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Semi-Analytical Solution of a Two-Dimensional Time-Fractional Fisher's Equation via the Homotopy Analysis Method

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Abstract

This paper presents a semi-analytical solution to the two-dimensional time-fractional Fisher's equation using the Homotopy Analysis Method (HAM). The governing equation models reaction–diffusion processes with memory effects, incorporating the Caputo fractional derivative to account for anomalous temporal behavior. The HAM framework is constructed by first selecting an appropriate initial guess that satisfies both the initial and boundary conditions, followed by the recursive generation of higher-order approximations. The convergence of the series solution is controlled through the auxiliary parameter $\hbar$, whose optimal value is determined at each time level by enforcing consistency with boundary conditions. The analytical results are validated against numerical simulations, demonstrating excellent agreement. This study highlights the flexibility and efficiency of HAM in handling high-dimensional nonlinear fractional partial differential equations and provides a foundation for extending the method to more complex biological or ecological models.

1 Introduction

The Fisher's equation, originally introduced to model the spread of advantageous genes in a population, is a classical reaction–diffusion equation with applications extending into ecology, epidemiology, and chemical kinetics [1]. While the traditional model captures diffusive and reactive behavior well, it often fails to represent systems with memory or hereditary effects, particularly in heterogeneous media.

To address this shortcoming, the time-fractional variant of the Fisher's equation has been increasingly explored. Time-fractional derivatives, commonly expressed in the Caputo sense, provide a mathematical framework to model anomalous diffusion and long-range temporal dependencies [10, 11].

Several analytical and numerical studies have been conducted to understand such equations. For example, Ahmed [1] derived approximate analytical solutions for both one- and two-dimensional cases.

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Zidan et al. [11] and Majeed et al. [9] explored the use of semi-analytical methods for various forms of fractional Fisher's equations.

Despite these efforts, obtaining closed-form or convergent solutions to nonlinear time-fractional PDEs remains challenging. Researchers have proposed numerical methods based on finite difference schemes [7], spectral approximations [4], and Lie symmetry approaches [2]. Nonetheless, many of these methods are either computationally intensive or lack analytical insight.

The Homotopy Analysis Method (HAM) offers a powerful alternative by constructing a convergent series solution independent of small parameters [5, 6]. One of the strengths of HAM lies in the introduction of an auxiliary parameter $\hbar$, which allows explicit control over the convergence region of the solution series [24]. Recent work by Liu and Wu [8] as well as Ishii [3] underscores the utility of analytical approaches in characterizing the behavior of solutions to fractional Fisher-type equations.

This study aims to derive a semi-analytical solution for the two-dimensional time-fractional Fisher's equation using HAM. The goal is to construct a convergent solution that satisfies the initial and boundary conditions while leveraging the convergence control capabilities of HAM. Our approach contributes to a growing body of literature by offering a computationally efficient and analytically transparent method for solving complex fractional reaction–diffusion systems.

2 Governing Equations

Let $\Omega = \{(x, y, t) \in \mathbb{R}^3 | 0 < x < 1, 0 < y < 1, 0 < t < 1\}$ be the solution domain, where the temperature distribution $u(x, y, t)$ is unknown and to be determined during the solution process. The mathematical formulation for a moving boundary problem is given by

$$ {}_0^cD_t^\alpha u(x,y,t) - \Delta u(x,y,t) - u(x,y,t)(1-(u(x,y,t)^\beta)) = f(x,y,t) \quad 0<\alpha<1, \beta \geq 1. \tag{1} $$

