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Intuitionistic Fuzzy Riemann-Liouville and Hadamard Fractional ⊛Integrals

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Abstract

In this study, we introduce intuitionistic fuzzy Riemann-Liouville and Hadamard fractional ⊛integrals of intuitionistic fuzzy valued functions and obtain some of their basic properties. We also give some examples to illustrate the obtained results.

1 Introduction

For $f \in L_1[a,b]$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, the Riemann-Liouville fractional integrals $\mathcal{I}^{\alpha}_{a^+} f(x)$ and $\mathcal{I}^{\alpha}_{b^-} f(x)$ are defined by

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a), \\
\mathcal{I}^{\alpha}_{b^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \quad (x < b).
\end{aligned}$$

The Hadamard fractional integrals $\mathcal{H}^{\alpha}_{a^+} f(x)$ and $\mathcal{H}^{\alpha}_{b^-} f(x)$ are defined by

$$\begin{aligned}
\mathcal{H}^{\alpha}_{a^+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1} f(t)dt \quad (0 < a < x), \\
\mathcal{H}^{\alpha}_{b^-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{1}{t}(\ln t - \ln x)^{\alpha-1} f(t)dt \quad (0 < x < b).
\end{aligned}$$

Riemann-Liouville and Hadamard fractional integrals are two extensions of the concept of integral of integer order to non-integer(fractional) order and they have caught the attention of many researchers since their introduction. Authors have studied these two fractional integrals in different topics and obtained various results. In multiplicative calculus [1, 2], Abdeljawad and Grossman [3] defined multiplicative Riemann-Liouville, Letnikov fractional integrals and introduced geometric fractional calculus. Also, they

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defined the corresponding multiplicative fractional derivatives in geometric fractional calculus. On the other hand, Zhou and Du [4] introduced multiplicative Hadamard k-fractional integrals and obtained Hermite-Hadamard and Bullen type multiplicative fractional integral inequalities with application to multiplicative differential equations, quadrature formulas, special means for real numbers. Recently, Bas *et al.* [5] defined multiplicative(*bigeometric*) Riemann-Liouville fractional integrals and derivatives by using multiplicative(*bigeometric*) gamma and beta functions and examined their fundamental properties. See also [6–11].

In this paper, we aim to study the concepts of Riemann-Liouville and Hadamard fractional integrals in $^{\oplus}$calculus and $^{\otimes}$calculus(or shortly $^{\circledast}$*calculus*) [12]. We introduce intuitionistic fuzzy Riemann-Liouville and Hadamard fractional $^{\circledast}$integrals and obtain some of their properties in intuitionistic fuzzy $^{\circledast}$calculus. We give representations of these intuitionistic fuzzy fractional $^{\circledast}$integrals by using the multiplicative ones given in [3–5] and provide some examples for selected intuitionistic fuzzy valued functions(IFVFs).

2 Preliminaries

In this section we give preliminaries for multiplicative calculus, intuitionistic fuzzy values, $^{\oplus}$calculus and $^{\otimes}$calculus.

2.1 Multiplicative calculus

We give some preliminaries for multiplicative calculus [1, 2, 13].

Let $u \in \mathbb{R}^{+}$. Then, absolute value of u in the multiplicative sense is

$$|u|^{*} = \begin{cases} u, & u \geq 1, \\ \frac{1}{u}, & u < 1. \end{cases}$$

Definition 2.1. *Let $f : [a,b] \to \mathbb{R}^{+}$. Then f is said to be Riemann *integrable on $[a,b]$ if there exists L such that for any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for any points $c_k \in [x_k, x_{k+1}]$, we have*

$$\lim_{\|\mathcal{P}\|\to 0} \prod_{k=0}^{n-1} f(c_k)^{\Delta x_k} = L.$$

In that case, we write $L = \int_a^b f(x)^{dx}$.

Theorem 2.2. *If $f : [a,b] \to \mathbb{R}^{+}$ is *integrable, then*

$$\int_a^b f(x)^{dx} = \exp\left(\int_a^b \ln(f(x))\, dx\right).$$

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}^+$ is said to be *bounded if there exists $M > 1$ such that $|f(x)|^* < M$ for all $x \in I$. Otherwise, it is said to be *unbounded.

