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Single and Multivalued Extended Interpolative Berinde Weak Type F-Contraction Mapping Theorem

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Abstract

In this paper we introduce the notion of an extended interpolative single and multivalued Berinde weak type F-contraction, and obtain some fixed point theorems for such mappings. An example is given to illustrate the main result.

1 Introduction and Preliminaries

Definition 1.1. *[1] Let (X,d) be a metric space. The mapping $S : X \mapsto X$ is called an interpolative Kannan type contraction, if there are constants $\alpha \in [0,1)$ and $c_1 \in (0,1)$ such that*

$$d(Sp,Sq) \leq \alpha d(p,Sp)^{c_1} d(q,Sq)^{1-c_1},$$

for all $p,q \in X \backslash Fix(S)$, where $Fix(S) = \{p \in X : Sp = p\}$.

Definition 1.2. *[2] Let (X,d) be a metric space. A mapping $S : X \mapsto X$ is said to be an interpolative Reich-Rus-Ciric type contraction, if there are constants $\alpha \in [0,1)$ and $c_1, c_2 \in (0,1)$ such that*

$$d(Sp,Sq) \leq \alpha d(p,q)^{c_1} d(p,Sp)^{c_2} d(q,Sq)^{1-c_1-c_2},$$

for all $p,q \in X \backslash Fix(S)$.

Definition 1.3. *[3] Let (X,d) be a metric space. We say that the self-mapping $S : X \mapsto X$ is an interpolative Hardy-Rogers type contraction if there exists $\alpha \in [0,1)$ and $c_1, c_2, c_3 \in (0,1)$ with $c_1+c_2+c_3 < 1$, such that*

$$d(Sp,Sq) \leq \alpha d(p,q)^{c_1} d(p,Sp)^{c_2} d(q,Sq)^{c_3} \left[\frac{d(p,Sq)+d(q,Sp)}{2}\right]^{1-c_1-c_2-c_3},$$

for all $p,q \in X \backslash Fix(S)$.

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Notation 1.4. *[4] Ω will denote the class of all functions $F : \mathbb{R}^+ \mapsto \mathbb{R}$ satisfying the following properties*

(a) *F is strictly increasing,*

(b) *for each sequence $\{p_n\} \subset \mathbb{R}^+$ of positive numbers $\lim_{n\to\infty} p_n = 0 \iff \lim_{n\to\infty} F(p_n) = -\infty$,*

(c) *there exists $m \in (0,1)$ such that $\lim_{p\to\ 0^+} p^m F(p) = 0$.*

Example 1.5. *[4] The following functions, $F : \mathbb{R}^+ \mapsto \mathbb{R}$, belong to Ω*

(a) *$F(p) = ln(p^2 + p)$, $p > 0$,*

(b) *$F(p) = \frac{-1}{\sqrt{p}}$, $p > 0$,*

(c) *$F(p) = ln(p)$, $p > 0$.*

Definition 1.6. *[5] Let (X, d) be a metric space. A mapping $S : X \mapsto X$ is called an extended interpolative Ciric-Reich-Rus type F-contraction if there exists $c_1, c_2 \in [0, 1)$ with $c_1 + c_2 < 1$, $\tau > 0$ and $F \in \Omega$ such that*

$$\tau + F(d(Sp, Sq)) \leq c_1 F(d(p, q)) + c_2 F(d(p, Sp)) + (1 - c_1 - c_2)F(d(q, Sq)),$$

for all $p, q \in X \backslash Fix(S)$ with $d(Sp, Sq) > 0$.

Definition 1.7. *[6] Let (X, d) be a metric space. A mapping $S : X \mapsto X$ is called an extended interpolative Hardy-Rogers type F-contraction, if there exists $c_1, c_2, c_3 \in [0, 1)$ with $c_1 + c_2 + c_3 < 1$, $\tau > 0$, and $F \in \Omega$ such that*

$$\tau + F(d(Sp, Sq)) \leq c_1 F(d(p, q)) + c_2 F(d(p, Sp)) + c_3 F(d(q, Sq)) + (1 - c_1 - c_2 - c_3) F\left(\frac{d(p, Sq) + d(q, Sp)}{2}\right),$$

for all $p, q \in X \backslash Fix(S)$ with $d(Sp, Sq) > 0$.

