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A Collection of Trigonometric Inequalities via Functional Estimates

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Abstract

In this article, a stream of inequalities involving trigonometric functions is presented. All inequalities are rigorously proved and derived via a specific collection of functional estimates. Most of the results that are presented in the literature rely on monotonicity properties or power series expansions to derive inequalities, but in this work the authors heavily rely on functional estimates involving Lebesgue norms. The inequalities derived are non-trivial and they enrich the current works in the mathematical literature.

1 Introduction

In this article, inequalities involving trigonometric functions are derived using functional estimates which involve Lebesgue norms. The are various inequalities involving trigonometric functions in the literature [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [13], [14], [15] and usually monotonicity properties or series expansions are used to derive the results. This work diverges from the literature, as it employs functional inequalities to obtain the results and introduces new machinery to derive inequalities. This machinery is something that is very rare in the literature, as functional inequalities are mainly used in Partial Differential Equations to examine the behavior of the solutions. This work is motivated by the work [16] where the authors use functional estimates to derive multiple inequalities and use Pade approximation to obtain estimates in terms of polynomials. Another work that motivated this article is the one by Chesneau [12], where basic integral inequalities are used to derive trigonometric ones. He obtained a stream of inequalities which are completely different compared to the inequalities presented in this current article. All the inequalities presented here are different from the ones in the literature as they involve nested quantities. The format of these inequalities is different with the format that appears in other works. Before delving into the main body of the work, the article is structured as follows: At first, there

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are some functional estimates obtained and these are accompanied by rigorous proofs. The next section contains the main results, where all the functional estimates are employed and there are detailed proofs. In the next section, there is graphical illustration of all the inequalities, as verification of the rigorous theoretical results. The last section contains all the conclusions and remarks of some the inequalities that are interesting according to the author. This article enriches the literature of inequalities and promotes new machinery to derive those.

2 Preliminary Estimates

In this section, some preliminary estimates are derived that will be used later on. There are rigorous proofs for all ten estimates.

Theorem 2.1. *Let $f : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$\Big| \int_a^b f(t)dt \Big| \leq |bf(b) - af(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f'(t)||_{L_2((a,b))},\ 0 < a < b. \tag{2.1}$$

Proof.

$$\begin{aligned}
\Big| \int_a^b f(t)dt \Big| &= \Big| \int_a^b (t)' f(t)dt \Big| \\
&= \Big|\ bf(b) - af(a) - \int_a^b tf'(t)dt \Big| \\
&\leq \Big|\ bf(b) - af(a)\ \Big| + \Big| \int_a^b tf'(t)dt \Big| \\
&\leq \Big|\ bf(b) - af(a)\ \Big| + \int_a^b |tf'(t)|dt \\
&\leq \Big|\ bf(b) - af(a)\ \Big| + \left(\int_a^b t^2 dt \right)^{\frac{1}{2}} \left(\int_a^b f'(t)^2 dt \right)^{\frac{1}{2}} \\
&= \Big|\ bf(b) - af(a)\ \Big| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3} \left(\int_a^b f'(t)^2 dt \right)^{\frac{1}{2}} \\
&= |bf(b) - af(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f'(t)||_{L_2((a,b))},\ 0 < a < b
\end{aligned}$$

using integration by parts once, then employing the triangle and the Schwarz-Cauchy inequality. □

Theorem 2.2. *Let $f : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$\Big| \int_a^b f'(t)dt \Big| \leq |bf'(b) - af'(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f''(t)||_{L_2((a,b))},\ 0 < a < b. \tag{2.2}$$

Proof.

$$
\begin{aligned}
\Big|\int_a^b f'(t)dt\Big| &= \Big|\int_a^b (t)' f'(t)dt\Big| \\
&= \Big| bf'(b) - af'(a) - \int_a^b tf''(t)dt\Big| \\
&\leq \Big| bf'(b) - af'(a) \Big| + \Big|\int_a^b tf''(t)dt\Big| \\
&\leq \Big| bf'(b) - af'(a) \Big| + \int_a^b |tf''(t)|dt \\
&\leq \Big| bf'(b) - af'(a) \Big| + \left(\int_a^b t^2 dt\right)^{\frac{1}{2}} \left(\int_a^b f''(t)^2 dt\right)^{\frac{1}{2}} \\
&= \Big| bf'(b) - af'(a) \Big| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3} \left(\int_a^b f''(t)^2 dt\right)^{\frac{1}{2}} \\
&= |bf'(b) - af'(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f''(t)||_{L_2((a,b))}, \ 0 < a < b
\end{aligned}
$$

employing integration by parts once, then employing the triangle and the Schwarz-Cauchy inequality. □

Theorem 2.3. *Let $f : \Omega = (a, b) \subset]0, +\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$
\Big|\int_a^b f''(t)dt\Big| \leq |bf''(b) - af''(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f'''(t)||_{L_2((a,b))}, \ 0 < a < b. \tag{2.3}
$$

Proof.

$$
\begin{aligned}
\Big|\int_a^b f''(t)dt\Big| &= \Big|\int_a^b (t)' f''(t)dt\Big| \\
&= \Big| bf''(b) - af''(a) - \int_a^b tf'''(t)dt\Big| \\
&\leq \Big| bf''(b) - af''(a) \Big| + \Big|\int_a^b tf'''(t)dt\Big| \\
&\leq \Big| bf''(b) - af''(a) \Big| + \int_a^b |tf'''(t)|dt \\
&\leq \Big| bf''(b) - af''(a) \Big| + \left(\int_a^b t^2 dt\right)^{\frac{1}{2}} \left(\int_a^b f'''(t)^2 dt\right)^{\frac{1}{2}} \\
&= \Big| bf''(b) - af''(a) \Big| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3} \left(\int_a^b f'''(t)^2 dt\right)^{\frac{1}{2}} \\
&= |bf''(b) - af''(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f'''(t)||_{L_2((a,b))}, \ 0 < a < b
\end{aligned}
$$

employing the integration by parts, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.4. *Let $f : \Omega = (a, b) \subset]0, +\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$\Big| \int_a^b f'''(t)dt \Big| \leq |bf'''(b) - af'''(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f''''(t)||_{L_2((a,b))}, \ 0 < a < b. \tag{2.4}$$

Proof.

