) = \sum_{i=1}^{m} c_i f_{l_i}(z),$$

where the coefficients satisfy $0 \leq c_i \leq 1$ and $\sum_{i=1}^{m} c_i = 1$. Under these conditions, the function $t_i(z)$ also belongs to the class $\overline{E_p^l}(\alpha)$.

Proof. By the definition of $t_i(z)$, we formulate as follows:

$$t_i(z) = z^p - \sum_{s=2}^{\infty}\left(\sum_{i=1}^{m} c_i \left|a_{s+p-1,i}\right|\right) z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty}\left(\sum_{i=1}^{m} c_i \left|b_{s+p-1,i}\right|\right) \bar{z}^{s+p-1}.$$

Moreover, given that $f_{l_i}(z)$ belongs to $\overline{E_p^l}(\alpha)$ for each $i = 1, 2, 3, \ldots, m$, then

$$\begin{aligned}
&\sum_{s=2}^{\infty}\Big(|s+p-l-1| - \alpha p\Big) C_{(s+p-1)l}\left(\sum_{i=1}^{m} c_i \left|a_{s+p-1,i}\right|\right) \\
&\qquad + \sum_{s=1}^{\infty}\Big(|s+p-l-1| + \alpha p\Big) C_{(s+p-1)l}\left(\sum_{i=1}^{m} c_i \left|b_{s+p-1,i}\right|\right) \\
&= \sum_{i=1}^{m} c_i \left(\sum_{s=2}^{\infty}\left(|s+p-l-1| - \alpha p\right) C_{(s+p-1)l} \left|a_{s+p-1,i}\right| \right. \\
&\qquad\qquad \left. + \sum_{s=1}^{\infty}\left(|s+p-l-1| + \alpha p\right) C_{(s+p-1)l} \left|b_{s+p-1,i}\right|\right) \\
&\le \sum_{i=1}^{m} c_i (1-\alpha)p \\
&\le (1-\alpha)p.
\end{aligned}$$

Hence, by Theorem (2.3), we conclude that $t_i(z) \in \overline{E_p^l}(\alpha)$. This completes the proof. □

5 Extreme Points

In this part, we evaluate the extreme points for the class $\overline{E_p^l}(\alpha)$.

Theorem 5.1. *Let the harmonic multivalent function f_l be defined as in (1.6). Then, $f_l \in \overline{E_p^l}(\alpha)$, if and only if it can be expressed in the form*

$$f_l(z) = \sum_{s=1}^{\infty}\left[X_{s+p-1} h_{s+p-1}(z) + Y_{s+p-1} g_{l_{s+p-1}}(z)\right], \tag{5.1}$$

where

$$h_p(z) = z^p, \quad h_{s+p-1}(z) = z^p - \frac{(1-\alpha)p}{\Big(|s+p-l-1| - \alpha p\Big) C_{(s+p-1)l}} z^{s+p-1}, \ l = 2, 3, 4, \cdots,$$

and

$$g_{l_{s+p-1}}(z) = z^p + (-1)^l \frac{(1-\alpha)p}{\Big(|s+p-l-1| + \alpha p\Big) C_{(s+p-1)l}} \bar{z}^{s+p-1}, \ l = 1, 2, 3, \cdots,$$

and the coefficients X_{s+p-1} *and* Y_{s+p-1} *are non-negative and satisfy the normalization condition:*

$$\sum_{s=1}^{\infty}[X_{s+p-1}+Y_{s+p-1}]=1, \quad \textit{where} \quad X_{s+p-1}\geq 0, \quad \textit{and} \quad Y_{s+p-1}\geq 0.$$

Specifically, the functions $h_{s+p-1}(z)$ *and* $g_{l_{s+p-1}}(z)$ *serve as the extreme points of the class* $\overline{E_p^l}(\alpha)$.