Definition 1.8. *([5], [6]) Let (X, d) be a metric space, and let $CB(X)$ be the collection of all nonempty bounded and closed subsets of X. The Hausdorff metric induced by the metric d of X is defined as*

$$H(A, B) = \max\{\sup_{p\in A} d(p, B), \sup_{q\in B} d(q, A)\},$$

for every $A, B \in CB(X)$ with $d(p, B) = \inf\{d(p, q) : q \in B\}$.

Definition 1.9. *([5], [6]) Let S be a mapping from X to $CB(X)$. If $p \in Sp$, then $p \in Q$ is called a fixed point of the multivalued mapping S.*

Definition 1.10. *[5] Let (X,d) be a metric space. A mapping $S : X \mapsto CB(X)$ is called an extended interpolative multivalued Ciric-Reich-Rus type F-contraction if there exists $c_1, c_2 \in [0,1)$ with $c_1 + c_2 < 1$, $\tau > 0$ and $F \in \Omega$ such that*

$$\tau + H(d(Sp, Sq)) \leq c_1 F(d(p,q)) + c_2 F(d(p, Sp)) + (1 - c_1 - c_2) F(d(q, Sq)),$$

for all $p, q \in X \backslash Fix(S)$ with $H(Sp, Sq) > 0$.

Definition 1.11. *[6] Let (X,d) be a metric space. A mapping $S : X \mapsto CB(X)$ is called an extended interpolative multivalued Hardy-Rogers type F-contraction, if there exists $c_1, c_2, c_3 \in [0,1)$ with $c_1+c_2+c_3 < 1$, $\tau > 0$, and $F \in \Omega$ such that*

$$\tau + H(d(Sp, Sq)) \leq c_1 F(d(p,q)) + c_2 F(d(p, Sp)) + c_3 F(d(q, Sq)) + (1 - c_1 - c_2 - c_3) F\left(\frac{d(p, Sq) + d(q, Sp)}{2}\right),$$

for all $p, q \in X \backslash Fix(S)$ with $H(Sp, Sq) > 0$.

Definition 1.12. *[7] Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x,y)^{\alpha} d(x, Tx)^{1-\alpha},$$

where $\lambda \in [0,1)$ and $\alpha \in (0,1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.13. *[7] Let (X,d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x,y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in [0,1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

2 Main Result

Definition 2.1. *Let (X,d) be a metric space. We will call the self mapping T on X an extended interpolative Berinde weak type F-contraction if there exists $\alpha \in (0,1)$, $\tau > 0$, and $F \in \Omega$ such that*

$$\tau + F(d(Tx, Ty)) \leq \alpha F(d(x,y)) + (1 - \alpha) F(d(x, Tx)),$$

for all $x, y \in X \backslash Fix(T)$ with $d(Tx, Ty) > 0$.

Theorem 2.2. *An extended interpolative Berinde weak type F-contraction self mapping on a complete metric space admits a fixed point in X.*

Proof. Let $\theta_0 \in X$. Define the sequence $\{\theta_n\}$ by $\theta_n = T^n(\theta_0)$ for each positive integer n. If there exists n_0 so that $\theta_{n_0} = \theta_{n_0+1}$, then θ_{n_0} is a fixed point of T. Suppose that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. From Definition 2.1, we have

$$\begin{aligned}\tau + F(d(\theta_n, \theta_{n+1})) &= \tau + F(d(T(\theta_n), T(\theta_{n-1}))) \\ &\leq \alpha F(d(\theta_n, \theta_{n-1})) + (1-\alpha)F(d(\theta_n, T\theta_n)) \\ &= \alpha F(d(\theta_n, \theta_{n-1})) + (1-\alpha)F(d(\theta_n, \theta_{n+1})).\end{aligned}$$

