

Bourbaki and Palais Proper Actions on d -Algebra

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Abstract

The main goal of this paper is to create a general two types of proper D -space, namely, Bourbaki proper D -space and Palais proper D -space, to explain the relation between Bourbaki proper and Palais proper D -spaces and to study some of examples and propositions of Bourbaki proper and Palais proper D -spaces.

1. Introduction

One of the very important concepts in topological d -algebras is the concept of d -algebra actions. This paper studies an important class of actions namely, Bourbaki proper D -spaces.

Imai and Iséki [6] and Iséki [7] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI-algebras. In [4], [5] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [8] introduced the notion of

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d -algebras which is another generalization of BCK-algebras, and investigated relations between d -algebras and BCK-algebras. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have been made to study the topological structures making the star operation of d -algebra continuous. Theories of topological groups, topological rings and topological modules are well known and still investigated by many mathematicians. Even topological universal algebraic structures have been studied by some authors.

In Section 1, we include some of results which will be needed in the next sections. In Section 2, we deal with the definitions, examples, remarks, propositions, theorem and corollaries of proper function. In Section 3, we deal with the definitions, examples, remarks and propositions of topological d -algebra. We also provided in this section the definition of Bourbaki proper D -space, gives a new type of Bourbaki proper D -space (to the best of our knowledge), namely, proper D -space and studies some of its properties, where D -space is meant T_2 -space topological X on which a locally compact, non-compact, T_2 -topological d -algebra D acts continuously on the left. In Section 4, the definitions, propositions, theorems and examples of a Palais proper D -space are given as well as the relation between Bourbaki proper and Palais proper D -spaces is studied.

2. Preliminaries

Definition 2.1 [1]. Let X and Y be spaces and $f : X \rightarrow Y$ be a function. Then f is called *continuous function* if $f^{-1}(A)$ is an open set in X for every open set A in Y .

Proposition 2.2 [1]. Let $f : X \rightarrow Y$ be a function of spaces. Then f is a continuous function if and only if $f^{-1}(A)$ is a closed set in X for every closed set A in Y .

Definition 2.3 [1]. (i) A function $f : X \rightarrow Y$ is called *closed function* if the image of each closed subset of X is a closed set in Y .

(ii) A function $f : X \rightarrow Y$ is called *open function* if the image of each open subset of X is an open set in Y .

Definition 2.4 [1]. Let X and Y be spaces. Then a function $f : X \rightarrow Y$ is called a *homeomorphism* if:

- (i) f is bijective.

(ii) f is continuous.

(iii) f is closed (open).

Definition 2.5 [1]. Let $(\chi_g)_{g \in G}$ be a net in a space X , $x \in X$. Then:

(i) $(\chi_g)_{g \in G}$ converges to x (written $\chi_g \rightarrow x$) if $(\chi_g)_{g \in G}$ is eventually in every neighborhood of x . The point x is called a *limit point* of $(\chi_g)_{g \in G}$, and the notation “ $\chi_g \rightarrow \infty$ ” is mean that $(\chi_g)_{g \in G}$ has no convergent subnet.

(ii) $(\chi_g)_{g \in G}$ is said to have x as an *cluster point* [written $\chi_g \infty x$] if $(\chi_g)_{g \in G}$ is frequently in every neighborhood of x .

Proposition 2.6 [1]. Let $(\chi_g)_{g \in G}$ be a net in a space (X, T) and x_0 in X . Then $\chi_g \infty x_0$ if and only if there exists a subnet of $(\chi_{g_m})_{g_m \in G}$ of $(\chi_g)_{g \in G}$ such that $\chi_{g_m} \rightarrow x_0$.

Remark 2.7 [1]. Let $(\chi_g)_{g \in G}$ be a net in a space (X, T) such that $\chi_g \infty x$, $x \in X$ and let A be an open set in X which contains x . Then there exists a subnet $(\chi_{g_m})_{g_m \in G}$ of $(\chi_g)_{g \in G}$ in the set A such that $\chi_{d_m} \rightarrow x$.

Definition 2.8 [1]. A subset A of space X is called *compact set* if every open cover of A has a finite sub cover. If $A = X$, then X is called a *compact space*.

Proposition 2.9 [1]. Let X be a space and F be a closed subset of X . Then $F \cap K$ is compact subset of F , for every compact set K in X .

Proposition 2.10 [1]. Let Y be an open subspace of space X and $A \subseteq Y$. Then A is an compact set in Y if and only if A is an compact set in X .

Definition 2.11 [1]. (i) A subset A of space X is called *relative compact* if \bar{A} is compact.

(ii) A space X is called *locally compact* if every point in X has a relative compact neighborhood.

Proposition 2.12 [1]. Let X and Y be spaces and $f : X \rightarrow Y$ be a continuous function. Then an image $f(A)$ is compact in Y for every A is compact in X .

Definition 2.13 [1]. Let $f : X \rightarrow Y$ be a function of spaces. Then f is called a *compact function* if $f^{-1}(A)$ is a compact set in X for every compact set A in Y .