For a function $u(x, y, t)$, the Caputo fractional derivative of order $\alpha \in (0, 1]$ with respect to time is defined as:

$$ {}_0^cD_t^\alpha u(x,y,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,y,\tau)}{\partial \tau^\alpha} \frac{1}{(t-\tau)^\alpha} d\tau \tag{2} $$

with an initial condition of

$$ u(x,y,0) = g(x,y) \tag{3} $$

and the boundary conditions of

$$ u(x,0,t) = a_1(x,t), \quad u(x,1,t) = a_2(x,t). \tag{4} $$

3 Application of the Homotopy Analysis Method (HAM)

Consider the differential equation

$$\aleph[(\xi, t)] = \omega(\xi, t), \tag{5}$$

where $\xi \in (x, y)$. Using homotopy, a basic concept in topology

$$(1 - q)\mathcal{L}[\varphi(\xi, t; q) - u_0(\xi, t)] = \hbar q \mathcal{H}(\xi, t)\aleph[\varphi(\xi, t; q) - \omega(\xi, t)], \tag{6}$$

where $\mathcal{L}$ is an auxiliary linear operator with the property

$$\mathcal{L}[0] = 0, \tag{7}$$

$\aleph$ - the nonlinear operator related to the original equation (5),
$q \in [0, 1]$ - the embedding parameter in topology (called the homotopy parameter),
$\varphi(\xi, t; q)$ - the solution for equation (6) for $q \in [0, 1]$,
$u_0(\xi, t)$ – the initial guess for $u(\xi, t)$,
$\hbar \neq 0$ - the convergence control parameter,
$\mathcal{H}(\xi, t)$ – an auxiliary function that is non-zero almost everywhere.

When $q = 0$ due to the property $\mathcal{L}[0] = 0$, equation (3) becomes

$$\varphi(\xi, t; 0) = u_0(\xi, t). \tag{8}$$

When $q = 1$, with $\hbar \neq 0$ and $\mathcal{H}(\xi, t) \neq 0$ equation (2) becomes equivalent to the original nonlinear equation (6) so that we have

$$\varphi(\xi, t; 1) = u(\xi, t), \tag{9}$$

where $u(\xi, t)$ is the solution to equation (5). As the homotopy parameter q increases from 0 to 1, the solution $\varphi(\xi, t; q)$ of equation (6) varies (or deforms) continuously from the initial guess $u_0(\xi, t)$ to the solution $u(\xi, t)$ of the original equation (5). This is why equation (6) is called the *zeroth-order deformation* equation.

If $\mathcal{L}, \mathcal{H}(\xi, t)$ and $\hbar$ are properly chosen so that the solution $\varphi(\xi, t; q)$ of the *zeroth-order deformation* equation (6) always exists for $q \in [0, 1]$ and it is analytic at $q = 0$; the Maclaurin series solution for $\varphi(\xi, t; q)$ with respect to q, i.e.

$$\varphi(\xi, t; q) = u_0(\xi, t) + \sum_{m=1}^{\infty} u_m(\xi, t) q^m \tag{10}$$

converges at $q = 1$. Then, due to equation (9), we have the approximation series

$$u(\xi, t) = u_0(\xi, t) + \sum_{m=1}^{\infty} u_m(\xi, t). \tag{11}$$

Substituting the series equation (10) into the *zeroth-order deformation* equation (6), we have the high-order approximation equations for $u_m(\xi,t)$ called the *mth-order deformation* equation

$$\mathcal{L}[u_m(\xi,t) - \chi_m u_{m-1}(\xi,t)] = \hbar\mathcal{H}(\xi,t)R_m(u_{m-1}(\xi,t)), \tag{12}$$

where

$$R_m(u_{m-1}(\xi,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\frac{\partial\varphi(\xi,t;q)}{\partial t} - (\aleph[\varphi(\xi,t;q)] - \omega(\xi,t))\right]\Bigg|_{q=0} \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1. \end{cases} \tag{14}$$

For our governing equations, equation (1) – equation (4)

$$R_m(u_{m-1}(\xi,t)) = \frac{\partial^\alpha u_{m-1}(\xi,t)}{\partial t^\alpha} - \frac{\partial u_{m-1}(\xi,t)}{\partial x^2} - \frac{\partial u_{m-1}(\xi,t)}{\partial y^2} - u_{m-1}(\xi,t)(1-(u_{m-1}(\xi,t))^\beta) - f(\xi,t). \tag{15}$$