Definition 2.3. *[14] Let $f : [a,b) \to \mathbb{R}^+$ be *integrable on interval $[a,t]$ for each $t \in [a,b)$ and *unbounded on the left point of b. Then, improper *integral of type 2 of the function f on $[a,b]$ is defined as*

$$\int_a^b f(x)^{dx} = \lim_{t\to b^-} \int_a^t f(x)^{dx}.$$

*In the case $f : (a,b] \to \mathbb{R}^+$ is *unbounded on the right point of a, it is defined as*

$$\int_a^b f(x)^{dx} = \lim_{t\to a^+} \int_t^b f(x)^{dx}.$$

*If the limit exists in $\mathbb{R}^+$, then the integral is said to be *convergent. Otherwise, it is said to be *divergent.*

Definition 2.4. *[3] Let $f > 0$. Then, the multiplicative Riemann-Liouville fractional integrals ${}^*\mathcal{I}^\alpha_{a^+} f(x)$ and ${}^*\mathcal{I}^\alpha_{b^-} f(x)$ are defined by*

$${}^*\mathcal{I}^\alpha_{a^+} f(x) = \exp\left\{\mathcal{I}^\alpha_{a^+} (\ln \circ f)(x)\right\}, \quad {}^*\mathcal{I}^\alpha_{b^-} f(x) = \exp\left\{\mathcal{I}^\alpha_{b^-} (\ln \circ f)(x)\right\}.$$

Definition 2.5. *[4, 5] Let $f > 0$. Then, the multiplicative Hadamard fractional integrals ${}^*\mathcal{H}^\alpha_{a^+} f(x)$ and ${}^*\mathcal{H}^\alpha_{b^-} f(x)$ are defined by*

$${}^*\mathcal{H}^\alpha_{a^+} f(x) = \exp\left\{\mathcal{H}^\alpha_{a^+} (\ln \circ f)(x)\right\}, \quad {}^*\mathcal{H}^\alpha_{b^-} f(x) = \exp\left\{\mathcal{H}^\alpha_{b^-} (\ln \circ f)(x)\right\}.$$

2.2 Intuitionistic fuzzy values

Now we give definitions concerning intuitionistic fuzzy values.

Let U be a non-empty set. Then, an Atanassov's intuitionistic fuzzy set [15] has the following form: $A = \{\langle u, \mu_A(u), \nu_A(u)\rangle | u \in U\}$, where $\mu : U \to [0,1]$ is called membership function and $\nu : U \to [0,1]$ is called non-membership function. For any $u \in U$, $0 \le \mu_A(u) + \nu_A(u) \le 1$. In special case $\mu_A(u) + \nu_A(u) = 1$, A-IFS degenerates to fuzzy set [16]. Authors [17–19] used the notation $u = (u_\mu, u_\nu)$ for an intuitionistic fuzzy value(IFV) where $u_\mu \in [0,1]$, $u_\nu \in [0,1]$, and $0 \le u_\mu + u_\nu \le 1$. The set of all IFVs is denoted by $\mathcal{L}$. Also, we give the following sets

$$\mathcal{L}^\oplus = \{u \in \mathcal{L} : u <_L (1,0)\}, \qquad \mathcal{L}^\otimes = \{u \in \mathcal{L} : (0,1) <_L u\},$$

where the order $<_L$ is defined by $u <_L v \Longleftrightarrow u_\mu < v_\mu \ \wedge \ u_\nu > v_\nu$.

Now we give basic operations of IFVs.

Definition 2.6. *[19–21] Let $u = (u_\mu, u_\nu)$ and $v = (v_\mu, v_\nu)$ be two IFVs and $c \ge 0$. Then,* *(i)* $u \oplus v = (1 - (1 - u_\mu)(1 - v_\mu), u_\nu v_\nu)$,

(ii) $u \otimes v = (u_\mu v_\mu, 1 - (1 - u_\nu)(1 - v_\nu))$,

(iii) $cu = (1 - (1 - u_\mu)^c, u_\nu^c)$, *where* $u <_L (1,0)$ *(scalar multiplication)*,

(iv) $u^c = (u_\mu^c, 1 - (1 - u_\nu)^c)$, *where* $u >_L (0,1)$ *(scalar exponentiation)*.

2.3 $^\oplus$Calculus for intuitionistic fuzzy values

$^\oplus$Calculus [12] is defined on the set $\mathcal{L}^\oplus = \{u \in \mathcal{L} : u <_L (1,0)\}$.