Proof. The function $f_l(z)$ form (5.1), we have

$$\begin{aligned}
f_l(z) &= \sum_{s=1}^{\infty}\left[X_{s+p-1}h_{s+p-1}(z)+Y_{s+p-1}g_{l_{s+p-1}}(z)\right] \\
&= \sum_{s=1}^{\infty}X_{s+p-1}\left[z^p-\sum_{s=2}^{\infty}\frac{(1-\alpha)p}{\Big(|s+p-l-1|-\alpha p\Big)C_{(s+p-1)l}}z^{s+p-1}\right] \\
&\quad +\sum_{s=1}^{\infty}Y_{s+p-1}\left[z^p+(-1)^l\sum_{s=1}^{\infty}\frac{(1-\alpha)p}{\Big(|s+p-l-1|+\alpha p\Big)C_{(s+p-1)l}}\bar{z}^{s+p-1}\right] \\
&= z^pX_p-\sum_{s=2}^{\infty}\left[\frac{(1-\alpha)p}{\Big(|s+p-l-1|-\alpha p\Big)C_{(s+p-1)l}}\right]X_{s+p-1}z^{s+p-1} \\
&\quad +(-1)^l\sum_{s=1}^{\infty}\left[\frac{(1-\alpha)p}{\Big(|s+p-l-1|+\alpha p\Big)C_{(s+p-1)l}}\right]Y_{s+p-1}z^{s+p-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{s=2}^{\infty}\frac{\Big(|s+p-l-1|-\alpha p\Big)C_{(s+p-1)l}\,|a_{s+p-1}|}{(1-\alpha)p} \\
&\qquad +(-1)^l\sum_{s=1}^{\infty}\frac{\Big(|s+p-l-1|+p\alpha\Big)C_{(s+p-1)l}\,|b_{s+p-1}|}{p(1-\alpha)} \\
&= \sum_{s=2}^{\infty}X_{s+p-1}+\sum_{s=1}^{\infty}Y_{s+p-1} \\
&= 1-X_p \\
&\leq 1.
\end{aligned}$$

By Theorem 2.3, this implies that $f_l\in\overline{E_p^l}(\alpha)$.

Conversely, suppose $f_l\in\overline{E_p^l}(\alpha)$ and define

$$X_{s+p-1}=\frac{\Big(|s+p-l-1|-\alpha p\Big)C_{(s+p-1)l}}{(1-\alpha)p}\,|a_{s+p-1}|\,,\ 0\leq X_{s+p-1}\leq 1,\ s=2,3,4,\ldots\ ,$$

$$Y_{s+p-1} = \frac{\Big(|s+p-l-1| + \alpha p\Big) C_{(s+p-1)l}}{(1-\alpha)p} \left|b_{s+p-1}\right|, \ 0 \leq Y_{s+p-1} \leq 1, \ s = 1, 2, 3, 4, \ldots$$

and

$$X_p = 1 - \left(\sum_{s=2}^{\infty} X_{s+p-1} + \sum_{s=1}^{\infty} Y_{s+p-1} \right).$$

So, f_l can be expressed as

$$\begin{aligned} f_l(z) &= z^p - \sum_{s=2}^{\infty} \left|a_{s+p-1}\right| z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty} \left|b_{s+p-1}\right| \bar{z}^{s+p-1} \\ &= z^p - \sum_{s=2}^{\infty} \left[\frac{(1-\alpha)p}{\Big(|s+p-l-1| - \alpha p\Big) C_{(s+p-1)l}} \right] X_{s+p-1} z^{s+p-1} \\ &\quad + (-1)^l \sum_{s=1}^{\infty} \left[\frac{(1-\alpha)p}{\Big(\Big|s+p-l-1\Big| + \alpha p\Big) C_{(s+p-1)l}} \right] Y_{s+p-1} z^{s+p-1} \\ &= X_p h_p(z) + \sum_{s=2}^{\infty} X_{s+p-1} h_{s+p-1}(z) + \sum_{s=1}^{\infty} Y_{s+p-1} g_{l_{s+p-1}}(z) \\ &= \sum_{s=1}^{\infty} X_{s+p-1} h_{s+p-1}(z) + \sum_{s=1}^{\infty} Y_{s+p-1} g_{l_{s+p-1}}(z). \end{aligned}$$

This completes the proof of Theorem (5.1). □

6 Integral Operator

In this part, we examine the generalized Bernardi-Libera-Livingston integral operator [2] introduce by Bernardi in 1969 for the class $\overline{E_p^l}(\alpha)$.