Suppose that $d(\theta_{n-1}, \theta_n) < d(\theta_n, \theta_{n+1})$ for some $n \geq 1$. Then from the above inequality we have, $\tau + F(d(\theta_n, \theta_{n+1})) \leq F(d(\theta_n, \theta_{n+1}))$, which is a contradiction. Therefore, $d(\theta_n, \theta_{n+1}) \leq d(\theta_{n-1}, \theta_n)$ for all $n \geq 1$. Thus, we have, $\tau + F(d(\theta_n, \theta_{n+1})) \leq F(d(\theta_{n-1}, \theta_n))$. Consequently we have,

$$F(d(\theta_n, \theta_{n+1})) \leq F(d(\theta_{n-1}, \theta_n)) - \tau \leq \cdots \leq F(d(\theta_0, \theta_1)) - n\tau,$$

for all $n \geq 1$. Therefore, $d(\theta_n, \theta_{n+1}) < d(\theta_{n-1}, \theta_n)$ for all $n \geq 1$. Now taking limits in the above inequality as $n \to \infty$ we get that $\lim_{n\to\infty} F(d(\theta_n, \theta_{n+1})) = -\infty$. From Notation 1.4(b), we have $\lim_{n\to\infty} d(\theta_n, \theta_{n+1}) = 0$. Put $\gamma_n = d(\theta_n, \theta_{n+1})$, Thus, $\lim_{n\to\infty} \gamma_n = 0$ Then for any $n \in \mathbb{N}$, we have $\gamma_n^k(F(\gamma_n) - F(\gamma_0)) \leq -\gamma_n^k n\tau < 0$. Thus, $\lim_{n\to\infty} \gamma_n^k n = 0$. So there is $N \in \mathbb{N}$, so that, $\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq N$. Now for any $m, n \in \mathbb{N}$ with $m > n$ we get

$$d(\theta_n, \theta_m) \leq \sum_{i=n}^{m-1} d(\theta_i, \theta_{i+1}) = \sum_{i=n}^{m-1} \gamma_n \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}}.$$

Since the last term of the above inequality tends to zero as $m, n \to \infty$, we have $d(\theta_n, \theta_m) \to 0$ as $m, n \to \infty$, that is $\{\theta_n\}$ is a Cauchy sequence. Since X is complete, there is $\theta \in X$ such that $\theta_n \to \theta$ as $n \to \infty$. Now we show that θ is a fixed point of T. Suppose to the contrary that $\theta \neq T\theta$.We consider two cases

Case 1: There is a subsequence $\{\theta_{n_k}\}$ such that $T\theta_{n_k} = T\theta$ for all $k \in \mathbb{N}$.

In this case

$$d(\theta, T\theta) = \lim_{k\to\infty} d(\theta_{n_k+1}, T\theta) = \lim_{k\to\infty} d(T\theta_{n_k}, T\theta) = 0.$$

Case 2: There is a natural number N such that $T\theta_n \neq T\theta$ for all $n \geq N$.

In this case, from Definition 2.1, we have

$$\begin{aligned}\tau + F(d(\theta_{n+1}, T\theta)) &= \tau + F(d(T\theta_n, T\theta)) \\ &\leq \alpha F(d(\theta_n, \theta)) + (1-\alpha)F(d(\theta_n, \theta_{n+1})).\end{aligned}$$

Letting $n \to \infty$ in the above inequalty, we get that $\lim_{n\to\infty} F(d(\theta_{n+1}, T\theta)) = -\infty$ and so $\lim_{n\to\infty} d(\theta_{n+1}, T\theta) = 0$. Therefore,

$$d(\theta, T\theta) = \lim_{n\to\infty} d(\theta_{n+1}, T\theta) = \lim_{n\to\infty} d(T\theta_n, T\theta) = 0.$$

