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Common Fixed Point Theorem for Berinde Weak Type Contraction Via Interpolation

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Abstract

In this paper, we introduce the notion of an interpolative Berinde weak pair contraction, and obtain a common fixed point theorem in the setting of metric spaces. We illustrate the main result of the paper with an example.

1 Introduction and Preliminaries

Theorem 1.1. *[1] Let (X, d) be a complete metric space. Suppose $T : X \mapsto X$ is a mapping satisfying the following condition*

$$d(Tx, Ty) \le \lambda[d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point in X.

Definition 1.2. *[2] Let (X, d) be a metric space. A self mapping $T : X \mapsto X$ is said to be an interpolative Kannan type contraction if there exists a constant $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \le \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha},$$

for all $x, y \in X \backslash Fix(T)$.

Theorem 1.3. *[2] Let (X, d) be a complete metric space and $T : X \mapsto X$ be an interpolative Kannan type contraction mapping. Then T has a unique fixed point in X.*

Theorem 1.4. *[3] Let (X, d) be a complete metric space, $S, T : X \mapsto X$ be self-mappings. Assume that there are some $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ such that the condition*

$$d(Tp, Sq) \le \lambda d(p, Tp)^{\alpha} d(q, Sq)^{1-\alpha}$$

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is satisfied for all $p, q \in X$ such that $Tp \neq p$ whenever $Sq \neq q$. Then S and T have a unique common fixed point.

Definition 1.5. *[4] Let (X, d) be a metric space. A pair of mappings $T, S : X \mapsto X$ is said to be an interpolative Hardy-Rogers pair contraction if there exists $k \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$d(Tx, Ty) \leq kd(x, y)^{\beta} d(Tx, x)^{\gamma} d(Sy, y)^{\alpha} \left[\frac{1}{2}(d(Tx, y) + d(Sy, x))\right]^{1-\alpha-\beta-\gamma},$$

for all $x, y \in X$ such that $Tx \neq x$ whenever $Sy \neq y$.

Theorem 1.6. *[4] Suppose that (X, d) is a complete metric space, and (T, S) is an interpolative Hardy-Rogers pair contraction. Then S and T have a unique common fixed point.*

Definition 1.7. *[5] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha},$$

where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Alternatively, the interpolative Berinde weak operator is given as follows

Definition 1.8. *[5] Let (X, d) be a metric space. We say $T : X \mapsto X$ is an interpolative Berinde weak operator if it satisfies*

$$d(Tx, Ty) \leq \lambda d(x, y)^{\frac{1}{2}} d(x, Tx)^{\frac{1}{2}},$$

where $\lambda \in [0, 1)$, for all $x, y \in X$, $x, y \notin Fix(T)$.

Theorem 1.9. *[5] Let (X, d) be a metric space. Suppose $T : X \mapsto X$ is an interpolative Berinde weak operator. If (X, d) is complete, then the fixed point of T exists.*

2 Main Result

Definition 2.1. *Let (X, d) be a metric space, and $T, S : X \mapsto X$ be self mappings. We will call (T, S) an interpolative Berinde weak pair contraction if there exists $\lambda \in [0, 1)$, $\alpha \in (0, 1)$, such that the following inequality holds for all $x, y \in X \backslash \{Fix(T), Fix(S)\}$,*

$$d(Tx, Sy) \leq \lambda d(x, y)^{\alpha} d(x, Tx)^{1-\alpha}.$$

Theorem 2.2. *Let (X, d) be a complete metric space, and $T, S : X \mapsto X$ be an interpolative Berinde weak pair contraction. Then T and S have a unique common fixed point.*

Proof. Let $p_0 \in X$, and define the sequence $\{p_n\}$ by

$$p_{2n+1} = Tp_{2n}, \quad p_{2n+2} = Sp_{2n+1} \text{ for all } n \in \{0, 1, 2, \cdots\}.$$

If there exists $n \in \{0, 1, 2, \cdots\}$ such that $p_n = p_{n+1} = p_{n+2}$, then p_n is a common fixed point of S and T, and the proof is finished. Now assume that there does not exist three consecutive identical terms in the sequence $\{p_n\}$ and that $p_0 \neq p_1$. Since (T, S) is an interpolative Berinde weak pair contraction, we deduce the following

$$\begin{aligned} d(p_{2n+1}, p_{2n+2}) &= d(Tp_{2n}, Sp_{2n+1}) \\ &\leq \lambda d(p_{2n}, p_{2n+1})^{\alpha} d(p_{2n}, Tp_{2n})^{1-\alpha} \\ &= \lambda d(p_{2n}, p_{2n+1})^{\alpha} d(p_{2n}, p_{2n+1})^{1-\alpha} \\ &= \lambda d(p_{2n}, p_{2n+1}). \end{aligned}$$