$$\begin{aligned}
\Big| \int_a^b f'''(t)dt \Big| &= \Big| \int_a^b (t)' f'''(t)dt \Big| \\
&= \Big| \ bf'''(b) - af'''(a) - \int_a^b tf''''(t)dt \Big| \\
&\leq \Big| \ bf'''(b) - af'''(a) \ \Big| + \Big| \int_a^b tf''''(t)dt \Big| \\
&\leq \Big| \ bf'''(b) - af'''(a) \ \Big| + \int_a^b |tf''''(t)|dt \\
&\leq \Big| \ bf'''(b) - af'''(a) \ \Big| + \left(\int_a^b t^2 dt \right)^{\frac{1}{2}} \left(\int_a^b f''''(t)^2 dt \right)^{\frac{1}{2}} \\
&= \Big| \ bf'''(b) - af'''(a) \ \Big| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3} \left(\int_a^b f''''(t)^2 dt \right)^{\frac{1}{2}} \\
&= |bf'''(b) - af'''(a)| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3}||f''''(t)||_{L_2((a,b))}, \ 0 < a < b
\end{aligned}$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.5. *Let $f, g : \Omega = (a, b) \subset]0, +\infty[\to \mathbb{R}$, $f, g \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$\Big| \int_a^b f(t)g(t)dt \Big| \leq \Big| \ bf(b)g(b) - af(a)g(a) \Big| + \frac{\sqrt{3}}{3}\sqrt{b^3 - a^3} \ (||f'(t)g(t)||_{L_2((a,b))} + ||f(t)g'(t)||_{L_2((a,b))}) \tag{2.5}$$

$$0 < a < b.$$

Proof.

$$
\begin{aligned}
\Big|\int_a^b f(t)g(t)dt\Big| &= \Big|\int_a^b (t)'f(t)g(t)dt\Big| \\
&= \Big| bf(b)g(b) - af(a)g(a) - \int_a^b t(f(t)g(t))'dt\Big| \\
&= \Big| bf(b)g(b) - af(a)g(a) - \int_a^b tf'(t)g(t)dt \quad - \int_a^b tf(t)g'(t)dt\Big| \\
&\leq \Big| bf(b)g(b) - af(a)g(a)\Big| + \Big|\int_a^b tf'(t)g(t)dt\Big| + \Big|\int_a^b tf(t)g'(t)dt\Big| \\
&\leq \Big| bf(b)g(b) - af(a)g(a)\Big| + \int_a^b |tf'(t)g(t)|dt + \int_a^b |tf(t)g'(t)|dt \\
&\leq \Big| bf(b)g(b) - af(a)g(a)\Big| + \left(\int_a^b t^2dt\right)^{\frac{1}{2}} \left(\int_a^b (f'(t)g(t))^2dt\right)^{\frac{1}{2}} \\
&+ \left(\int_a^b t^2dt\right)^{\frac{1}{2}} \left(\int_a^b (f(t)g'(t))^2dt\right)^{\frac{1}{2}} \\
&= \Big| bf(b)g(b) - af(a)g(a)\Big| \\
&+ \frac{\sqrt{3}}{3}\sqrt{b^3-a^3}\ (||f'(t)g(t)||_{L_2((a,b))} + ||f(t)g'(t)||_{L_2((a,b))})
\end{aligned}
$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.6. *Let $f > 0 : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$
\Big|\int_a^b f'(t)dt\Big| \leq \Big|f(b)\ln(f(b)) - f(a)\ln(f(a))\Big| + ||\ln(f(t))||_{L_2((a,b))}\ ||f'(t)||_{L_2((a,b))}\ , 0 < a < b. \qquad (2.6)
$$

Proof.

$$\begin{aligned}
\Big|\int_a^b f'(t)dt\Big| &= \Big|\int_a^b \frac{f'(t)}{f(t)} f(t)dt\Big| \\
&= \Big|\int_a^b (\ln(f(t))\ )' f(t)dt\ \Big| \\
&= \Big|\ f(b)\ln(f(b)) - f(a)\ln(f(a)) - \int_a^b \ln(f(t))f'(t)dt\Big| \\
&\leq \Big|f(b)\ln(f(b)) - f(a)\ln(f(a))\Big| + \Big|\int_a^b \ln(f(t))f'(t)dt\Big| \\
&\leq \Big|f(b)\ln(f(b)) - f(a)\ln(f(a))\Big| + \int_a^b |\ln(f(t))f'(t)|dt \\
&\leq \Big|f(b)\ln(f(b)) - f(a)\ln(f(a))\Big| + \left(\int_a^b |\ln(f(t))|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b f'(t)^2 dt\right)^{\frac{1}{2}} \\
&= \Big|f(b)\ln(f(b)) - f(a)\ln(f(a))\Big| + ||\ln(f(t))||_{L_2((a,b))}\ ||f'(t)||_{L_2((a,b))},\ 0 < a < b
\end{aligned}$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.7. *Let $f : \Omega = (a, b) \subset]0, +\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions and $f' > 0$, in the domain Ω. Then,*

$$\Big|\int_a^b f''(t)dt\Big| \leq \Big|f'(b)\ln(f'(b)) - f'(a)\ln(f'(a))\Big| + ||\ln(f'(t))||_{L_2((a,b))}\ ||f''(t)||_{L_2((a,b))}\ , 0 < a < b. \quad (2.7)$$

Proof.

$$\begin{aligned}
\Big|\int_a^b f''(t)dt\Big| &= \Big|\int_a^b \frac{f''(t)}{f'(t)} f'(t)dt\Big| \\
&= \Big|\int_a^b (\ln(f'(t))\ )' f(t)dt\ \Big| \\
&= \Big|\ f'(b)\ln(f'(b)) - f'(a)\ln(f'(a)) - \int_a^b \ln(f'(t))f''(t)dt\Big| \\
&\leq \Big|f'(b)\ln(f'(b)) - f'(a)\ln(f'(a))\Big| + \Big|\int_a^b \ln(f'(t))f''(t)dt\Big| \\
&\leq \Big|f'(b)\ln(f'(b)) - f'(a)\ln(f'(a))\Big| + \int_a^b |\ln(f'(t))f''(t)|dt \\
&\leq \Big|f'(b)\ln(f'(b)) - f'(a)\ln(f'(a))\Big| + \left(\int_a^b |\ln(f'(t))|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b f''(t)^2 dt\right)^{\frac{1}{2}} \\
&= \Big|f'(b)\ln(f'(b)) - f'(a)\ln(f'(a))\Big| + ||\ln(f'(t))||_{L_2((a,b))}\ ||f''(t)||_{L_2((a,b))},\ 0 < a < b
\end{aligned}$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.8. *Let $f : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^{\infty}(\Omega)$, where $C^{\infty}(\Omega)$ denotes the class of smooth functions. Then,*

$$||f(t)||^2_{L_2((a,b))} \leq \Big|bf^2(b) - af^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3}\ ||f(t)f'(t)||_{L_2((a,b))},\ 0 < a < b. \tag{2.8}$$

Proof.

