**Earthline Journal of Mathematical Sciences**
E-ISSN: 2581-8147
Volume 15, Number 6, 2025, Pages 1051-1062
https://doi.org/10.34198/ejms.15625.10511062

# Properties of Generalized Strongly Close-to-convex Functions

Khalida Inayat Noor[1] and Imtiaz Waheed[2,*]

[1] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
e-mail: khalidan@gmail.com

[2] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
e-mail: imtiazwaheed96@gmail.com

**Abstract**

This paper defines certain subclasses of analytic functions and various properties including necessary conditions, distortion result, inclusion properties are investigated. In addition radius problems are discussed. Several known consequences of our investigations are also pointed out.

## 1 Introduction

Let $\mathcal{A}$ be the class of analytic functions $f$ in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and be given by

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n. \tag{1.1}$$

Also, we denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ the subclasses of $\mathcal{A}$ which contains univalent, starlike, convex and close-to-convex in $\mathcal{U}$, respectively.

For $f, g \in \mathcal{A}$, the convolution (Hadamard product) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^n,$$

with $g(z) = z + \sum_{k=2}^{\infty} b_k z^n$ and $f(z)$ is given by (1.1).

Received: July 22, 2025; Accepted: September 8, 2025; Published: September 15, 2025
2020 Mathematics Subject Classification: 30C45, 30C50.
Keywords and phrases: analytic functions, generalized Janowski functions, close-to-convex functions, bounded boundary rotation.

Using the techniques from convolution theory, the authors [1] introduced an integral operator $\mathcal{O}_N\left[\left(f_j, g_j, h_j\right)(z)\right] : \mathcal{A}^N \to \mathcal{A}$ as follows:

$$\mathcal{F}_N(z) = \mathcal{O}_N\left(f_j, g_j, h_j\right)(z) = \int_0^z \prod_{j=1}^{N} \left[f_j(t) * g_j(t)\right]^{\alpha_j} \left(\frac{h_j(t)}{t}\right)^{\beta_j} dt, \tag{1.2}$$

where $f_j, g_j, h_j \in \mathcal{A}$ with $f_j(z) * g_j(z) \neq 0$, $\alpha_j$, $\beta_j \geq 0$ for $j = 1, 2, 3, ..., N$.

If $f \prec g$, then $f(z) = g(w(z))$, where $\prec$ denote subordination and $w$ is a Schwartz function with the properties $w(0) = 0$ and $|w(z)| < |z|$. Further, if the function $g$ is univalent in $\mathcal{U}$, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$. Then $p \in \mathcal{P}$, if $Re\left(p(z)\right) > 0$ in $\mathcal{U}$. Such functions are known as Caratheodory functions. Now, we generalize the class $\mathcal{P}$ as following.

**Definition 1.1.** Let $p_1$, $p_2 \in \mathcal{P}$. Then $h \in \mathcal{P}_m\left(\phi\right)$, $m \geq 2$ if and only if

$$h(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_1(z), \tag{1.3}$$

where $p_i \prec \phi$, $i = 1, 2$ and $\phi$ is convex univalent function in $\mathcal{U}$.

We choose $\phi(z) = \left(\frac{1+Xz}{1+Yz}\right)^{\alpha}$, $(\alpha \in (0, 1]; -1 \leq Y < X \leq 1)$, in Definition 1.1.

First, we show that $\phi(z) = \left(\frac{1+Xz}{1+Yz}\right)^{\alpha}$ is convex univalent. Expanding $\phi(z)$ as

$$\phi(z) = 1 + \alpha\left(X - Y\right) z - \left[\alpha\left(X - Y\right) - \frac{1}{2}\alpha\left(\alpha - 1\right)\left(X - Y\right)^2\right] z^2 + ....$$

Also

$$\phi'(z) = \left(\frac{1 + Xz}{1 + Yz}\right)^{\alpha} \frac{\alpha\left(X - Y\right)}{\left(1 + Xz\right)\left(1 + Yz\right)}.$$

Simple calculations can show that

$$\Re\left\{\phi'(z)\right\} \geq \alpha\left|X - Y\right| \frac{1 - |X|^{\alpha-1}}{1 + |Y|^{\alpha+1}} > 0, \quad \text{for all } z \in \mathcal{U}.$$

This shows $\phi(z)$ is close-to-convex and hence univalent, see [5].