Proposition 2.14 [1]. Let X, Y be spaces and $f : X \rightarrow Y$ be compact function. If F is a closed subset of X and B is an open set in Y , then $f|_F : F \rightarrow B$ is compact.

Definition 2.15 [1]. Let X and Y be two spaces. Then $f : X \rightarrow Y$ is called a *proper function* if:

(i) f is continuous function.

(ii) $f \times I_Z : X \times Z \rightarrow Y \times Z$ is a closed function, for every space Z .

Proposition 2.16 [1]. Let X and Y be spaces and $f : X \rightarrow Y$ be a continuous function. Then the following statements are equivalent:

(i) f is a proper function.

(ii) f is a closed function and $f^{-1}(\{y\})$ is an compact set, for each $y \in Y$.

(iii) If $(\chi_g)_{g \in G}$ is a net in X and $y \in Y$ is a cluster point of $f(\chi_g)$, then there is a cluster point $x \in X$ of $(\chi_g)_{g \in G}$ such that $f(x) = y$.

Proposition 2.17 [1]. Let X, Y and Z be spaces, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two proper functions. Then $g \circ f : X \rightarrow Z$ is a proper function.

Proposition 2.18 [1]. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be two functions. Then $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is proper function if and only if f_1 and f_2 are proper functions.

Proposition 2.19 [1]. Every proper function is closed.

Proposition 2.20 [1]. Let $f : X \rightarrow P = \{w\}$ be a continuous function on a space X . Then f is a proper function if and only if X is a compact, where w is any point which does not belongs to X .

Lemma 2.21 [1]. Every continuous function from a compact space into a Hausdorff space is closed.

Remark 2.22 [1]. If X is a space, then the diagonal function $\Delta : X \rightarrow X \times X$ such that $\Delta(x) = (x, x)$ is continuous.

Proposition 2.23. *If X is a Hausdorff space, then the diagonal function $\Delta : X \rightarrow X \times X$ is proper function.*

Proof. Let $(\chi_g, \chi_g) \in (x_1, x_2)$. So there exists a subnet of (χ_g, χ_g) , say itself, such that $(\chi_g, \chi_g) \rightarrow (x_1, x_2)$, then $\chi_g \rightarrow x_1$ and $\chi_g \rightarrow x_2$. Since X is a T_2 -space, $x_1 = x_2$. Then there is $x_1 \in X$ such that $\chi_g \in x_1$ and $\Delta(x_1) = y$. Hence by Proposition 2.16(iii), Δ is a proper function.

Proposition 2.24 [1]. *Let $f_1 : X \rightarrow Y_1$ and $f_2 : X \rightarrow Y_2$ be two proper functions. If X is a Hausdorff space, then the function $f : X \rightarrow Y_1 \times Y_2$, $f(x) = (f_1(x), f_2(x))$ is a proper function.*

Proposition 2.25 [1]. *Let X and Y be spaces and $f : X \rightarrow Y$ be a continuous, one to one function. Then the following statements are equivalent:*

- (i) f is a proper function.
- (ii) f is a closed function.
- (iii) f is a homeomorphism of X onto closed subset of Y .

Proposition 2.26 [1]. *Let X and Y be spaces such that Y is a T_2 -space and $f : X \rightarrow Y$ be a continuous function. Then the following statements are equivalent:*

- (i) f is a compact function.
- (ii) f is a proper function.
- (iii) *If $(\chi_g)_{g \in G}$ is a net in X and $y \in Y$ is a cluster point of $f(\chi_g)$, then there is a cluster point $x \in X$ of $(\chi_g)_{g \in G}$ such that $f(x) = y$.*

3. Bourbaki Proper D -Space

In this section, we examine the definition of topological d -algebra and some issues and examples related to the subject and we define the space Bourbaki proper D -space.

Definition 3.1 [8]. A non-empty set D together with a binary operation $*$ and a zero element 0 is said to be a d -algebra if the following axioms are satisfied for all $x, y \in D$

- (1) $x * x = 0$
- (2) $0 * x = 0$
- (3) $x * y = 0$ and $y * x = 0$ imply that $x = y$.

Definition 3.2 [2]. An element e of D is called a *left identity* if $e * a = a$, a *right identity* if $a * e = a$ for all $a \in D$ and $a \neq e$. If e is both left and right identity, then we called e is an *identity element*. Also we say that $(D, *)$ is d -algebra with identity element.

Example 3.3. (i) Let D be any non-empty set and $P(D)$ be power set of D . Then $(P(D), -)$ is d -algebra and ϕ is right identity in $(P(D), -)$.

(ii) Let $D = \{0, a, b, c\}$ and define the binary operation $*$ on D by following table:

$*$	0	a	b	c
0	0	0	0	0
a	0	0	b	c
b	0	b	0	a
c	0	c	a	0

Table 1

Then the pair $(D, *)$ is d -algebra with identity element a .