$$\begin{aligned}
||f(t)||^2_{L_2((a,b))} &= \Big|\int_a^b f(t)^2 dt\Big| \\
&= \Big|\int_a^b (t)' f(t)^2 dt\Big| \\
&= \Big|bf^2(b) - af^2(a) - \int_a^b t\left(f(t)^2\right)' dt\ \Big| \\
&= \Big|bf^2(b) - af^2(a) - 2\int_a^b tf(t)f'(t)dt\Big| \\
&\leq \Big|bf^2(b) - af^2(a)\Big| + 2\Big|\int_a^b tf(t)f'(t)dt\Big| \\
&\leq \Big|bf^2(b) - af^2(a)\Big| + 2\int_a^b |tf(t)f'(t)|dt \\
&\leq \Big|bf^2(b) - af^2(a)\Big| + 2\left(\int_a^b t^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |f(t)f'(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \Big|bf^2(b) - af^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3} \left(\int_a^b |f(t)f'(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \Big|bf^2(b) - af^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3}\ ||f(t)f'(t)||_{L_2((a,b))},\ 0 < a < b
\end{aligned}$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.9. *Let $f : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^{\infty}(\Omega)$, where $C^{\infty}(\Omega)$ denotes the class of smooth functions. Then,*

$$||f'(t)||^2_{L_2((a,b))} \leq \Big|bf'^2(b) - af'^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3}\ ||f'(t)f''(t)||_{L_2((a,b))},\ 0 < a < b. \tag{2.9}$$

Proof.

$$
\begin{aligned}
||f'(t)||^2_{L_2((a,b))} &= \Big|\int_a^b f'(t)^2 dt\Big| \\
&= \Big|\int_a^b (t)' f'(t)^2 dt\Big| \\
&= \Big|bf'^2(b) - af'^2(a) - \int_a^b t\left(f(t)^2\right)' dt\ \Big| \\
&= \Big|bf'^2(b) - af'^2(a) - 2\int_a^b tf'(t)f''(t)dt\Big| \\
&\leq \Big|bf'^2(b) - af'^2(a)\Big| + 2\Big|\int_a^b tf'(t)f''(t)dt\Big| \\
&\leq \Big|bf'^2(b) - af'^2(a)\Big| + 2\int_a^b |tf'(t)f''(t)|dt \\
&\leq \Big|bf'^2(b) - af'^2(a)\Big| + 2\left(\int_a^b t^2 dt\right)^{\frac{1}{2}}\left(\int_a^b |f'(t)f''(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \Big|bf'^2(b) - af'^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3}\left(\int_a^b |f'(t)f''(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \Big|bf'^2(b) - af'^2(a)\Big| + \frac{2\sqrt{3}}{3}\sqrt{b^3 - a^3}\ ||f'(t)f''(t)||_{L_2((a,b))},\ 0 < a < b
\end{aligned}
$$

using the integration by parts once, the triangle inequality and the Schwarz-Cauchy inequality. □

Theorem 2.10. *Let $f : \Omega = (a,b) \subset]0,+\infty[\to \mathbb{R}$, $f \in C^\infty(\Omega)$, where $C^\infty(\Omega)$ denotes the class of smooth functions. Then,*

$$
\begin{aligned}
\Big|\int_a^b f(t)\sinh(t)dt\Big| &\leq \Big|bf(b)\sinh(b) - af(a)\sinh(a)\Big| + \\
&+ \frac{\sqrt{2}}{2}||tf'(t)||_{L_2((a,b))}\sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) - (b-a)} \\
&+ \frac{\sqrt{2}}{2}||tf(t)||_{L_2((a,b))}\sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) + (b-a)},\ 0 < a < b. \qquad (2.10)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\Big| \int_a^b f(t)\sinh(t)dt \Big| = \Big| \int_a^b (t)' f(t)\sinh(t)dt \Big| \\
&= \Big| bf(b)\sinh(b) - af(a)\sinh(a) - \int_a^b t(f(t)\sinh(t))' dt \Big| \\
&= \Big| bf(b)\sinh(b) - af(a)\sinh(a) - \int_a^b t\ f'(t)\sinh(t)dt - \int_a^b tf(t)\cosh(t)dt \Big| \\
&\le \Big| bf(b)\sinh(b) - af(a)\sinh(a) \Big| + \Big| \int_a^b t\ f'(t)\sinh(t)dt \Big| \\
&+ \Big| \int_a^b tf(t)\cosh(t)dt \Big| \\
&\le \Big| bf(b)\sinh(b) - af(a)\sinh(a) \Big| + \int_a^b |t\ f'(t)\sinh(t)| dt \\
&+ \int_a^b |tf(t)\cosh(t)| dt \\
&\le \Big| bf(b)\sinh(b) - af(a)\sinh(a) \Big| + \\
&+ \left(\int_a^b |tf'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b |\sinh(t)|^2 dt \right)^{\frac{1}{2}} \\
&+ \left(\int_a^b |tf(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b |\cosh(t)|^2 dt \right)^{\frac{1}{2}} \\
&= \Big| bf(b)\sinh(b) - af(a)\sinh(a) \Big| + \\
&+ \frac{\sqrt{2}}{2} ||tf'(t)||_{L_2((a,b))} \sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) - (b-a)} \\
&+ \frac{\sqrt{2}}{2} ||tf(t)||_{L_2((a,b))} \sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) + (b-a)},\ 0 < a < b
\end{aligned}
$$

using integration by parts, the triangle inequality, the Schwarz-Cauchy inequality and exploiting the Lebesgue norms of the hyberbolic functions

$$
\begin{aligned}
||\ \sinh(t)||_{L_2((a,b))} &= \frac{\sqrt{2}}{2} \sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) - (b-a)}, \\
||\ \cosh(t)||_{L_2((a,b))} &= \frac{\sqrt{2}}{2} \sqrt{\sinh(b)\cosh(b) - \sinh(a)\cosh(a) + (b-a)}.
\end{aligned}
$$

□

3 Main Results

In the current section, the main results are presented. There are ten inequalities involving trigonometric functions, stemming from the estimates involving the Lebesgue norms in the first section.