Now, using definition of convexity, we have

$$\frac{\left(z\phi'(z)\right)'}{\phi'(z)} = \frac{X\alpha z - XYz^2 - \alpha Yz + 1}{\left(1 + Xz\right)\left(1 + Yz\right)}.$$

Since $t(r) = 1 - \alpha(X - Y)r - XYr^2$ is decreasing in $[0, 1)$ and $t(0) = 1$. It can easily be seen that

$$\Re\left(\frac{(z\phi'(z))'}{\phi'(z)}\right) \geq 0.$$

Thus, taking $\phi(z) = \left(\frac{1+Xz}{1+Yz}\right)^{\alpha}$ in Definition 1.1, we obtain the class $\mathcal{P}_m[X, Y; \alpha]$.

We note the following as special cases:

(i) For $m = 2$, $\alpha = 1$, $X = 1$ and $Y = -1$, we have the class $\mathcal{P}$. For the class $\mathcal{P}$, we refer to [3].

(ii) Let $m = 2$ and $\alpha = 1$. Then the class $\mathcal{P}_2[X, Y; 1] = \mathcal{P}[X, Y]$ is well-known class of Janowski functions, see [4].

(iii) When we take $\alpha = 1$, $X = 1$, $Y = -1$ this class reduces to the class $\mathcal{P}_m$ which was introduced and studied by Pinchuk [8].

(iv) $\mathcal{P}_m[X, Y; \alpha] \subset \mathcal{P}_m(\rho)$, $\rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}$, where $\rho$ is called the order of the function $p \in \mathcal{P}_m[X, Y; \alpha]$. We note that $p \in \mathcal{P}_2(\rho) = \mathcal{P}(\rho)$ implies $\Re(p(z)) > \rho$, $(0 \leq \rho < 1)$. For the class $\mathcal{P}_m(\rho)$, we refer to [7]. If $h \in \mathcal{P}_m(\rho)$, then we can write

$$h(z) = (1 - \rho)\,p_1(z) + \rho, \;\; p_1 \in \mathcal{P}_m.$$

(v) $\mathcal{P}_{2,\alpha}[1, -1] = \widetilde{\mathcal{P}}_{\alpha}$, and $\widetilde{\mathcal{P}}_{\frac{1}{2}}$ is related to the right-half of the Lemniscate of Bernoulli (see [6]) enclosing the region

$$\mathcal{D} = \{w \in \mathcal{C} : \Re(w) > 0, \; \left|w^2 - 1\right| < 1\}.$$

Also, it is obvious that, in this case $|\arg w| < \frac{\pi}{4}$.

**Definition 1.2.** Let $f \in \mathcal{A}$ and be locally univalent with $f'(z) \neq 0$, $z \in \mathcal{U}$. Then $f \in \mathcal{V}_m[X, Y; \alpha]$ if and only if

$$\frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_m[X, Y; \alpha].$$

We note some of the special cases as follows:

(i) $\mathcal{V}_m[1, -1; 1] = \mathcal{V}_m$ is the class of functions $f$ with bounded boundary rotation, see [2, 8].

(ii) $\mathcal{V}_m[X, Y; 1] = \mathcal{V}_m[X, Y]$, see [4].

(iii) $\mathcal{V}_2\left[X,Y;\alpha\right]=\mathcal{C}_\alpha\left[X,Y\right]\subset\mathcal{C}\left(\rho\right)\subset\mathcal{C}$, where $\mathcal{C}$ is the well-known class of convex univalent functions and $\rho=\left(\frac{1-X}{1-Y}\right)^\alpha$.

(iv) $\mathcal{V}_m\left[X,Y;\alpha\right]\subset\mathcal{V}_m(\rho)$, see [7].

The class $\mathcal{R}_m\left[X,Y;\alpha\right]$ related with functions of bounded radius rotation is defined by using Alexander type relation as given below.

$$\mathcal{V}_m\left[X,Y;\alpha\right]=\left\{f\in\mathcal{A}:zf'\in\mathcal{R}_m\left[X,Y;\alpha\right]\right\}.$$

In particular, we note that $\mathcal{R}_2\left[1,-1;1\right]=\mathcal{S}^*$, the class of starlike univalent functions.