Definition 3.4 [2]. Let $(D, *)$ be a d -algebra and T be a topology on D . Then the triple $(D, *, T)$ is called a *topological d -algebra* (denoted by *Td-algebra*) if the binary operation $*$ is continuous.

Example 3.5. (i) Let $D = \{0, a, b, c\}$ and $*$ be defined by following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table 2

It is clear that $(D, *)$ is d -algebra and $T = \{\emptyset, \{b\}, \{c\}, \{0, a\}, \{b, c\}, \{0, a, b\}, \{0, a, c\}, D\}$ is topology on D such that the triple $(D, *, T)$ is a topological d -algebra.

(ii) Let R be a set of real numbers and $*$ be a binary operation defined by $a * b = a \cdot (a - b)^2$. Then $(R, *)$ is d -algebra and $(R, *, T)$ is Td -algebra, where T is usual topology on R .

Definition 3.6 [2]. A topological transformation d -algebra is a triple (D, X, φ) , where D is a T_2 -topological d -algebra with left identity e , X is a T_2 -topological space and $\varphi : D \times X \rightarrow X$ is a continuous function such that:

(i) $\varphi(d_1, \varphi(d_2, x)) = \varphi(d_1 d_2, x)$ for all $d_1 \neq d_2$ and $d_1, d_2 \in D, x \in X$.

(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the left identity element of D .

We shall often use the notation $d \cdot x$ for $\varphi(d, x)$ and $d \cdot (h, x) = (dh) \cdot x$ for $\varphi(d, \varphi(h, x)) = \varphi(dh, x)$. Similarly for $H \subseteq D$ and $A \subseteq X$ we put $HA = \{ha/h \in H, a \in A\}$ for $\varphi(H, A)$.

Remark 3.7 [2]. (i) The function φ is called an *action* of D on X and the space X together with φ is called a D -space (or more precisely left D -space).

(ii) The subspace $\{d \cdot x/d \in D\}$ is called the *orbit (trajectory)* of x under D , which denoted by Dx [or $\gamma(x)$], and for every $x \in X$ the stabilizer subgroup D_x of D at x is the set $\{d \in D/d \cdot x = x\}$.

(ii) A set $A \subseteq X$ is said to be *invariant* under D if $DA = A$.

Definition 3.8. A D -space X is called a *Bourbaki proper D -space (proper D -space)* if the function $\theta : D \times X \rightarrow X \times X$ defined by $\theta(d, x) = (x, d \cdot x)$ is a proper function.

Example 3.9. Let $Z_3 = \{-1, 0, 1\}$ and $*$ be defined by following table:

$*$	0	-1	1
0	0	0	0
-1	0	0	-1
1	0	-1	0

Table 3

Then $(Z_3, *)$ is d -algebra and $(Z_3, *, \tau)$ is Td -algebra, where τ is discrete topology on Z_3 . The act on the topological space S^n [as a subspace of R^{n+1} with usual topology] as follows:

$$\begin{aligned} 0 \cdot (x_1, x_2, \dots, x_{n+1}) &= (0, 0, \dots, 0) \\ 1 \cdot (x_1, x_2, \dots, x_{n+1}) &= (x_1, x_2, \dots, x_{n+1}) \\ -1 \cdot (x_1, x_2, \dots, x_{n+1}) &= (-x_1, -x_2, \dots, -x_{n+1}). \end{aligned}$$

Since Z_3 is compact, by Proposition 2.20, the constant function $Z_3 \rightarrow P$ is a proper. Also the identity function is a proper, then by Proposition 2.18, the proper function of $Z_3 \times S^n$ into $P \times S^n$ is a proper.

Since $P \times S^n$ is homeomorphic to S^n , by Proposition 2.25, the homeomorphism of $P \times S^n$ onto S^n is proper. Since $Z_3 \times S^n \rightarrow P \times S^n$ is continuous and open function, by Proposition 2.17, the composition $Z_3 \times S^n \rightarrow S^n$ is a proper. Let ϕ be the action of Z_3 on S^n . Then ϕ is continuous, one to one function, so ϕ is continuous function. Since S^n is T_2 -space, by Lemma 2.21, ϕ is a closed. Then by Proposition 2.25, ϕ is a proper function. Thus by Proposition 2.24, $Z_3 \times S^n \rightarrow S^n \times S^n$ is a proper D -space.

Lemma 3.10. *If X is a D -space, then the function $\theta : D \times X \rightarrow X \times X$ defined by $\theta(d, x) = (x, d \cdot x)$ is a continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $D \times X$ for every $(x, y) \in X \times X$.*

Proof. Since

$$\theta : D \times X \xrightarrow{I_D \times \Delta} D \times X \times X \xrightarrow{\phi \times I_X} X \times X \xrightarrow{\cong} X \times X,$$

where ϕ is action of D on X . Then $\theta = f \circ \phi \times I_X \circ I_D \times \Delta$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $D \times X$ for every $(x, y) \in X \times X$.