Theorem 3.1. *The following inequality holds*

$$\frac{\sin(x)}{x} \leq \cos(x) + \frac{\sqrt{6}}{6}\sqrt{x}\ \sqrt{x - \sin(x)\cos(x)}, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{3.1}$$

Proof. The result follows by applying (2.1). Pick $a = 0, b = x, f(t) = \cos(t)$, then the left hand side and the right hand side becomes

$$L(x) = \sin(x), \quad x \in \left(0, \frac{\pi}{2}\right)$$

$$R(x) = x\ \cos(x) + \frac{\sqrt{6}}{6}\ x^{\frac{3}{2}}\ \sqrt{x - \sin(x)\cos(x)},\ x \in \left(0, \frac{\pi}{2}\right)$$

employing the Lebesgue norm

$$||f'(t)||_{L_2((0,x))} = \left(\int_0^x \sin^2(t)dt\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}\sqrt{x - \sin(x)\ \cos(x)}.$$

Consequently

$$\sin(x) \leq x\ \cos(x) + \frac{\sqrt{6}}{6}\ x^{\frac{3}{2}}\ \sqrt{x - \sin(x)\cos(x)}$$

$$\Rightarrow \frac{\sin(x)}{x} \leq \cos(x) + \frac{\sqrt{6}}{6}\sqrt{x}\ \sqrt{x - \sin(x)\cos(x)}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

$\square$

Theorem 3.2.

$$1 - x\sin(x) - \frac{\sqrt{6}}{6}x^{\frac{3}{2}}\ \sqrt{x + \sin(x)\cos(x)} \leq \cos(x),\ x \in \left(0, \frac{\pi}{2}\right). \tag{3.2}$$

Proof. The result is derived by direct application of (2.2). Pick $a = 0, b = x, f(t) = \sin(t)$, then the left hand side and the right hand side becomes

$$L(x) = 1 - \cos(x), \quad x \in \left(0, \frac{\pi}{2}\right)$$

$$R(x) = x\ \sin(x) + \frac{\sqrt{6}}{6}\ x^{\frac{3}{2}}\ \sqrt{x + \sin(x)\cos(x)},\ x \in \left(0, \frac{\pi}{2}\right).$$

Consequently, this yields

$$1 - \cos(x) \leq x\ \sin(x) + \frac{\sqrt{6}}{6}\ x^{\frac{3}{2}}\ \sqrt{x + \sin(x)\cos(x)}$$

$$\Rightarrow 1 - x\ \sin(x) - \frac{\sqrt{6}}{6}\ x^{\frac{3}{2}}\ \sqrt{x + \sin(x)\cos(x)}\ \leq \cos(x),\ x \in \left(0, \frac{\pi}{2}\right).$$

$\square$

Theorem 3.3.

$$1 - 9x^2 + x\sin(x) - \frac{\sqrt{6}}{6}\, x^{\frac{3}{2}}\, \sqrt{73x + \sin(x)\cos(x) - 24\sin(x)}$$
$$\leq \cos(x) \leq$$
$$1 + 3x^2 - x\sin(x) + \frac{\sqrt{6}}{6}\, x^{\frac{3}{2}}\, \sqrt{73x + \sin(x)\cos(x) - 24\sin(x)}, \tag{3.3}$$
$$x \in \left(0, \frac{\pi}{2}\right).$$

Proof. The result follows after direct application of (2.3). Pick $a = 0, b = x, f(t) = t^3 + \sin(t)$. Also consider the Lebesgue norm

$$||f'''(t)||_{L_2((0,x))} = \left(\int_0^x (6 - \cos(t))^2 dt\right)^{\frac{1}{2}}$$
$$= \frac{\sqrt{2}}{2}\sqrt{73x + \sin(x)\cos(x) - 24\sin(x)}.$$

The left hand side and the right hand side becomes

$$L(x) = |3x^2 + \cos(x) - 1|, x \in \left(0, \frac{\pi}{2}\right)$$
$$R(x) = x(6x - \sin(x)) + \frac{\sqrt{6}}{6}x^{\frac{3}{2}}\sqrt{73x + \sin(x)\cos(x) - 24\sin(x)},$$
$$= 6x^2 - x\sin(x) + \frac{\sqrt{6}}{6}x^{\frac{3}{2}}\sqrt{73x + \sin(x)\cos(x) - 24\sin(x)}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Removing the absolute value and further working, the desired estimate is obtained

$$1 - 9x^2 + x\sin(x) - \frac{\sqrt{6}}{6}\, x^{\frac{3}{2}}\, \sqrt{73x + \sin(x)\cos(x) - 24\sin(x)}$$
$$\leq \cos(x) \leq$$
$$1 + 3x^2 - x\sin(x) + \frac{\sqrt{6}}{6}\, x^{\frac{3}{2}}\, \sqrt{73x + \sin(x)\cos(x) - 24\sin(x)},$$
$$x \in \left(0, \frac{\pi}{2}\right).$$

□

Theorem 3.4.

$$|1 - \sin(x) - \cos(x)| \leq x|\sin(x) - \cos(x)| + \frac{\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{x - \frac{\cos(2x)}{2} + \frac{1}{2}}, \; x \in \left(0, \frac{\pi}{2}\right). \tag{3.4}$$

Proof. The inequality is derived using the estimate (2.4). Pick $a = 0, b = x, f(t) = \sin(t) + \cos(t)$. Consider the Lebesgue norm

$$||f''''(t)||_{L_2((0,x))} = \left(\int_0^x (\sin(t) + \cos(t))^2 dt\right)^{\frac{1}{2}} = \sqrt{x - \frac{\cos(2x)}{2} + \frac{1}{2}}, \; x \in \left(0, \frac{\pi}{2}\right).$$

As a consequence, the left hand side and the right hand side of the inequality are written below

$$L(x) = |1 - \sin(x) - \cos(x)|, \ x \ \in \left(0, \frac{\pi}{2}\right)$$

$$R(x) = x \ |\sin(x) - \cos(x)| + \frac{\sqrt{3}}{3} x^{\frac{3}{2}} \sqrt{x - \frac{\cos(2x)}{2} + \frac{1}{2}}, \ x \ \in \left(0, \frac{\pi}{2}\right).$$

Combining the above, the desired inequality is obtained

$$|1 - \sin(x) - \cos(x)| \leq x|\sin(x) - \cos(x)| + \frac{\sqrt{3}}{3} x^{\frac{3}{2}} \sqrt{x - \frac{\cos(2x)}{2} + \frac{1}{2}}, \ x \ \in \left(0, \frac{\pi}{2}\right).$$

□

Theorem 3.5.

$$\begin{aligned} |1 - \cos(2x)| &\leq 4|x \ \sin(x) \ \cos(x) \ | \\ &\quad + \frac{2\sqrt{3}}{3} \ x^{\frac{3}{2}} \left(\sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} + \sin(2x)} + \sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} - \sin(2x)} \right), \\ &\quad x \ \in \left(0, \frac{\pi}{2}\right). \end{aligned} \tag{3.5}$$