**Definition 1.3.** Let $f\in\mathcal{A}$. Then $f\in\mathcal{K}_m\left[X,Y;\alpha\right]$ if and only if there exists $g\in\mathcal{V}_m$ such that

$$\frac{f'}{g'}\in\mathcal{P}_m\left[X,Y;\alpha\right],\quad z\in\mathcal{U}.$$

The class $\mathcal{K}_m\left[1,-1;1\right]=\mathcal{T}_m$ has been introduced and studied in [9]. Also, it can easily be seen that $\mathcal{K}_2\left[1,-1;1\right]=\mathcal{K}$ is the familiar class of close-to-convex univalent functions first introduced by Kaplan [5].

# 2 Main Results

## 2.1 Necessary Conditions

**Theorem 2.1.** *Let $p\in\mathcal{P}\left(\rho\right)$, $0\le\rho<1$. Then, for $z=re^{i\theta}$, $0\le\theta_1<\theta_2\le2\pi$, we have*

$$\begin{aligned}\max_{p\in\mathcal{P}(\rho)}\left|\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zp'(z)}{p(z)}\right\}d\theta\right| &\le 2\sin^{-1}\left\{\frac{2\left(1-\rho\right)r}{1-\left|1-2\rho\right|r^2}\right\}\\ &= \pi-2\cos^{-1}\left\{\frac{2\left(1-\rho\right)r}{1-\left|1-2\rho\right|r^2}\right\}.\end{aligned}$$

*Proof.* We can write, [3], for $z=re^{i\theta}$

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg p\left(re^{i\theta}\right) &= \frac{\partial}{\partial\theta}\Re\left\{-i\ln p(re^{i\theta})\right\}\\ &= \Re\left\{\frac{re^{i\theta}p'\left(re^{i\theta}\right)}{p\left(re^{i\theta}\right)}\right\}=\Re\left\{\frac{zp'(z)}{p(z)}\right\}.\end{aligned}$$

Consequently,

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}p'\left(re^{i\theta}\right)}{p\left(re^{i\theta}\right)}\right\} d\theta = \arg p\left(re^{i\theta_2}\right) - \arg p\left(re^{i\theta_1}\right).$$

Hence,

$$\max_{p\in\mathcal{P}(\rho)}\left|\int_{\theta_1}^{\theta_2} \Re\left\{\frac{re^{i\theta}p'\left(re^{i\theta}\right)}{p\left(re^{i\theta}\right)}\right\} d\theta\right| = \max_{p\in\mathcal{P}(\rho)}\left|\arg p\left(re^{i\theta_2}\right) - \arg p\left(re^{i\theta_1}\right)\right|.$$

Since we can write

$$p(z) = (1-\rho)\, p_1(z) + \rho, \quad p_1 \in \mathcal{P}.$$

We have well-known [3] result as

$$\left|p_1(z) - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}.$$

From this, we have

$$\left|p_1(z) - \frac{1+(1-2\rho)\, r^2}{1-r^2}\right| \le \frac{2\,(1-\rho)\, r}{1-r^2}.$$

Thus the values of $p(z)$ are contained in the circle of Apollonius whose diameter is the line segment from $\frac{1+(1-2\rho)r^2}{1-r^2}$ to $\frac{1-(1-2\rho)r^2}{1-r^2}$. The circle is centered at the point $\frac{1+(1-2\rho)r^2}{1-r^2}$ and has the radius $\frac{2(1-\rho)r}{1-r^2}$. So $|\arg p(z)|$ attains its maximum at points where a ray from the origin is tangent to the circle, that is, when

$$\arg p(z) = \pm \sin^{-1}\left\{\frac{2\,(1-\rho)\, r}{1-(1-2\rho)\, r^2}\right\},$$

and this completes the proof. □

**Theorem 2.2.** *Let $g \in \mathcal{V}_m\left[X, Y; \alpha\right]$. Then, for $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} d\theta > -(1-\rho)\left(\frac{m}{2}-1\right)\pi,$$

*where $m \ge 2$ and $\rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}$.*

*Proof.* By the definition, we observe that

$$\mathcal{V}_m\left[X, Y; \alpha\right] \subset \mathcal{V}_m(\rho), \quad \rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}. \tag{2.1}$$