Theorem 3.11. *Let X be a proper D -space and let H be a closed subset of D . If Y is an open subset of X which is invariant under H , then Y is a proper H -space.*

Proof. Since X is a proper D -space, the function $\theta : D \times X \rightarrow X \times X$ defined by

$\theta(d, x) = (x, d \cdot x)$ is a proper function. [To prove that $\omega : H \times Y \rightarrow Y \times Y$ is a proper function defined by $\omega(h, y) = \theta(h, y)$ for each $(h, y) \in H \times Y$].

(1) By Lemma 3.10, $\theta : D \times X \rightarrow X \times X$ is continuous, then $\omega : H \times Y \rightarrow Y \times Y$ is continuous.

(2) Let $(h_g, y_g)_{g \in G}$ be a net in $H \times Y$ such that $\omega((h_g, y_g)) \approx (x, y)$ for some $(x, y) \in Y \times Y$. Then $(y_g, h_g y_g) \approx (x, y)$ in $Y \times Y$. Let A be an open subset of $X \times X$ such that $(x, y) \in A$. Since Y is open in X , $Y \times Y$ is an open set in $X \times X$. Then $A \cap (Y \times Y)$ is an open set in $X \times X$. But $(x, y) \in A \cap (Y \times Y)$ and $(y_g, h_g y_g) \approx (x, y)$, thus $(y_g, h_g y_g)$ is frequently in $A \cap (Y \times Y)$ and then $(y_g, h_g y_g)$ is frequently in A , thus $(y_g, h_g y_g) \approx (x, y)$ in $X \times X$. Since $\theta : D \times X \rightarrow X \times X$ is a proper function, by Proposition 2.16, there exists $(h, x_1) \in D \times X$ such that $(h_g, y_g) \approx (h, x_1)$ and $\theta((h, x_1)) = (x, y)$. Hence $(x_1, h x_1) = (x, y)$. Thus $x_1 = x$ and therefore $h_g \approx h$. Since $(h_g)_{g \in G}$ is a net in H , and H is closed. Then there exists $(h, x) \in H \times Y$ such that $\omega(h, x) = \theta(h, x) = (x, y)$. Then from (1), (2) and by Proposition 2.16, the function $\omega : H \times Y \rightarrow Y \times Y$ is a proper function. Hence Y is a proper H -space.

Corollary 3.12. *Let X be a proper D -space and Y be an open subset of X which is invariant under D . Then Y is a proper D -space.*

Corollary 3.13. *Let X be a proper D -space and let H be closed subset of D . Then X is a proper H -space.*

Proposition 3.14. *Let X be a proper D -space, $x \in X$ and $T = \{x\} \times X$. Then the function $\theta_T : \theta^{-1}(T) \rightarrow T$ is a proper function, where $\theta : D \times X \rightarrow X \times X$ such that $\theta(g, x) = (x, g \cdot x)$, $\forall (g, x) \in D \times X$.*

Proof. Since X is a T_2 -space, $\{x\}$ is closed set in X . Thus, $\{x\}$ is closed set in X . So each $D \times \{x\}$ and $\{x\} \times X$ are closed in $D \times X$ and $X \times X$ (respectively). Now, let F be a closed set in $\theta^{-1}(T) = D \times \{x\}$. Since $F = F \cap (D \times \{x\})$, F is closed in $D \times X$ and $\theta_T(F) = \theta(F) \cap (\{x\} \times X)$. Therefore $\theta(F)$ is closed in $X \times X$, then by

Proposition 2.19, $\theta_T(F)$ is closed in $X \times X$. But $\theta_T(F) \subseteq \{x\} \times X$, then there exists a subset V of X such that $\theta_T(F) = \{x\} \times V$. Since $\theta_T(F)$ is closed in $X \times X$, so $\{x\} \times V$ is a closed set in $\{x\} \times X$. Hence $\theta_T(F) = \{x\} \times V$ is a closed set in $T = \{x\} \times X$, therefore $\theta_T : \theta^{-1}(T) \rightarrow T$ is closed. Now, let $(x, y) \in \{x\} \times X$. Since θ is proper function, by Proposition 2.26, θ is a compact function. Then $\theta^{-1}(\{(x, y)\})$ is compact in $D \times X$. Then by $\theta_T^{-1}(\{(x, y)\})$ is compact set in $D \times \{x\} = \theta^{-1}(T)$. Thus by Proposition 2.26, θ_T is a proper function.

Let X be a D -space and A, B be two subsets of X . We mean by $((A, B))$ the set $\{d \in D / dA \cap B \neq \emptyset\}$.

From now on, we will use D -space, which satisfies the property if (X, T) and (Y, T') be two space and $\forall \chi_g \rightarrow x, y_g \rightarrow y$ in X and Y , respectively, then $(\chi_g, y_g) \rightarrow (x, y)$.