Proof. This result occurs naturally from the estimate (2.5). Pick $a = 0, b = x, f(t) = \sin(t), g(t) = \cos(t)$. Then the left member and the right member of the inequality is

$$\begin{aligned} L(x) &= \frac{1}{4}|1 - \cos(2x)|, \ x \ \in \left(0, \frac{\pi}{2}\right) \\ R(x) &= |x \ \sin(x) \ \cos(x)| \\ &\quad + \frac{\sqrt{3}}{6} \ x^{\frac{3}{2}} \left(\sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} + \sin(2x)} + \sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} - \sin(2x)} \right), \ x \ \in \left(0, \frac{\pi}{2}\right) \end{aligned}$$

exploiting the Lebesgue norms

$$\begin{aligned} ||f'(t)g(t)||_{L_2((0,x))} &= \frac{1}{2} \left(\int_0^x (\cos(t) \ \cos(t))^2 dt \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left(\int_0^x \cos^4(t) dt \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} + \sin(2x)}, \\ ||f(t)g'(t)||_{L_2((0,x))} &= \frac{1}{2} \left(\int_0^x (\sin(t) \sin(t))^2 dt \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \left(\int_0^x \sin^4(t) dt \right)^{\frac{1}{2}} \\ &= \frac{1}{2} \sqrt{\frac{3x}{2} + \frac{\sin(4x)}{8} - \sin(2x)}. \end{aligned}$$

This yields

$$\begin{aligned}
&\frac{1}{4}|1-\cos(2x)| \leq |x\ \sin(x)\ \cos(x)| \\
&\qquad + \frac{\sqrt{3}}{6}\ x^{\frac{3}{2}} \left(\sqrt{\frac{3x}{2}+\frac{\sin(4x)}{8}+\sin(2x)} + \sqrt{\frac{3x}{2}+\frac{\sin(4x)}{8}-\sin(2x)} \right) \\
&\Rightarrow |1-\cos(2x)| \leq 4|x\ \sin(x)\ \cos(x)\ | \\
&\qquad + \frac{2\sqrt{3}}{3}\ x^{\frac{3}{2}} \left(\sqrt{\frac{3x}{2}+\frac{\sin(4x)}{8}+\sin(2x)} + \sqrt{\frac{3x}{2}+\frac{\sin(4x)}{8}-\sin(2x)} \right), \\
&\ x\ \in \left(0, \frac{\pi}{2}\right)
\end{aligned}$$

and the proof is complete. □

Theorem 3.6.

$$\sin(x) \leq \ln\left(1+\exp(\sin(x))\sin(x)+\frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}(\exp(\sin(x))-1)^{\frac{1}{2}}(\exp(\sin(x))+1)^{\frac{1}{2}}\right), \quad x\ \in \left(0, \frac{\pi}{2}\right). \tag{3.6}$$

Proof. Pick the estimate (2.6) and adjust the upper bound of integration to $b = b(x) = \sin(x), x \in (0, \frac{\pi}{2})$. The lower bound is $a = 0$. Then the estimate is of the form

$$\begin{aligned}
\Big| \int_0^{b(x)=\sin(x)} f'(t)dt\Big| &\leq \Big|f(b(x))\ln(f(b(x))) - f(0)\ln(f(0))\Big| \\
&\quad + ||\ln(f(t))||_{L_2((0,b(x)))}\ ||f'(t)||_{L_2((0,b(x)))}\ .
\end{aligned}$$

Pick $f(t) = \exp(t)$. Then,working out the above estimate yields the following

$$\begin{aligned}
&\exp(\sin(x)) - 1 \leq \exp(\sin(x))\sin(x) \\
&\quad +\frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}(\exp(\sin(x))-1)^{\frac{1}{2}}(\exp(\sin(x))+1)^{\frac{1}{2}} \\
&\Rightarrow \exp(\sin(x)) \leq 1+\frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}(\exp(\sin(x))-1)^{\frac{1}{2}}(\exp(\sin(x))+1)^{\frac{1}{2}} \\
&\quad +\exp(\sin(x))\sin(x) \\
&\Rightarrow \sin(x) \leq \\
&\ln\left(1+\exp(\sin(x))\sin(x)+\frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}(\exp(\sin(x))-1)^{\frac{1}{2}}(\exp(\sin(x))+1)^{\frac{1}{2}}\right), \\
&x\ \in \left(0, \frac{\pi}{2}\right)
\end{aligned}$$

and the proof is complete. $\square$

Theorem 3.7.

$$T(x) \leq \ln\left(1 + T(x)\exp(T(x)) + \frac{\sqrt{6}}{6}\, T^{\frac{3}{2}}(x)(\exp(T(x)) - 1)^{\frac{1}{2}}(\exp(T(x)) + 1)^{\frac{1}{2}}\right), \quad (3.7)$$
$$T(x) = 1 + \sin(x) + \sin^2(x),$$
$$x \in \left(0, \frac{\pi}{2}\right).$$

Proof. The proof is similar in fashion to the inequality (3.6). Pick $a = 0, b = b(x) = T(x) = 1 + \sin(x) + \sin^2(x), x \in (0, \frac{\pi}{2}),\ f(t) = \exp(t)$ in the estimate (2.7). Then by thorough workout, this yields

$$\exp(T(x)) - 1 \leq \exp(T(x))T(x) + \frac{\sqrt{6}}{6}(T(x))^{\frac{3}{2}}(\exp(T(x)) - 1)^{\frac{1}{2}}(\exp(T(x)) + 1)^{\frac{1}{2}}$$
$$\Rightarrow \exp(T(x)) \leq 1 + \frac{\sqrt{6}}{6}(T(x))^{\frac{3}{2}}(\exp(T(x)) - 1)^{\frac{1}{2}}(\exp(T(x)) + 1)^{\frac{1}{2}} + \exp(T(x))T(x)$$
$$\Rightarrow T(x) \leq \ln\left(1 + \exp(T(x))T(x) + \frac{\sqrt{6}}{6}(T(x))^{\frac{3}{2}}(\exp(T(x)) - 1)^{\frac{1}{2}}(\exp(T(x)) + 1)^{\frac{1}{2}}\right),$$
$$x \in \left(0, \frac{\pi}{2}\right)$$

and the proof is complete. $\square$

Theorem 3.8.

$$x - \sin(x)\cos(x) \leq 2x\sin^2(x) + \frac{2\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}},\ x \in \left(0, \frac{\pi}{2}\right). \quad (3.8)$$

Proof. This inequality is a direct consequence of the estimate (2.8). Let $f(t) = \sin(t)$, $a = 0$ and $b = x$ where $x \in (0, \frac{\pi}{2})$. Then, it follows

$$\frac{(x - \sin(x)\cos(x))}{2} \leq x\sin^2(x) + \frac{\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}}$$
$$\Rightarrow x - \sin(x)\cos(x) \leq 2x\sin^2(x) + \frac{2\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}},\ x \in \left(0, \frac{\pi}{2}\right).$$

$\square$

Theorem 3.9.

$$x + \sin(x)\cos(x) \leq 2x\cos^2(x) + \frac{2\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}}, \ x \ \in \left(0, \frac{\pi}{2}\right). \tag{3.9}$$