It is known [7] that, for $g \in \mathcal{V}_m(\rho)$ we have $g_1 \in \mathcal{V}_m$ such that

$$g'(z) = \left(g_1'(z)\right)^{1-\rho}. \tag{2.2}$$

Also, for $g_1 \in \mathcal{V}_m$, Brannan [2] has proved that

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zg_1''(z)}{g_1'(z)}\right\} d\theta > -\left(\frac{m}{2}-1\right), \; z = re^{i\theta}. \tag{2.3}$$

Taking logarithmic differentiation of (2.2), we get

$$1+\frac{zg''(z)}{g'(z)}=(1-\rho)\left[1+\frac{zg_1''(z)}{g_1'(z)}\right]. \tag{2.4}$$

Using (2.2) to (2.4), we obtain the required result and the proof is complete. □

In particular, when $\rho = 0$, we obtain a known result [2] for $f \in \mathcal{V}_m$.

**Theorem 2.3.** *Let $f \in \mathcal{K}_m[1,-1;\alpha]$. Then, $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\left(\frac{m}{2}-1+\alpha\right)\pi.$$

*Proof.* We can write

$$\frac{f'(z)}{g'(z)} = h(z), \quad g \in \mathcal{V}_m, \;\; h \in \mathcal{P}_m[1,-1;\alpha] = \mathcal{P}_\alpha. \tag{2.5}$$

For $h \in \mathcal{P}_\alpha$, we have $h(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha$. Therefore, it follows that we can write

$$h(z) = p^\alpha(z), \quad p \in \mathcal{P}. \tag{2.6}$$

Using (2.6) in (2.5) and differentiating logarithmically, we get

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \alpha\frac{zp'(z)}{p(z)}.$$

Now, the result follows from Theorem 2.1 and Theorem 2.2 with $\rho = 0$. □

From Theorem 2.3, we have the following special cases:

**Corollary 2.4.** *Let $f \in \mathcal{T}_m$. Then, for $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} d\theta > -\frac{m}{2}\pi.$$

This result has been proved in [9]. Furthermore, for $m = 2$, we obtain a necessary (and sufficient) condition for close-to-convex functions which has been proved in [5].

**Theorem 2.5.** *Let $\mathcal{F}_N(z)$ be defined by the operator (1.2). Also, let $g_j = \frac{z}{(1-z)^2}$, $f_j \in \mathcal{K}_m[X,Y;\alpha]$ and $h_j \in \mathcal{R}_m[X,Y;\alpha]$ for all $j = 1,2,3,...,N$, $\alpha_j$, $\beta_j \ge 0$. Then, for $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$ and $\left(1-\sum_{j=1}^N(\alpha_j+\beta_j)=0\right)$, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{1+\frac{z\mathcal{F}_N''(z)}{\mathcal{F}_N'(z)}\right\} d\theta > -\left[\sum_{j=1}^{N}\left(\alpha_j\left(\frac{m}{2}+\alpha-1\right)+\beta_j(1-\rho)\left(\frac{m}{2}-1\right)\right)\right]\pi,$$

*where $\rho = \left(\frac{1-X}{1-Y}\right)^\alpha$.*

*Proof.* From (1.2), we have

$$\mathcal{F}'_N(z) = \prod_{j=1}^{N} \left[ f_j(z) * \frac{z}{(1-z)^2} \right]^{\alpha_j} \left( H'_j(z) \right)^{\beta_j}, \quad zH'_j(z) = h_j(z). \tag{2.7}$$

Using convolution property $f_j(z) * \frac{z}{(1-z)^2} = zf'_j(z)$ and Alexander relation $H_j \in \mathcal{V}_m[X, Y; \alpha] \subset \mathcal{V}_m(\rho)$, with $\rho = \left(\frac{1-A}{1-B}\right)^{\alpha}$, we can write (1.2) as

$$\mathcal{F}'_N(z) = \prod_{j=1}^{N} \left( f'_j(z) \right)^{\alpha_j} \left( H'_j(z) \right)^{\beta_j}. \tag{2.8}$$