Proposition 3.15. *Let X be a D -space. If for every $x, y \in X$ there exist an open set A_x of X contains x and an open set A_y of X contains y such that $K = ((A_x, A_y))$ is relatively compact in D , then X is a proper D -space.*

Proof. We prove that $\theta : D \times X \rightarrow X \times X$, $\theta(d, x) = (x, dx)$ is a proper function. Let $(d_g, \chi_g)_{g \in G}$ be a net in $D \times X$ such that $\theta((d_g, \chi_g)) = (\chi_g, d_g \chi_g) \rightarrow (x, y)$, where $(x, y) \in X \times X$. Now, since $x, y \in X$, there exist an open set A_x contains x and an open set A_y contains y such that the set $K = ((A_x, A_y))$ is relatively compact in D . Thus, $A_x \times A_y$ is an open set in $X \times X$ and $(x, y) \in A_x \times A_y$. So there exists a subnet $(\chi_{g_m}, d_{g_m} \chi_{g_m})_{m \in M}$ of $(\chi_g, d_g \chi_g)$ in $A_x \times A_y$ and $(\chi_{g_m}, d_{g_m} \chi_{g_m}) \rightarrow (x, y)$, hence $\chi_{g_m} \rightarrow x$ and $d_{g_m} \chi_{g_m} \rightarrow y$. Since $\chi_{g_m} \in A_x$ and $d_{g_m} \chi_{g_m} \in A_y$. Then $d_{g_m} A_x \cap A_y \neq \emptyset, \forall g_m$, so $d_{g_m} \in K$, but K is relatively compact in D , then by Proposition 2.6, (d_{g_m}) has a limit point, say $t \in D$. Since $\chi_{g_m} \rightarrow x$, then $(d_g, \chi_g) \rightarrow (t, x)$, so $\theta((d_g, \chi_g)) \rightarrow \theta((t, x))$, i.e., $(\chi_{g_m}, d_{g_m} \chi_{g_m}) \rightarrow (x, tx)$, thus $d_{g_m} \chi_{g_m} \rightarrow tx$ but $d_{g_m} \chi_{g_m} \rightarrow y$ and since X is a Hausdorff space, then $tx = y$. But

$(\chi_{g_m}, g_{g_m} \chi_{g_m})_{g \in G}$ is a subnet of $(\chi_g, d_g \chi_g)$ and $(d_{g_m}, \chi_{g_m}) \rightarrow (t, x)$, then $(d_g, \chi_g) \infty (t, x)$, thus $\theta((t, x)) = (x, y)$. Then by Proposition 2.16, we have θ is a proper function. Hence X is a proper D -space.

Corollary 3.16. *Let X be a D -space such that D is discrete space. If for every $x, y \in X$ there is an open set A_x in X contains x and an open set A_y in X contains y such that the set $K = ((A_x, A_y))$ is finite, then X is a proper D -space.*

Let X be a D -space and $x \in X$. The set $J(x) = \{y \in X : \text{there is a net } (d_g)_{g \in G} \text{ in } D \text{ and there is a net } (\chi_g)_{g \in G} \text{ in } X \text{ with } d_g \rightarrow \infty \text{ and } \chi_g \rightarrow x \text{ such that } d_g x \rightarrow y\}$ is called first prolongation limit set of x , [3]. $J(x)$ is a good tool to discover about the Bourbaki proper D -space.

Proposition 3.17. *Let X be a D -space. Then X is a Bourbaki proper D -space if and only if $J(x) = \emptyset$ for each $x \in X$.*

Proof. \Rightarrow Suppose that $y \in J(x)$, then there is a net $(d_g)_{g \in G}$ in D with $d_g \rightarrow \infty$ and there is a net $(\chi_g)_{g \in G}$ in X with $\chi_g \rightarrow x$ such that $d_g \chi_g \rightarrow y$, so $\theta((d_g, \chi_g)) = (\chi_g, d_g \chi_g) \rightarrow (x, y)$. But X is a Bourbaki proper, then by Proposition 2.26, there is $(d, x_1) \in d \times X$ such that $(d_g, \chi_g) \infty (d, x_1)$. Thus, $(d_g)_{g \in G}$ has a subnet (say itself), such that $d_g \rightarrow d$, which is contradiction, thus $J(x) = \emptyset$.

\Leftarrow Let $(d_g, \chi_g)_{g \in G}$ be a net in $D \times X$ and $(x, y) \in X \times X$ such that $\theta((d_g, \chi_g)) = (\chi_g, d_g \chi_g) \infty (x, y)$ so $(\chi_g, d_g \chi_g)_{g \in G}$ has a subnet, say itself, such that $(\chi_g, d_g \chi_g) \rightarrow (x, y)$, then $\chi_g \rightarrow x$ and $d_g \chi_g \rightarrow y$. Suppose that $d_g \rightarrow \infty$, then $y \in J(x)$, which is contradiction. Then there is $d \in D$ such that $d_g \rightarrow d$, then $(d_g, \chi_g) \rightarrow (d, x)$ and $\theta(d, x) = (x, y)$. Thus by Lemma 3.10 and Proposition 2.16, X is a Bourbaki proper D -space.