From the above inequality we deduce the following

$$d(p_{2n+1}, p_{2n+2}) \leq \lambda d(p_{2n}, p_{2n+1}) \leq \lambda^2 d(p_{2n-1}, p_{2n}) \leq \cdots \leq \lambda^{2n+1} d(p_0, p_1).$$

Similarly,

$$\begin{aligned} d(p_{2n+1}, p_{2n}) &= d(Tp_{2n}, Sp_{2n-1}) \\ &\leq \lambda d(p_{2n}, p_{2n-1})^{\alpha} d(p_{2n}, Tp_{2n})^{1-\alpha} \\ &= \lambda d(p_{2n}, p_{2n-1})^{\alpha} d(p_{2n}, p_{2n+1})^{1-\alpha}. \end{aligned}$$

From the above inequality we have

$$d(p_{2n+1}, p_{2n}) \leq \lambda^{\frac{1}{\alpha}} d(p_{2n}, p_{2n-1}) \leq \lambda d(p_{2n}, p_{2n-1}).$$

Hence,

$$d(p_{2n+1}, p_{2n}) \leq \lambda d(p_{2n-1}, p_{2n}) \leq \lambda^2 d(p_{2n-2}, p_{2n-1}) \leq \cdots \leq \lambda^{2n} d(p_0, p_1).$$

Since, $d(p_{2n+1}, p_{2n+2}) \leq \lambda^{2n+1} d(p_0, p_1)$, and $d(p_{2n+1}, p_{2n}) \leq \lambda^{2n} d(p_0, p_1)$, we have

$$d(p_n, p_{n+1}) \leq \lambda^n d(p_0, p_1).$$

Now we will prove that the sequence $\{p_n\}$ is a Cauchy sequence using the inequality immediately above. For this, let $m, r \in \{0, 1, 2, \cdots\}$, and observe we have

$$\begin{aligned} d(p_m, p_{m+r}) &\leq d(p_m, p_{m+1}) + d(p_{m+1}, p_{m+2}) + \cdots + d(p_{m+r-1}, p_{m+r}) \\ &\leq [\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+r+1}] d(p_0, p_1) \\ &\leq [\lambda^m + \lambda^{m+1} + \cdots + \lambda^{m+r+1} + \cdots] d(p_0, p_1) \\ &= \frac{\lambda^m}{1-\lambda} d(p_0, p_1). \end{aligned}$$

Now letting $n \to \infty$, we deduce that $\{p_n\}$ is a Cauchy sequence. As X is complete, there exists $u \in X$ such that $\lim_{n\to\infty} p_n = u$. Using the continuity of the metric in both its variables we prove that u is a common fixed point of T and S. For this, observe we have

$$\begin{aligned} d(Tu, p_{2n+2}) &= d(Tu, Sp_{2n+1}) \\ &\leq \lambda d(u, p_{2n+1})^{\alpha} d(u, Tu)^{1-\alpha}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get $d(Tu, u) = 0$, that is $Tu = u$. Similarly,

$$\begin{aligned} d(p_{2n+1}, Su) &= d(Tp_{2n}, Su) \\ &\leq \lambda d(p_{2n}, u)^{\alpha} d(p_{2n}, p_{2n+1})^{1-\alpha}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality we get $d(u, Su) = 0$, that is $u = Su$. So, u is a common fixed point of S and T. Now we prove that u is the unique common fixed point of S and T. For this let v be another common fixed point of S and T, and observe we have

$$\begin{aligned} d(u, v) &= d(Tu, Sv) \\ &\leq \lambda d(u, v)^{\alpha} d(u, Tu)^{1-\alpha} \\ &= \lambda d(u, v)^{\alpha} d(u, u)^{1-\alpha} \\ &= 0. \end{aligned}$$

So $d(u, v) = 0$, that is $u = v$, and the proof is finished. □

Example 2.3. *Let $X = \{p, q, z, w\}$, define a metric d on X as follows*

$$d(p, p) = d(q, q) = d(z, z) = d(w, w) = 0,$$

$$\begin{aligned} d(p, q) &= d(q, p) = 3, \\ d(z, p) &= d(p, z) = 4, \\ d(q, z) &= d(z, q) = \frac{3}{2}, \\ d(w, p) &= d(p, w) = \frac{5}{2}, \\ d(w, q) &= d(q, w) = 2, \\ d(w, z) &= d(z, w) = \frac{3}{2}. \end{aligned}$$

Define self-mappings T and S as follows

$$T(p) = p, \; T(q) = w, \; T(z) = w, \; T(w) = w,$$

$$S(p) = p, \ S(q) = w, \ S(z) = w, \ S(w) = w.$$

It is clear that (S, T) *is an interpolative Berinde weak pair contraction with* $\lambda = \frac{9}{10}$ *and* $\alpha = \frac{1}{2}$*, and* S *and* T *have a unique common fixed point* p*.*

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