On logarithmic differentiation of (2.8) together with the given condition, we have

$$1 + \frac{z\mathcal{F}''_N(z)}{\mathcal{F}'_N(z)} = \sum_{j=1}^{N} \alpha_j \left( 1 + \frac{zf''_j(z)}{f'_j(z)} \right) + \sum_{j=1}^{N} \beta_j \left( 1 + \frac{zH''_j(z)}{H'_j(z)} \right),$$

and now using Theorem 2.2 and Theorem 2.3, it follows that

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{z\mathcal{F}''_N(z)}{\mathcal{F}'_N(z)} \right\} d\theta &= \int_{\theta_1}^{\theta_2} \Re \left[ \sum_{j=1}^{N} \alpha_j \left( 1 + \frac{zf''_j(z)}{f'_j(z)} \right) \right] d\theta \\ &\quad + \int_{\theta_1}^{\theta_2} \Re \left[ \sum_{j=1}^{N} \beta_j \left( 1 + \frac{zH''_j(z)}{H'_j(z)} \right) \right] d\theta \\ &> -\left[ \sum_{j=1}^{N} \alpha_j \left( \frac{m}{2} + \alpha - 1 \right) + \sum_{j=1}^{N} \beta_j (1-\rho) \left( \frac{m}{2} - 1 \right) \right] \pi \\ &= -\left[ \sum_{j=1}^{N} \left\{ \alpha_j \left( \frac{m}{2} + \alpha - 1 \right) + \beta_j (1-\rho) \left( \frac{m}{2} - 1 \right) \right\} \right] \pi. \end{aligned}$$

This completes the proof. □

**Corollary 2.6.** *Let $\alpha_j = 0$ for all $j$. Then $\mathcal{F}_N$, defined by (1.2), in Theorem 2.5, satisfies the necessary condition as*

$$\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{z\mathcal{F}''_N(z)}{\mathcal{F}'_N(z)} \right\} d\theta > -\sum_{j=1}^{N} \beta_j (1-\rho) \left( \frac{m}{2} - 1 \right) \pi,$$

*where $z = re^{i\theta}$, $0 \le \theta_1 < \theta_2 \le 2\pi$ and $\rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}$.*

**Corollary 2.7.** *If we take $\sum_{j=1}^{N} \beta_j = 1$ in Corollary 2.6, then*

$$\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{z\mathcal{F}''_N(z)}{\mathcal{F}'_N(z)} \right\} d\theta > -(1-\rho) \left( \frac{m}{2} - 1 \right) \pi.$$

*Moreover, when $\rho = 0$, this gives necessary condition for $\mathcal{F}_N \in \mathcal{V}_m$. Also, $\mathcal{F}_N$ is close-to-convex for $2 \le m \le \frac{2(2-\rho)}{1-\rho}$.*

## 2.2 Distortion Result

**Theorem 2.8.** *Let $f \in \mathcal{K}_m[1,-1;\alpha]$. Then, $z = re^{i\theta}$*

$$\frac{(1-r)^{\frac{m}{2}+\alpha-1}}{(1+r)^{\frac{m}{2}+\alpha+1}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{m}{2}+\alpha-1}}{(1-r)^{\frac{m}{2}+\alpha+1}}. \tag{2.9}$$

*The equality is attained for the function $f_0 \in \mathcal{K}_m[1,-1;\alpha]$ defined by*

$$f_0'(z) = \frac{(1+\delta_1 z)^{\frac{m}{2}+\alpha-1}}{(1-\delta_2 z)^{\frac{m}{2}+\alpha+1}}, \quad |\delta_1| = |\delta_2| = 1.$$

*Proof.* The proof is immediate when we use the distortion theorem for $g \in \mathcal{V}_m$, see [8], and for $h = p^\alpha$, $p \in \mathcal{P}$ given as ( see [3]):

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r}.$$

□

Special cases:

(i) for $f \in \mathcal{K}_m[1,-1;1]$ implies $f \in \mathcal{T}_m$ and in this case (2.9) reduces to the following bounds

$$\frac{(1-r)^{\frac{m}{2}}}{(1+r)^{\frac{m}{2}+2}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{m}{2}}}{(1-r)^{\frac{m}{2}+2}}.$$

We can obtain the bounds (2.9) for other permissable values $m \geq 2$, $\alpha \in (0,1]$ and $f \in \mathcal{K}_m[1,-1;\alpha]$. By taking $m = 2$ and $\alpha = 1$, we obtain distortion result for the class $\mathcal{K}$ of close-to-convex functions and it is sharp, see [3].

(ii) When $m = 2$, $f \in \mathcal{K}_m[1,-1;\alpha]$ is strongly close-to-convex and have

$$\frac{(1-r)^{\alpha}}{(1+r)^{\alpha+2}} \leq |f'(z)| \leq \frac{(1+r)^{\alpha}}{(1-r)^{\alpha+2}}.$$

## 2.3 Inclusion Property

The following lemma is required to investigate our result.