Theorem 6.1. *Let the function $f_l(z)$, as defined in equation (1.6), belong to the class $\overline{E_p^l}(\alpha)$. For any real number d such that $d > -p$, define the operator $L_d\left[f_l(z)\right]$ as follows [2]:*

$$L_d\left[f_l(z)\right] = \frac{d+p}{z^p} \int_0^z t^{(d-1)} f_l(t) dt.$$

Under these conditions, the transformed function $L_d\left[f_l(z)\right]$ also belongs to the class $\overline{E_p^l}(\alpha)$.

Proof. From definition of $L_d\left[f_l(z)\right]$, we formulate as follows:

$$\begin{aligned}
L_d\left[f_l(z)\right] &= \frac{d+p}{z^p}\int_0^z t^{(d-1)} f_l(t)dt \\
&= \frac{d+p}{z^p}\int_0^z t^{(d-1)}\left[t^p - \sum_{s=2}^{\infty}\left|a_{s+p-1}\right| t^{s+p-1} + (-1)^l \sum_{s=1}^{\infty}\left|b_{s+p-1}\right| \bar{t}^{s+p-1}\right] \\
&= z^p - \sum_{s=2}^{\infty}\frac{p+d}{d+s+p-1}\left|a_{s+p-1}\right| z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty}\frac{p+d}{d+s+p-1}\left|b_{s+p-1}\right| \bar{z}^{s+p-1} \\
&= z^p - \sum_{s=2}^{\infty}\left|A_{s+p-1}\right| z^{s+p-1} + (-1)^l \sum_{s=1}^{\infty}\left|B_{s+p-1}\right| \bar{z}^{s+p-1}.
\end{aligned}$$

Now, we compare above equation with (1.6), then we get

$$\begin{aligned}
\left|a_{s+p-1}\right| = \left|A_{s+p-1}\right| &= \frac{p+d}{d+s+p-1}\left|a_{s+p-1}\right|, \\
\left|b_{s+p-1}\right| = \left|B_{s+p-1}\right| &= \frac{p+d}{d+s+p-1}\left|b_{s+p-1}\right|.
\end{aligned}$$

In the order to show $L_d\left[f_l(z)\right] \in \overline{E_p^l}(\alpha)$, we have to prove that condition (2.4) is satisfied. Now consider,

$$\begin{aligned}
&\sum_{s=2}^{\infty}\frac{\Big(\left|s+p-l-1\right| - \alpha p\Big)C_{(s+p-1)l}}{(1-\alpha)p}\frac{p+d}{d+s+p-1}\left|a_{s+p-1}\right| z^{s+p-1} \\
&\quad + \sum_{s=1}^{\infty}\frac{\Big(\left|s+p-l-1\right| + \alpha p\Big)C_{(s+p-1)l}}{(1-\alpha)p}\frac{p+d}{d+s+p-1}\left|b_{s+p-1}\right| \bar{z}^{s+p-1} \\
&\leq \sum_{s=2}^{\infty}\frac{\Big(\left|s+p-l-1\right| - \alpha p\Big)C_{(s+p-1)l}}{(1-\alpha)p}\left|a_{s+p-1}\right| \\
&\quad\quad + \sum_{s=1}^{\infty}\frac{\Big(\left|s+p-l-1\right| + \alpha p\Big)C_{(s+p-1)l}}{(1-\alpha)p}\left|b_{s+p-1}\right| \\
&\leq 1.
\end{aligned}$$

Since $f_l(z) \in \overline{E_p^l}(\alpha)$, it follows from Theorem 2.3 that $L_d\left[f_l(z)\right] \in \overline{E_p^l}(\alpha)$. This completes the proof. □

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