Now the solution of *mth-order deformation* equation (12) for $m \ge 1$ reads

$$u_m(\xi,t) = \chi_m u_{m-1}(\xi,t) + \hbar J_t^\alpha[\mathcal{H}(\xi,t)R_m(u_{m-1}(\xi,t))] + c, \tag{16}$$

where c is the integration constant which is determined by the initial condition $u_0(\xi,t)$ and

$$J_t^\alpha[\mathcal{H}(\xi,t)R_m(u_{m-1}(\xi,t))] = \frac{1}{\Gamma(\alpha)}\int_0^t [\mathcal{H}(\xi,\tau)R_m(u_{m-1}(\xi,\tau))](t-\tau)^{\alpha-1}d\tau. \tag{17}$$

Now from equation (16) the values $u_m(\xi,t)$ for $m = 1,2,3,\ldots$ can be obtained and the series solutions are thus gained. Finally, the approximate solution is gained by truncating the series as

$$u_m(\xi,t) = \sum_{i=0}^{m} u_i(\xi,t). \tag{18}$$

It is clear from equation (18) that $u_m(\xi,t)$ contains the convergence control parameter $\hbar$, which determines the convergence region and rate of the homotopy series solution.

Unlike the conventional Homotopy Analysis Method (HAM) where $\hbar$ is found by using $\hbar$-level curves or by finding minimizing the residual square of the governing equation; we find $\hbar$ by setting

$$(u(x,1,t))_{\text{HAM}} = a_2(x,t) \tag{19}$$

for every time step.

4 HAM Solutions to Some Examples

Consider equations (1)-(5), equation (18) and equation (19).

Example 1

$$g(x, y) = 0, \tag{20}$$

$$a_1(x, t) = 0, \tag{21}$$

$$a_2(x, t) = t^{3+\alpha} \sin x \sin(1), \tag{22}$$

$$f(x, y, t) = t^3 \left(2t^\alpha + \frac{\Gamma(4+\alpha)}{6}\right) \sin(x)\sin(y) - t^{3+\alpha}\sin(x)\sin(y)\left(1 - t^{3+\alpha}\sin(x)\sin(y)\right), \tag{23}$$

$$\alpha = 0.2, \beta = 1. \tag{24}$$

To effectively use HAM the following properties are used where the initial guess is

$$u_0(x, y, t) = g(x, y). \tag{25}$$

And the convergence control parameter $\hbar$ is found by setting

$$(u(x, 1, t))_{\text{HAM}} = a_2(x, t). \tag{26}$$

The exact solution is

$$u(x, y, t) = t^{3+\alpha} \sin(x)\sin(y). \tag{27}$$

Using the methodology discussed in Section 3, we have the following error analysis for $u(x, y, t)$ when $x = 0.75$ and $y = 0.80$

$$\text{Relative Error for } u(x, y, t) = \frac{|(u(x, y, t))_{\text{exact}} - (u(x, y, t))_{\text{HAM}}|}{(u(x, y, t))_{\text{exact}}}. \tag{28}$$

Table 1: Error analysis of $u(x,y,t)$ when $\alpha = 0.2, x = 0.75$ and $y = 0.80$

t	$\hbar$	RE for $u(x,y,t)$
0.1	-0.4772267968	0.00001750625097
0.2	-0.4545453524	0.0001760078459
0.3	-0.4396617980	0.0006757516595
0.4	-0.4269004525	0.001744962992
0.5	-0.4140243192	0.003613927804
0.6	-0.3997845570	0.006486282271
0.7	-0.3834472895	0.01050738623
0.8	-0.3646811798	0.01573529739
0.9	-0.3435214730	0.02212249429
1.0	-0.3203255966	0.02951530841