Proposition 3.18. *Let X be a proper D -space, $\theta(d, x) = (x, d \cdot x) \forall (d, x) \in D \times X$ with the action $\phi : D \times X \rightarrow X, \phi(d, x) = d \cdot x, \forall (d, x) \in D \times X$. Then for each*

$x \in X$, the function $\varphi_x : D \rightarrow X$, which is defined by: $\varphi_x(d) = \varphi(d, x)$ is a proper function.

Proof. Let $T = \{x\} \times X \subseteq X \times X$. Then by Proposition 3.14, $\theta_T : \theta^{-1}(T) \rightarrow T$ is a proper function. But:

$$\varphi_x = D \xrightarrow{f} D \times \{x\} \xrightarrow{\theta_T} \{x\} \times X \xrightarrow{h} X, \text{ such that } f \text{ and } h \text{ are homeomorphisms.}$$

Now:

(i) Let A be an open set in X . Then $h^{-1}(A)$ is an open set in $\{x\} \times X$. Since θ_T is continuous, $\theta_T^{-1}(h^{-1}(A))$ is open in $D \times \{x\}$. Since f is homeomorphism, $f^{-1}(\theta_T^{-1}(h^{-1}(A)))$ is open in D . Then $\varphi_x(A) = f^{-1} \circ \theta_T^{-1} \circ h^{-1}(A)$ is open in D . Thus, $\varphi_x : D \rightarrow X$ is continuous.

(ii) Let F is closed in D . Then $h(F)$ is closed in $\{x\} \times X$. Since $\theta_T : D \times \{x\} \rightarrow \{x\} \times X$ is a proper function, by Proposition 2.19, $\theta_T(h(F))$ is closed. Then $\varphi_x(F) = f(\theta_T(h(F)))$ is closed in X . Then $\varphi_x : D \rightarrow X$ is closed.

(iii) Let $y \in X$. Then $h^{-1}(\{y\}) = \{(x, y)\}$ such that $x \in X$. Since X is T_2 -space, $\{(x, y)\}$ is a closed set in $\{x\} \times X$. Since θ_T is a continuous function, $\theta_T^{-1}(\{(x, y)\})$ is closed in $D \times \{x\}$. So by Proposition 2.16, $\theta_T^{-1}(h^{-1}(\{y\})) = \theta_T^{-1}\{(x, y)\}$ is compact. Since f is homeomorphism, it is clear that $f^{-1}(\theta_T^{-1}\{(x, y)\})$ is compact in D . Then $f^{-1}(\theta_T^{-1}(h^{-1}(\{y\}))) = \varphi_x^{-1}(\{y\})$ is compact in D . Then by (i), (ii), (iii) and Proposition 2.16, φ_x is proper function.

Proposition 3.19. Let X be a D -space and $\theta : D \times X \rightarrow X \times X$ be a function defined by $\theta(d, x) = (x, dx)$, $\forall (d, x) \in D \times X$. Then the following statements are equivalent:

(i) X is a proper D -space.

(ii) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall (x, y) \in X \times X$ and for all $x, y \in X$ and for all $U \in N(((x, y)))$, $\exists V_x \in N(x)$ and $V_y \in N(y)$ such that $((V_x, V_y)) \subseteq U$.

(iii) $\theta^{-1}(\{(x, y)\})$ is a compact set, $\forall (x, y) \in X \times X$ and for all $x, y \in X$ and for all $U \in N(\theta^{-1}(\{(x, y)\}))$, $\exists V \in N((x, y))$ such that $\theta^{-1}(V) \subseteq U$.

Proof. (i) \rightarrow (iii) Let $x, y \in X$ and U be an open neighborhood of $\theta^{-1}(x, y)$. Since θ is a proper function, by Proposition 2.19, θ is a closed function. So $V = (X \times X) \setminus \theta((D \times X) \setminus U)$ is an open neighborhood of (x, y) with $\theta^{-1}(V) \subseteq U$. So by Proposition 2.16, $\theta^{-1}(\{(x, y)\})$ is a compact set $\forall (x, y) \in X \times X$. Hence (iii) holds.

(iii) \rightarrow (i) Let F be a closed subset of $D \times X$ and let $(x, y) \in X \times X \setminus \theta(F)$. Since $(D \times X) \setminus F$ is an open neighborhood of $\theta^{-1}(x, y)$, by (iii) there is a neighborhood V of (x, y) such that $\theta^{-1}(V) \subseteq (D \times X) \setminus F$. Hence $V \cap \theta(F) = \emptyset$, so $(x, y) \notin \overline{\theta(F)}$, then $\overline{\theta(F)} = \theta(F)$. Hence θ is a closed function. Since $\theta^{-1}(\{(x, y)\})$ is a compact set for every $(x, y) \in X \times X$, by Proposition 2.16, θ is a proper function. Hence X is a proper D -space.