Proof. Employ the estimate (2.9). Let $f(t) = \sin(t)$, $a = 0$ and $b = x$ where $x \in (0, \frac{\pi}{2})$. Then, it follows

$$\frac{(x + \sin(x)\cos(x))}{2} \leq x\cos^2(x) + \frac{\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}}$$
$$\Rightarrow x + \sin(x)\cos(x) \leq 2x\cos^2(x) + \frac{2\sqrt{3}}{3}x^{\frac{3}{2}}\sqrt{\frac{x}{2} - \frac{\sin(4x)}{8}}, \ x \ \in \left(0, \frac{\pi}{2}\right).$$

□

Theorem 3.10.

$$\begin{aligned}|\sin(x)\ \cosh(\sin(x)) - \sinh(\sin(x))| &\leq \sin^2(x)\sinh(\sin(x)) \\ &+ \frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) - \sin(x)} \\ &+ \frac{\sqrt{10}}{10}(\sin(x))^{\frac{5}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) + \sin(x)}, \ x \ \in \left(0, \frac{\pi}{2}\right).\end{aligned} \tag{3.10}$$

Proof. This inequality stems from the estimate (2.10). Set $a = 0$, $b = b(x) = \sin(x), x \in (0, \frac{\pi}{2})$, $f(t) = t$. Then the left and right member of the inequality is

$$\begin{aligned}L(x) &= |\sin(x)\ \cosh(\sin(x)) - \sinh(\sin(x))|, \ x \ \in \left(0, \frac{\pi}{2}\right) \\ R(x) &= \sin^2(x)\sinh(\sin(x)) \\ &\quad + \frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) - \sin(x)} \\ &\quad + \frac{\sqrt{10}}{10}(\sin(x))^{\frac{5}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) + \sin(x)}, \ x \ \in \left(0, \frac{\pi}{2}\right).\end{aligned}$$

Combining the above, yields the desired inequality

$$\begin{aligned}|\sin(x)\ \cosh(\sin(x)) - \sinh(\sin(x))| &\leq \sin^2(x)\sinh(\sin(x)) \\ &+ \frac{\sqrt{6}}{6}(\sin(x))^{\frac{3}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) - \sin(x)} \\ &+ \frac{\sqrt{10}}{10}(\sin(x))^{\frac{5}{2}}\sqrt{\sinh(\sin(x))\cosh(\sin(x)) + \sin(x)}, \ x \ \in \left(0, \frac{\pi}{2}\right).\end{aligned}$$

□

4 Graphical Illustration of the Results

In this section, graphical illustration is provided for all the inequalities involving trigonometric functions. The graphical results verify the theoretical results derived via rigorous proofs. All the related graphs follow below.

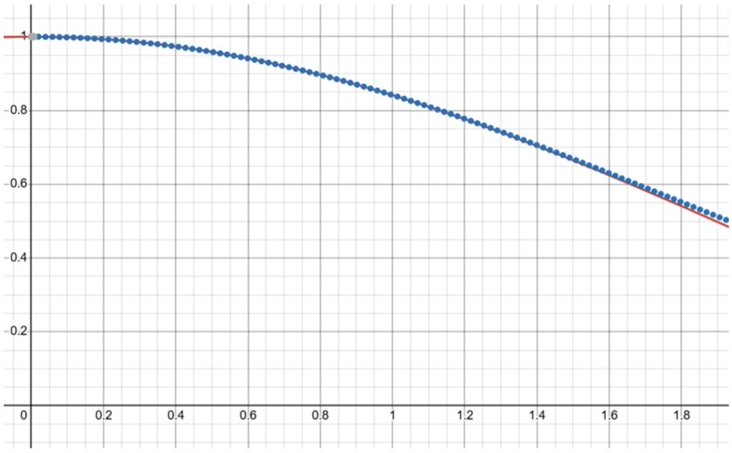

Figure 1: Graphical illustration of (3.1).

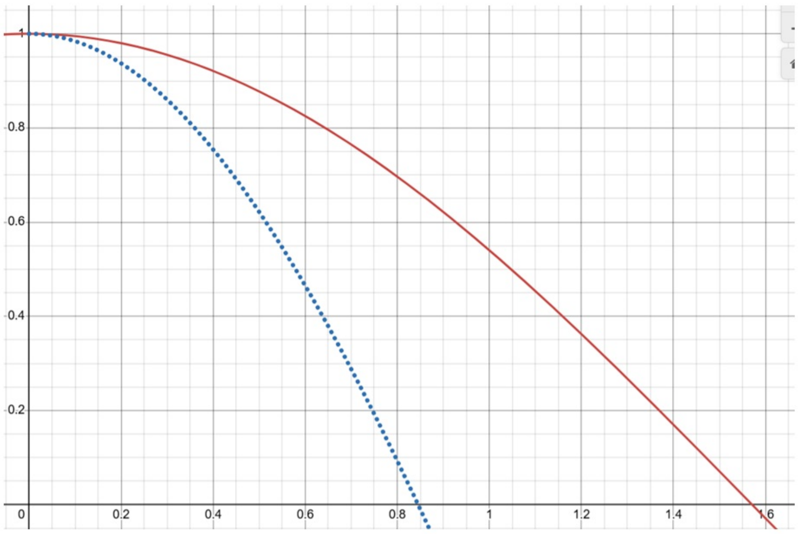

Figure 2: Graphical illustration of (3.2).

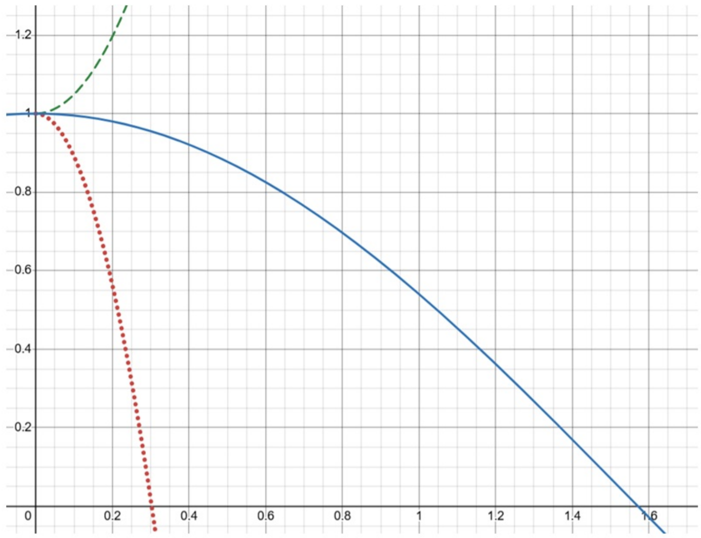

Figure 3: Graphical illustration of (3.3).