Example 2

$$g(x,y) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta}{4}\sqrt{\frac{(x+y)}{(\beta+2)}}\right)\right]^{\frac{2}{\beta}}, \tag{29}$$

$$a_1(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta}{4}\left(\sqrt{\frac{x}{(\beta+2)}} - (\beta+4)\sqrt{\frac{1}{(\beta+2)}t}\right)\right)\right]^{\frac{2}{\beta}}, \tag{30}$$

$$a_2(x,t) = \left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta}{4}\sqrt{\frac{x+1}{(\beta+2)}} - (\beta+4)\sqrt{\frac{1}{(\beta+2)}}\,t\right)\right]^{\frac{2}{\beta}}, \tag{31}$$

$$f(x,y,t) = 0, \tag{32}$$

where

$$\alpha = 0.75, \beta = 2. \tag{33}$$

To effectively use HAM the following properties are used where the initial guess is

$$u_0(x,y,t) = g(x,y). \tag{34}$$

And the convergence control parameter $\hbar$ is found by setting

$$(u(x,1,t))_{\text{HAM}} = a_2(x,t) \tag{35}$$

when $\alpha = 1$, the exact solution is

$$u(x,y,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\beta}{4}\sqrt{\frac{1}{\beta+2}}(x+y) - (\beta+4)\sqrt{\frac{1}{\beta+2}}\,t\right)\right)^{\frac{2}{\beta}} \tag{36}$$

using the methodology discussed in Section 3, we have the following error analysis for $u(x, y, t)$ when $\alpha = 1, x = 0.25$ and $y = 0.75$.

Table 2: Error analysis of $u(x, y, t)$ when $\alpha = 1, x = 0.25$ and $y = 0.75$

t	$\hbar$	RE for $u(x, y, t)$
0.1	-3.577207146	0.01532350766
0.2	-3.800004555	0.008237957546
0.3	-3.672126597	0.02670271794
0.4	-3.380436450	0.03972295325
0.5	-3.033638864	0.04811163042
0.6	-2.695600733	0.05318164577
0.7	-2.394324712	0.05612387124
0.8	-2.136694378	0.05779039304
0.9	-1.920001664	0.05872125700
1.0	-1.738344080	0.05923713801

The exact solution when $0 < \alpha < 1$ is not known, then the Relative Error will be calculated as follows

$$\text{Relative Error for } u(x, y, t) = \frac{|u_m(x, y, t) - u_{m-1}(x, y, t)|}{u_m(x, y, t)}. \tag{37}$$

Using the methodology discussed in Section 3, we have the following error analysis for $u(x, y, t)$ when $\alpha = 0.75, x = 0.25$ and $y = 0.75$.

Table 3: Error analysis of $u(x, y, t)$ when $\alpha = 0.75, x = 0.25$ and $y = 0.75$

t	$\hbar$	RE for $u(x, y, t)$
0.1	-1.848796662	0.2696549594
0.2	-2.335536587	0.2326842920
0.3	-2.497715779	0.1650018804
0.4	-2.470772918	0.1079414949
0.5	-2.344506490	0.06623720107
0.6	-2.180411366	0.03887556120
0.7	-2.012809237	0.02218118065
0.8	-1.857205507	0.01244415453
0.9	-1.718728552	0.006913946042
1.0	-1.597646902	0.003820383035

5 Analysis and Conclusion

The numerical results obtained using the Homotopy Analysis Method (HAM) for the temperature distribution $u(x, y, t)$ show excellent agreement with known or exact solutions. A key innovation in this study was the use of the boundary condition $u(x, 1, t)$ to compute the convergence control parameter $\hbar$ at each time step. This dynamic, boundary-driven approach led to significantly faster convergence, reduced computational time, and minimized relative error (RE).

Unlike earlier applications of HAM, which commonly determine $\hbar$ using heuristic $\hbar$-curves or by minimizing the average residual error of the discretized solution—often applying a single $\hbar$ value across all time steps—our method adapts $\hbar$ at each time level based on physical boundary data. This customization enhances the accuracy and efficiency of the solution process.

This study presents a novel and elegant approach for determining the convergence control parameter $\hbar$, one that can be extended to a wide range of linear and nonlinear ordinary and partial differential equations. We have demonstrated that HAM is not only suitable for one-dimensional problems but is also a powerful and flexible tool for solving complex two-dimensional moving boundary problems with strong physical and mathematical consistency.

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