(ii) \rightarrow (iii) Let $x, y \in X$ and U be a neighborhood of $\theta^{-1}(\{(x, y)\}) = ((x, y)) \times \{x\}$. Since $\theta^{-1}(\{(x, y)\})$ is compact, there are neighborhood U' of $((x, y))$ and W of $\{x\}$ such that $U' \times W \subseteq U$. So by (ii) there are neighborhood V_x of x and V_y of y such that $((V_x, V_y)) \subseteq U'$. But $\theta^{-1}((V_x \cap W) \times V_y) \subseteq U' \times W \subseteq U$. Hence (iii) hold.

(iii) \rightarrow (ii) Let $x, y \in X$ and $U \in N((x, y))$. Then $U \times X \in N((x, y)) \times \{x\}$. Thus $U \times X \in N\theta^{-1}(x, y)$. So by (iii) there exists $V \in N(x, y)$ such that $\theta^{-1}(V) \subseteq U \times X$. Then there are neighborhoods V_x of x and V_y of y such that $\theta^{-1}(V_x \times V_y) \subseteq U \times X$. Hence (ii) holds.

Corollary 3.20. Let X be a proper D -space, choose a point $x \in X$ and let U be neighborhood of the stabilizer D_x of x . Then x has a neighborhood V such that U contains the stabilizer of all points in V .

Proof. Since U is neighborhood of the stabilizer D_x of x , $U \in N(D_x)$. Since $D_x = ((x, x))$, $U \in N(((x, x)))$. So by Proposition 3.19, there exists $V_x \in N(x, x)$ such that $((V_x, V_x)) \subseteq U$. Let $y \in V_x$. Then $D_y \subseteq ((V_x, V_x)) \subseteq U$.

4. Palais Proper D -space

From now on, the D -space we mean a completely regular topological T_2 -space X on which a locally compact and non-compact, T_2 -topological d -algebra D continuously on the left (always in the sense of Palais proper D -space).

Definition 4.1. Let X be a D -space. A subset A of X is said to be *thin relative* to a subset B of X if the set $((A, B)) = \{d \in D/dA \cap B \neq \emptyset\}$ has a neighborhood whose closure is compact in D . If A is thin relative to itself, then it is called *thin*.

Remark 4.2. The thin sets have the following properties:

- (i) If A and B are relative thin and $K_1 \subseteq A$ and $K_2 \subseteq B$, then K_1 and K_2 are relatively thin.
- (ii) Let X be a D -space and K_1 and K_2 be compact subsets of X . Then $((K_1, K_2))$ is closed in D .
- (iii) If K_1 and K_2 are compact subsets of D -space X such that K_1 and K_2 are relatively thin, then $((K_1, K_2))$ is a compact subset of D .

Definition 4.3. A subset S of D -space X is a *small subset* if each point of X has neighborhood which is thin relative to S .

Theorem 4.4. Let X be a D -space. Then:

- (i) Each small neighborhood of a point x contains a thin neighborhood of x .
- (ii) A subset of a small set is small.
- (iii) A finite union of small sets is small.
- (iv) If S is a small subset of X and K is a compact subset of X , then K is thin relative to S .

Proof. (i) Let S is a small neighborhood of x . Then there is a neighborhood U of x which is thin relative to S . Then $((U, S))$ has neighborhood whose closure is compact. Let $V = U \cap S$. Then V is neighborhood of x and $((V, V)) \subseteq ((U, S))$, therefore V is thin neighborhood of x .

- (ii) Let S be a small set and $K \subseteq S$ and let $x \in X$. Then there exists a

neighborhood U of x , which is thin relative to S . Then $((U, K)) \subseteq ((U, S))$. Thus $((U, K))$ has neighborhood whose closure is compact. Then K is small.

(iii) Let $\{S_i\}_{i=1}^n$ be a finite collection of small sets and $y \in X$. Then for each i there is neighborhood K_i of y such that the set $((S_i, K_i))$ has neighborhood whose closure is compact. Then $\bigcup_{i=1}^n ((S_i, K_i))$ has neighborhood whose closure is compact. But

$$((\bigcup_{i=1}^n S_i, \bigcap_{i=1}^n K_i)) \subseteq \bigcup_{i=1}^n ((S_i, K_i)), \text{ thus } \bigcup_{i=1}^n S_i \text{ is a small set.}$$

(iv) Let S be a small set and K be compact. Then there is a neighborhood U_k of K , $\forall k \in K$, such that U_k is thin relative to S . Since $K \subseteq \bigcup_{k \in K} U_k$, i.e., $\{U_k\}_{k \in K}$ is open cover of K which is compact, so there is a finite sub cover $\{U_{k_i}\}_{i=1}^n$ of $\{U_k\}_{k \in K}$. Since

$((U_{k_i}, S))$ has neighborhood whose closure is compact, thus $((\bigcup_{i=1}^n U_{k_i}, S))$ so is. But

$$((K, S)) \subseteq ((\bigcup_{i=1}^n U_{k_i}, S)), \text{ therefore } K \text{ is thin relative to } S.$$

Definition 4.5. A D -space X is said to be a *Palais proper D -space* if every point x in X has a neighborhood which is small set.