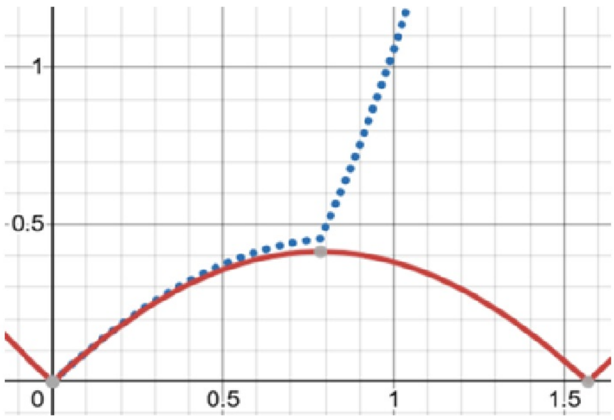

Figure 4: Graphical illustration of (3.4).

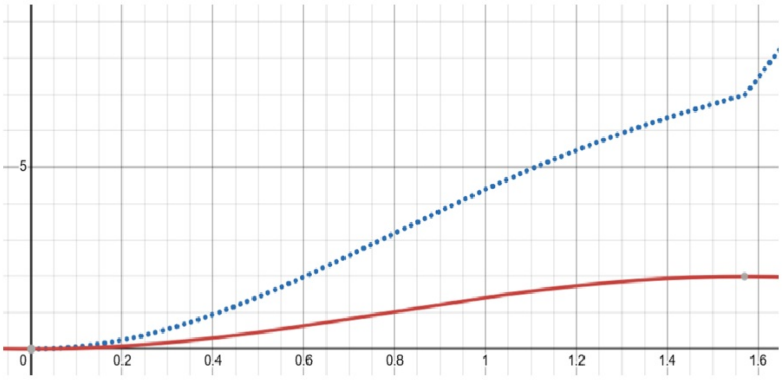

Figure 5: Graphical illustration of (3.5).

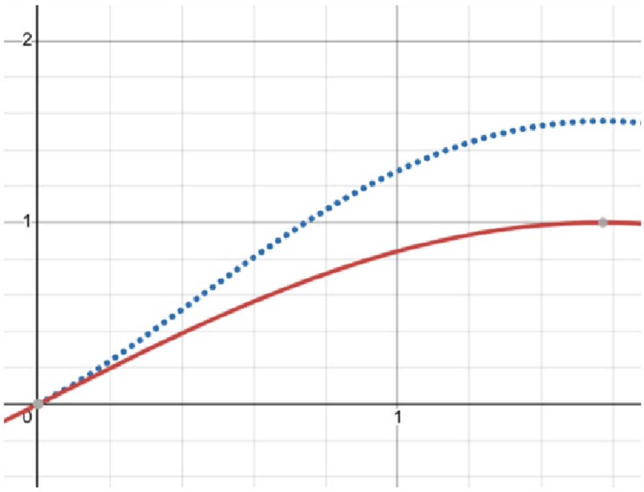

Figure 6: Graphical illustration of (3.6).

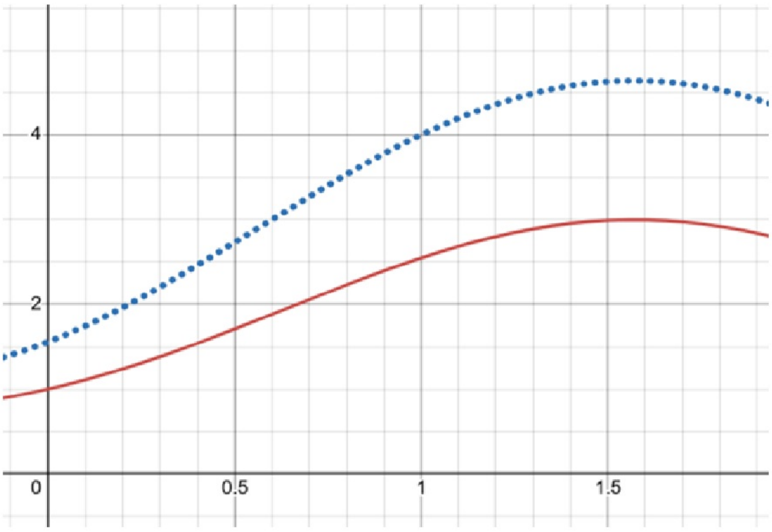

Figure 7: Graphical illustration of (3.7).

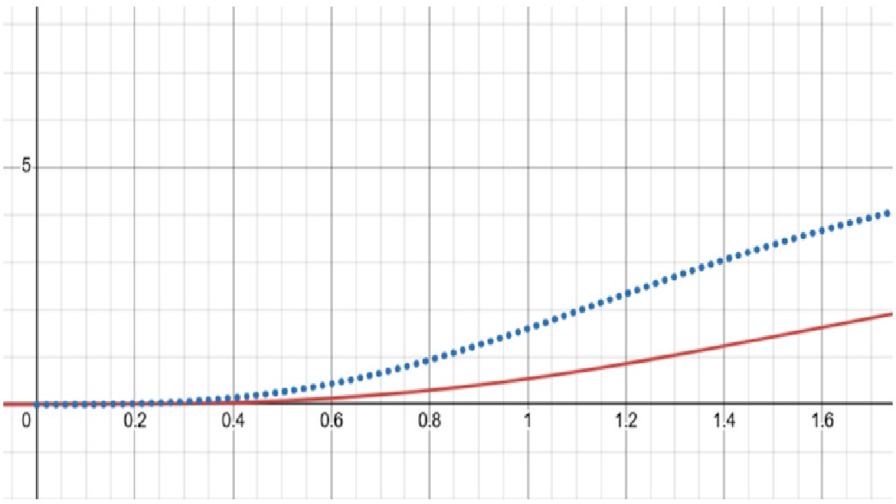

Figure 8: Graphical illustration of (3.8).

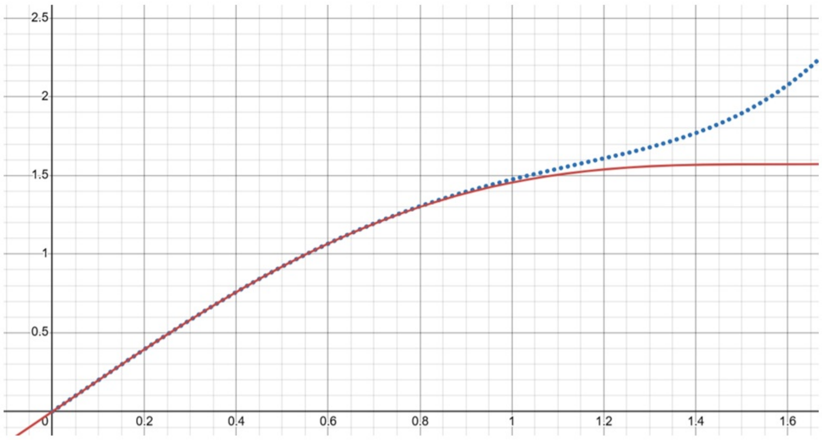

Figure 9: Graphical illustration of (3.9).