Thus, $d(\theta, T\theta) = 0$, and so $\theta = T\theta$, and the proof is finished. □

Definition 2.3. *Let (X, d) be a metric space. We will call the multivalued mapping $T : X \mapsto CB(X)$ an extended interpolative multivalued Berinde weak type F-contraction if there exists $\alpha \in (0, 1)$, $\tau > 0$, and $F \in \Omega$ such that*

$$\tau + F(H(Tx, Ty)) \leq \alpha F(d(x, y)) + (1 - \alpha)F(d(x, Tx)),$$

for all $x, y \in X\backslash Fix(T)$ with $H(Tx, Ty) > 0$.

Theorem 2.4. *Let (X, d) be a complete metric space, and T be an extended interpolative multivalued Berinde weak type F-contraction. Assume in addition that*

(H): $F(\inf A) = \inf F(A)$.

Then T has a fixed point.

Proof. Choose two arbitrary points $\theta_0 \in X$ and $\theta_1 \in T\theta_0$. If $\theta_0 \in T\theta_0$ or $\theta_1 \in T\theta_1$ we have nothing to prove. Let $\theta_0 \notin T\theta_0$ and $\theta_1 \notin T\theta_1$. Then $T\theta_0 \neq T\theta_1$. Now

$$\begin{aligned}
\frac{\tau}{2} + F(d(\theta_1, T\theta_1)) &< \tau + F(H(T\theta_0, T\theta_1)) \\
&\leq \alpha F(d(\theta_0, \theta_1)) + (1 - \alpha)F(d(\theta_0, T\theta_0)) \\
&\leq \alpha F(d(\theta_0, \theta_1)) + (1 - \alpha)F(d(\theta_0, \theta_1)) \\
&= F(d(\theta_0, \theta_1)).
\end{aligned}$$

From the above inequality and using (H), we can conclude that there is $\theta_2 \in T\theta_1$ so that

$$\frac{\tau}{2} + F(d(\theta_1, \theta_2)) < F(d(\theta_0, \theta_1)).$$

Continuing this process, we obtain a sequence $\{\theta_n\}$ in X such that $\theta_{n+1} \in T\theta_n$, $\theta_n \notin T\theta_n$, and,

$$\frac{\tau}{2} + F(d(\theta_n, \theta_{n+1})) < F(d(\theta_{n-1}, \theta_n)).$$

If there is n_0 so that $\theta_{n_0} = \theta_{n_0+1}$, then θ_{n_0} is a fixed point of T. So we assume that $\theta_n \neq \theta_{n+1}$ for all $n \geq 0$. Consequently,

$$F(d(\theta_n, \theta_{n+1})) \leq F(d(\theta_{n-1}, \theta_n)) - \frac{\tau}{2} \leq \cdots \leq F(d(\theta_0, \theta_1)) - n\left(\frac{\tau}{2}\right),$$

for all $n \geq 1$. Similar to Theorem 2.2, we find that $\{\theta_n\}$ is a Cauchy sequence. Since X is complete, there is $\theta \in X$ such that $\theta_n \to \theta$. Now we show that θ is a fixed point of T. Assume to the contrary that $\theta \notin T\theta$. We consider two cases.

Case 1: There is a subsequence $\{\theta_{n_k}\}$ such that $T\theta_{n_k} = T\theta$ for all $k \in \mathbb{N}$.

In this case

$$d(\theta, T\theta) = \lim d(\theta_{n_k+1}, T\theta) = \lim H(T\theta_{n_k}, T\theta) = 0.$$

Case 2: There is a natural number N such that $T\theta_n \neq T\theta$ for all $n \geq N$.