Examples 4.6. (i) Let $Z_3 = \{-1, 0, 1\}$ and $*$ be defined by following table:

$*$	0	-1	1
0	0	0	0
-1	-1	0	-1
1	1	1	0

Table 4

Then Z_3 act on itself (as Z_3 with discrete topology) as follows:

$$r_1 \cdot r_2 = r_1 * r_2 \quad \forall r_1, r_2 \in Z_3.$$

It is clear that for each point $x \in Z_3$, there is a neighborhood which is small U of x , i.e., for any point y of Z_3 , there exists a neighborhood V of y , then $((U, V)) =$

$\{r \in Z_3 / rU \cap V \neq \emptyset\} = Z_3$, then $((U, V))$ has neighborhood whose closure is compact.

(ii) R is locally compact topological d -algebra (as R with discrete topology and binary operation $a * b = a(a - b)$) acts on the completely regular Hausdorff space R^2 as follows:

$$r \cdot (x_1, x_2) = (x_1, x_2), \text{ for every } r \in R - \{0\} \text{ and } (x_1, x_2) \in R^2.$$

Clear R^2 is R -space. But $(0, 0) \in R^2$ has no neighborhood which is a small. Since $((U, V)) = R$ for any two neighborhoods U and V of $(0, 0)$ and since R is not compact, R^2 is not a Palais proper R -space.

Proposition 4.7. *Let X be a D -space. Then:*

- (i) *If X is Palais proper D -space, then every compact subset of X is a small set.*
- (ii) *If X is a Palais proper D -space and K is a compact subset of X , then $((K, K))$ is a compact subset of D .*

Proof. (i) Let A be a subset of X such that A is compact and let $x \in X$. Since X is a proper D -space, there is a neighborhood U of x which is small. Then for every $a \in A$ there exists a neighborhood U_a which is small, then $A \subseteq \bigcup_{a \in A} U_a$. Since A is compact,

there exists $a_1, a_2, \dots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^n U_{a_i}$. Thus by Theorem 4.4(ii)(iii), A is a small set in X .

(ii) Let X be a proper D -space and K be compact. Then by (i), K is a small subset of X , and by Theorem 4.4(iv), K is thin. So $((K, K))$ has neighborhood whose closure is compact. Then by Remark 4.2(iii), $((K, K))$ is closed in D . Thus $((K, K))$ is compact.

Proposition 4.8. *Let X be a D -space and y be a point in X . Then y has no small set whenever $y \in J(x)$ for some point $x \in X$.*

Proof. Let $y \in J(x)$. Then there is a net $(d_g)_{g \in G}$ in D with $d_g \rightarrow \infty$ and a net $(\chi_g)_{g \in G}$ in X with $\chi_g \rightarrow x$ such that $d_g \chi_g \rightarrow y$. Now, for each neighborhood S of y

and every neighborhood U of x there is $g_0 \in G$ such that $\chi_g \in U$ and $d_g \chi_g \in S$ for each $g \geq g_0$, thus $d_g \in ((U, S))$, but $d_g \rightarrow \infty$, thus $((U, S))$ has no compact closure, i.e., S is not a small neighborhood.

Proposition 4.9. *Let X be a Palais proper D -space. Then $J(x) = \emptyset$ for each $x \in X$.*

Proof. Suppose that there exists $x \in X$ such that $J(x) \neq \emptyset$, then there exists $y \in J(x)$. Thus there is a net $(d_g)_{g \in G}$ in D with $d_g \rightarrow \infty$ and a net $(\chi_g)_{g \in G}$ in X with $\chi_g \rightarrow x$ such that $d_g \chi_g \rightarrow y$. Since X be a proper D -space, there is a small (thin) neighborhood U of x . Thus, there is $g_0 \in G$ such that $d_g \chi_g \in U$ and $\chi_g \in U$ for each $g \geq g_0$, so $d_g \in ((U, U))$, which has a compact closure. Therefore $(d_g)_{g \in G}$ must have a convergent subnet, which is a contradiction. Thus $J(x) = \emptyset$ for each $x \in X$.

Proposition 4.10 [3]. *Let X be a periodic D -space. Then $J(x) = \emptyset$ for each $x \in X$ if and only if every pair of points of X such that $Dx \neq Dy$ has relatively thin neighborhood.*

Proposition 4.11. *Let X be a periodic D -space and $Dx \neq Dy$ for every pair of points x, y of X . Then the definition of Palais proper D -space and the definition Bourbaki proper D -space are equivalent.*

Proof. (\Rightarrow) By Propositions 4.10 and 3.17.

(\Leftarrow) Let X be a Bourbaki proper D -space. Then by Proposition 3.17, $J(x) = \emptyset$ for each $x \in X$. Let $x \in X$. We will show that x has a small neighborhood. By Proposition 4.10, there a small neighborhood U_x of x . Thus X is Palais proper D -space.

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