**Lemma 2.9.** *[10] Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and let $\psi(u,v)$ be a complex valued function satisfying conditions:*

*(i) $\psi(u,v)$ is continuous in a domain $\mathcal{D} \subset \mathcal{C}^2$.*

*(ii)* $(0,1) \in \mathcal{D}$ *and* $\psi(1,0) > 0$.

*(iii)* $\Re(\psi(iu_2, v_1)) \leq 0$, *whenever* $(iu_2, v_1) \in \mathcal{D}$ *and* $v_1 \leq -\frac{1}{2}\left(1+u_2^2\right)$.

*If* $h(z) = 1 + c_1 z + c_2 z^2 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n$ *is a function, analytic in* $\mathcal{U}$ *such that*

$$\left(h(z), zh'(z)\right) \in \mathcal{D} \text{ and } \Re\left\{\psi\left(h(z), zh'(z)\right)\right\} > 0, \text{ for } z \in \mathcal{U},$$

*then* $\Re(h(z)) > 0$ *in* $\mathcal{U}$.

**Theorem 2.10.** *Let* $f \in \mathcal{V}_2[X, Y; \alpha] \subset \mathcal{C}(\rho)$ *with* $\rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}$. *Then* $f \in \mathcal{S}^*(\rho_1)$, *where*

$$\rho_1 = \frac{(2\rho - 1) + \sqrt{(1-2\rho)^2 + 8}}{4},$$

*with* $\rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}$.

*Proof.* Let

$$\frac{zf'(z)}{f(z)} = (1-\rho_1)\, h(z) + \rho_1.$$

Then

$$zf'(z) = f(z)\left[(1-\rho_1)\, h(z) + \rho_1\right]$$

and so is

$$\frac{(zf'(z))'}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{(1-\rho_1)\, zh'(z)}{(1-\rho_1)\, h(z) + \rho_1}.$$

That is

$$\left[\frac{(zf'(z))'}{f'(z)} - \rho\right] = \left[(1-\rho_1)\, h(z) + \frac{(1-\rho_1)\, zh'(z)}{(1-\rho_1)\, h(z) + \rho_1} + \rho_1 - \rho\right].$$

We construct the functional $\psi(u, v)$ of Lemma 2.9 by taking $u = u_1 + iu_2 = h(z)$ and $v = v_1 + iv_2 = zh'(z)$. The first two conditions are obviously easy to verify. We proceed to check the condition (iii).

$$\begin{aligned}
\Re\psi(iu_2, v_1) &= \Re\left\{(1-\rho_1)\, iu_2 + \frac{(1-\rho_1)\, v_1}{(1-\rho_1)\, iu_2 + \rho_1} + \rho_1 - \rho\right\} \\
&= (\rho_1 - \rho) + \frac{\rho_1(1-\rho_1)\, v_1}{(1-\rho_1)^2 u_2^2 + \rho_1^2} \\
&\leq (\rho_1 - \rho) - \frac{\rho_1(1-\rho_1)\left(1+u_2^2\right)}{2\left\{(1-\rho_1)^2 u_2^2 + \rho_1^2\right\}}, \text{ for } v_1 \leq -\frac{\left(1+u_2^2\right)}{2} \\
&\leq 0,
\end{aligned}$$

when $\rho_1$ is calculated in terms of $\rho$ as,

$$\rho_1 = \frac{(2\rho-1)+\sqrt{(1-2\rho)^2+8}}{4}, \text{ with } \rho = \left(\frac{1-X}{1-Y}\right)^{\alpha}.$$

Thus condition (iii) is satisfied and applying Lemma 2.9, $\Re(h(z)) > 0$ in $\mathcal{U}$. Consequently, $f \in \mathcal{S}^*(\rho_1)$. □

As a special case, for $\rho = 0$ and so $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$ in $\mathcal{U}$. Furthermore, for $A = 0$, $B = -1$ and $\alpha = 1$ implies $\rho = 1/2$, we get $\rho_1 = \frac{1}{\sqrt{2}}$.