In this case we have

$$\begin{aligned}\tau + F(d(\theta_{n+1}, T\theta)) &= \tau + F(H(T\theta_n, T\theta)) \\ &\leq \alpha F(d(\theta_n, \theta)) + (1-\alpha)F(d(\theta_n, \theta_{n+1})).\end{aligned}$$

Now letting $n \to \infty$ in the above inequality, we find that $\lim_{n\to\infty} F(d(\theta_{n+1}, T\theta)) = -\infty$ and so $\lim_{n\to\infty} d(\theta_{n+1}, T\theta) = 0$. Therefore,

$$d(\theta, T\theta) = \lim_{n\to\infty} d(\theta_{n+1}, T\theta) \leq \lim_{n\to\infty} H(T\theta_n, T\theta) = 0.$$

Thus, $d(\theta, T\theta) = 0$ and hence $\theta = T\theta$, and the proof is finished. □

Example 2.5. *Let $X = \{-1, 0, 1\}$ be endowed with the metric*

$$d(\theta, v) = \begin{cases} 0 & \text{if } \theta = v \\ \frac{3}{2} & \text{if } (\theta, v) \in \{(1,-1), (-1,1)\} \\ 1 & \text{otherwise.} \end{cases}$$

Clearly (X, d) is complete. Take $T(0) = T(-1) = 0$ and $T(1) = -1$. First, letting $\theta = 0$ and $v = 1$, we have

$$F(d(T\theta, Tv)) = F(d(0,-1)) = F(1) \quad \text{and} \quad F(d(\theta, v)) = F(d(0,1)) = F(1).$$

Thus, we cannot find $\tau > 0$ such that $\tau + F(d(T\theta, Tv)) \leq F(d(\theta, v))$, that is, Theorem 1.9 [5] is not applicable. On the other hand, let $\theta, v \in X \backslash Fix(T)$ with $d(T\theta, Tv) > 0$. Hence, $\theta, v \in \{(1,-1), (-1,1)\}$. Without loss of generality take $(\theta, v) = (1,-1)$. Choose $\alpha = \frac{1}{2}$, $\tau = \frac{1}{3}ln\left(\frac{3}{2}\right)$, and $F(t) = ln(t)$. Observe we have $\tau + F(d(T\theta, Tv)) = \frac{1}{3}ln\left(\frac{3}{2}\right)$ and $\alpha F(d(\theta, v)) + (1-\alpha)F(d(\theta, T\theta)) = ln\left(\frac{3}{2}\right)$. It follows that T is an extended interpolative Berinde weak type F-contraction. Here, T admits a fixed point $(u = 0)$.

References

[1] Karapınar, E. (2018). Revisiting the Kannan type contractions via interpolation. *Advances in the Theory of Nonlinear Analysis and its Applications*, *2*(2), 85–87. https://doi.org/10.31197/atnaa.431135

[2] Karapınar, E., Agarwal, R., & Aydi, H. (2018). Interpolative Reich–Rus–Çirić type contractions on partial metric spaces. *Mathematics*, *6*(256), 1–7. https://doi.org/10.3390/math6110256

[3] Karapınar, E., Alqahtani, O., & Aydi, H. (2018). On interpolative Hardy–Rogers type contractions. *Symmetry*, *11*(8), 1–7. https://doi.org/10.3390/sym11010008

[4] Wardowski, D. (2012). Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory and Applications*, *2012*(94), 1–6. https://doi.org/10.1186/1687-1812-2012-94

[5] Mohammadi, B., Parvaneh, V., & Aydi, H. (2019). On extended interpolative Çirić–Reich–Rus type F-contractions and an application. *Journal of Inequalities and Applications*, *2019*, 1–11. https://doi.org/10.1186/s13660-019-2227-z

[6] Yildirim, İ. (2022). On extended interpolative single and multivalued F-contractions. *Turkish Journal of Mathematics*, *46*, 688–698. https://doi.org/10.55730/1300-0098.3116

[7] Ampadu, C. B. (2020). Some fixed point theory results for the interpolative Berinde weak operator. *Earthline Journal of Mathematical Sciences*, *4*(2), 253–271. https://doi.org/10.34198/ejms.4220.253271