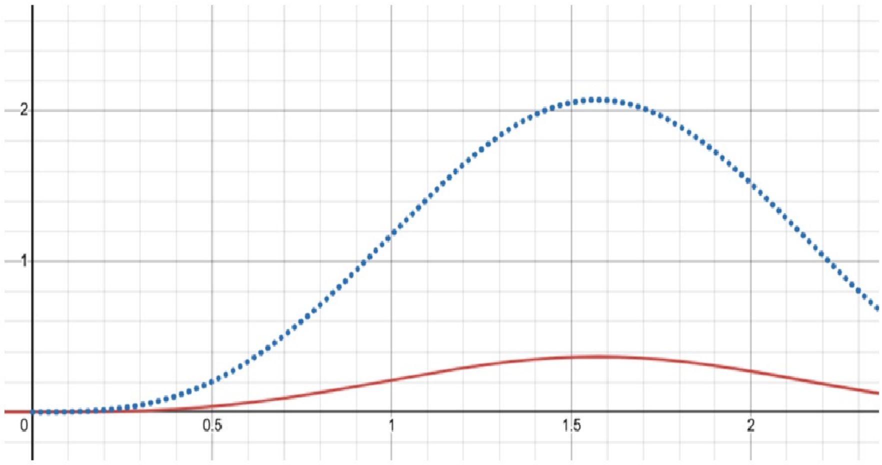

Figure 10: Graphical illustration of (3.10).

5 Conclusions and Remarks

New trigonometric inequalities have been derived and rigorously proved using functional inequalities, and avoiding using series expansions or monotonicity properties which is the most standard and most common technique in the literature. Using functional inequalities and other integral methods to derive trigonometric inequalities is a topic hardly seen in the literature. This article enriches the literature on inequalities involving trigonometric functions, serves as motivation for more research in analysis and as a pedagogical mean. This article presents a stream of inequalities and encourages the use of functional inequalities to derive further results. Additional inequalities using this machinery could be derived for other class of functions, such as special functions, polynomials, hyperbolic functions etc. A few remarks are made for some of the inequalities previously derived.

- Inequality (3.1):There are various inequalities for cardinal sine function such as Jordan's, Cusa-Huygens which motivated a large volume of works in analysis. This inequality is derived via functional estimate and it is new in the literature (to the best knowledge of the author). The inequality is sharp and more tight compared to Jordan's and Cusa-Huygens.
- Inequality (3.2): This inequality is new in the literature to the best knowledge of the author. The right hand side of the inequality is the cosine function and at the left hand side there is a combination of trigonometric, linear and nested quantities. This combination is novel in the literature.
- Inequalities (3.6), (3.7) These are implicit trigonometric inequalities. In the first case, the sine function is below a function which is the natural logarithm and its argument is a combination of exponential and trigonometric functions in some fractional power. In the second case, there is a trigonometric polynomial where it is bounded by a logarithmic function containing exponentials and the trigonometric polynomials in fractional power. This inequality is also novel.
- Inequality (3.10) This inequality is novel in the literature, containing trigonometric and hyperbolic functions. The arguments of the hyperbolic functions are trigonometric functions and there are also fractional powers of the trigonometric functions.
- The other inequalities are novel in the literature to the best knowledge of the author.

References

[1] Mitrinović, D. S. (1970). *Analytic inequalities*. Springer. https://doi.org/10.1007/978-3-642-99970-3

[2] Mitrinović, D. S., Pečarić, J. E., & Fink, A. M. (1991). *Inequalities involving functions and their integrals and derivatives*. Springer. https://doi.org/10.1007/978-94-011-3562-7_15

[3] Jaming, P., & Saba, C. (2023). From Ingham to Nazarov's inequality: A survey on some trigonometric inequalities. *arXiv preprint*, https://arxiv.org/abs/2311.17714.

[4] Bhayo, B. A., & Sándor, J. (2014). On classical inequalities of trigonometric and hyperbolic functions. *arXiv preprint*, https://arxiv.org/abs/1405.0934.

[5] Yang, Z.-H. (2014). Sharp Cusa type inequalities for trigonometric functions with two parameters. *arXiv preprint*, https://arxiv.org/abs/1408.2250.

[6] Yang, Z.-H. (2014). New sharp Cusa–Huygens type inequalities for trigonometric and hyperbolic functions. *arXiv preprint*, https://arxiv.org/abs/1408.2243.

[7] Yang, Z.-H., Jiang, Y.-L., Song, Y.-Q., & Chu, Y.-M. (2014). Sharp inequalities for trigonometric functions. *Abstract and Applied Analysis*, *2014*, Article ID 601839. https://doi.org/10.1155/2014/601839

[8] Bercu, G. (2021). Refinements of Wilker–Huygens–type inequalities via trigonometric series. *Symmetry, 13*(8), 1323. https://doi.org/10.3390/sym13081323

[9] Zhang, B., & Chen, C.-P. (2020). Sharp Wilker and Huygens type inequalities for trigonometric and inverse trigonometric functions. *Journal of Mathematical Inequalities*, *14*(3), 673–684. https://doi.org/10.7153/jmi-2020-14-43

[10] Zhang, L., & Ma, X. (2020). New refinements and improvements of some trigonometric inequalities based on Padé approximant. *Journal of Mathematics*, *2020*, Article ID 2753691, 7 pp. https://doi.org/10.1155/2020/2753691

[11] Qian, G., & Chen, X. D. (2023). Improved bounds of Mitrinović–Adamović–type inequalities by using two-parameter functions. *Journal of Inequalities and Applications, 2023*, 25. https://doi.org/10.1186/s13660-023-02940-2

[12] Chesneau, C. (2025). A collection of trigonometric inequalities using integral methods. *Advances in Analysis and Applied Mathematics, 2*(1), 1–20. https://doi.org/10.62298/advmath.22

[13] Mortici, C. (2014). A subtle analysis of Wilker inequality. *Applied Mathematics and Computation, 231*, 516–520. https://doi.org/10.1016/j.amc.2014.01.017

[14] Bhayo, B. A., & Yin, L. (2014). On the conjecture of generalized trigonometric and hyperbolic functions. *arXiv preprint*, https://arxiv.org/abs/1402.7331.

[15] Yang, Z.-H., Chu, Y.-M., & Zhang, X.-H. (2015). Sharp Cusa type inequalities with two parameters and their applications. *Applied Mathematics and Computation, 268*, 1177-1198. https://doi.org/10.1016/j.amc.2015.07.025

[16] Kyriakis, A., & Kyriakis, M. (2025). Some interpolation inequalities involving real valued functions of a single variable. *Earthline Journal of Mathematical Sciences*, *15*(5), 873–888. https://doi.org/10.34198/ejms.15525.873888