## 2.4 Radius Problem

**Theorem 2.11.** *Let $f \in \mathcal{K}_m[1,-1,\alpha]$. Then $f$ maps $|z| < r_1$ onto a convex domain, where*

$$r_1 = \frac{2}{(m+2)+\sqrt{(m+2)^2-4}}; \quad (m \geq 2).$$

*Proof.* Since $f \in \mathcal{K}_m[1,-1,\alpha]$, we can write

$$f'(z) = g'(z)h^{\alpha}(z), \quad g \in \mathcal{V}_m,\ h \in \mathcal{P}.$$

Logarithmic differentiation and some simple computation leads us to

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \alpha\frac{zh'(z)}{h(z)}.$$

It is known [8] that, for $g \in \mathcal{V}_m$,

$$\Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} \geq \frac{r^2-mr+1}{1-r^2},\ z = re^{i\theta},\ 0 \leq r < 1,$$

and

$$\left|\frac{zh'(z)}{h(z)}\right| \leq \frac{2r}{1-r^2}, \quad \text{for } h \in \mathcal{P}, \text{ (see [3])}.$$

Therefore using these inequalities in the following

$$\Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} \geq \Re\left\{\frac{(zg'(z))'}{g'(z)}\right\} - \alpha\left|\frac{zh'(z)}{h(z)}\right|,$$

this implies that

$$\begin{aligned}\Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} &\geq \frac{r^2-mr+1}{1-r^2} - \frac{2\alpha r}{1-r^2} \\ &= \frac{r^2-(m+2\alpha)r+1}{1-r^2}.\end{aligned}$$

The right hand side is positive for $r \leq r_1$, where

$$\begin{aligned} r_1 &= \frac{(m+2\alpha) - \sqrt{(m+2\alpha)^2 - 4}}{2} \\ &= \frac{2}{(m+2\alpha) + \sqrt{(m+2\alpha)^2 - 4}}. \end{aligned}$$

□

Special cases:

(i) For $\alpha = 1$ implies $f \in \mathcal{T}_m$, then we obtain radius of convexity given in [9] as

$$r_1 = \frac{2}{(m+2) + \sqrt{(m+2)^2 - 4}}.$$

(ii) Let $m = 2$. Then $f \in \mathcal{K}$ and this gives us well known [3] radius of convexity for close-to-convex functions. That is, in this case $r_1 = 1/\left(2 + \sqrt{3}\right)$. This radius is best possible for the Koebe function being the extremal one.

## References

[1] Arif, M., Noor, K. I., & Ghani, F. (2012). Some properties of an integral operator defined by convolution. *Journal of Inequalities and Applications*, *2012*(13), 1–13. https://doi.org/10.1186/1029-242X-2012-13

[2] Brannan, D. A. (1969). On functions of bounded boundary rotation I. *Proceedings of the Edinburgh Mathematical Society*, *16*, 339–347. https://doi.org/10.1017/S001309150001302X

[3] Goodman, A. W. (1983). *Univalent functions* (Vols. I & II). Polygonal Publishing House.

[4] Janowski, W. (1973). Some extremal problems for certain families of analytic functions. *Annales Polonici Mathematici*, *28*(3), 297–326. https://doi.org/10.4064/ap-28-3-297-326

[5] Kaplan, W. (1952). Close-to-convex schlicht functions. *Michigan Mathematical Journal*, *1*, 169–185. https://doi.org/10.1307/mmj/1028988895

[6] Kumar, S. S., Kumar, V., Ravichandran, V., & Cho, E. (2013). Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli. *Journal of Inequalities and Applications*, *2013*(176), 1–13. https://doi.org/10.1186/1029-242X-2013-176

[7] Padmanabhan, K. S., & Parvatham, R. (1975). Properties of a class of functions with bounded boundary rotation. *Annales Polonici Mathematici*, *31*(3), 311–323. https://doi.org/10.4064/ap-31-3-311-323

[8] Pinchuk, B. (1971). Functions with bounded boundary rotation. *Israel Journal of Mathematics*, *10*, 7–16. https://doi.org/10.1007/BF02771515

[9] Noor, K. I. (1983). On a generalization of close-to-convexity. *International Journal of Mathematics and Mathematical Sciences*, *6*, 327–334. https://doi.org/10.1155/S0161171283000289

[10] Ruscheweyh, S., & Sheil-Small, T. (1973). Hadamard product of schlicht functions and the Pólya-Schoenberg conjecture. *Commentarii Mathematici Helvetici*, *48*, 119–135. https://doi.org/10.1007/BF02566116