Definition 2.7. *[12] Let $f : I \subseteq \mathbb{R} \to \mathcal{L}^\oplus$ and c is a cluster point of I. We say that the $^\oplus$limit of f, as x approaches c, is $\xi \in \mathcal{L}^\oplus$ if for any IFV $\bar{\varepsilon} = (\varepsilon, 1 - \varepsilon) >_L (0,1)$ there exists $\delta > 0$ such that*

$$f(x) <_L \xi \oplus \bar{\varepsilon} \quad \text{and} \quad \xi <_L f(x) \oplus \bar{\varepsilon}$$

holds whenever $x \in I$ and $0 < |x - c| < \delta$. In this case, we write $^\oplus\lim\limits_{x\to c} f(x) = \xi$.

Theorem 2.8. *[12] Let $f : I \subseteq \mathbb{R} \to \mathcal{L}^\oplus$, $f(x) = (f_\mu(x), f_\nu(x))$ and $\xi \in \mathcal{L}^\oplus$. Then, $^\oplus\lim\limits_{x\to c} f(x) = \xi$ if and only if $\lim_{x\to c} f_\mu(x) = \xi_\mu$ and $\lim\limits_{x\to c} f_\nu(x) = \xi_\nu$.*

Definition 2.9. *[12] $f : [a,b] \to \mathcal{L}^\oplus$ is said to be Riemann $^\oplus$integrable on $[a,b]$ if there exists an IFV $\xi \in \mathcal{L}^\oplus$ such that for any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for any points $c_k \in [x_k, x_{k+1}]$, we have*

$$^\oplus\lim_{\|\mathcal{P}\|\to 0} \bigoplus_{k=0}^{n-1} f(c_k)\Delta x_k = \xi.$$

In that case, we write $\xi = {}^\oplus\!\int_a^b f(x)dx$.

Theorem 2.10. *[12] Let $f : [a,b] \to \mathcal{L}^\oplus$ and $f = (f_\mu, f_\nu)$. If f is $^\oplus$integrable on $[a,b]$, then*

$$^\oplus\!\int_a^b f(x)dx = \left(1 - \int_a^b (1 - f_\mu)^{dx}, \int_a^b f_\nu^{dx}\right). \tag{2.1}$$

$f : I \subset \mathbb{R} \to \mathcal{L}^\oplus$ is said to be $^\oplus$bounded if there exists $M \in \mathcal{L}^\oplus$ such that $f(x) <_L M$ for all $x \in I$. Otherwise, f is said to be $^\oplus$unbounded on I.

Definition 2.11. *Let $f : [a,b) \to \mathcal{L}^\oplus$ be $^\oplus$integrable on interval $[a,t]$ for each $t \in [a,b)$ and $^\oplus$unbounded on the left point of b. Then, improper $^\oplus$integral of type II of the function f on $[a,b]$ is defined as*

$$^\oplus\!\int_a^b f(x)dx = {}^\oplus\lim_{t\to b^-} {}^\oplus\!\int_a^t f(x)dx.$$

In the case $f:(a,b]\to\mathcal{L}^{\oplus}$ *is* $^{\oplus}$*unbounded on the right point of* a*, it is defined as*

$$^{\oplus}\!\int_a^b f(x)dx = {}^{\oplus}\!\lim_{t\to a^+}{}^{\oplus}\!\int_t^b f(x)dx.$$

If the $^{\oplus}$*limit exists in* $\mathcal{L}^{\oplus}$*, then the integral is said to be* $^{\oplus}$*convergent. Otherwise, it is said to be* $^{\oplus}$*divergent.*

We define set $L_1^{\oplus}[a,b]$ by

$$L_1^{\oplus}[a,b] = \left\{f\in\mathcal{L}^{\oplus} : f \text{ is } {}^{\oplus}\text{integrable on } [a,b] \text{ properly or improperly}\right\}.$$

Remark 2.12. *If* $f\in L_1^{\oplus}[a,b]$*, then* $\ln(1-f_\mu)\in L_1[a,b]$ *and* $\ln(f_\nu)\in L_1[a,b]$*.*

2.4 $^{\otimes}$Calculus for intuitionistic fuzzy values

$^{\otimes}$Calculus [12] is defined on the set $\mathcal{L}^{\otimes}=\{u\in\mathcal{L}:(0,1)<_L u\}$.

Definition 2.13. *[12] Let* $f: I\subseteq\mathbb{R}\to\mathcal{L}$ *and* c *is a cluster point of* I*. We say that the* $^{\otimes}$*limit of* f*, as* t *approaches* c*, is IFV* ξ *if for any IFV* $\bar{\varepsilon}=(1-\varepsilon,\varepsilon)<_L(1,0)$ *there exists* $\delta>0$ *such that*

$$\xi >_L f(x)\otimes\bar{\varepsilon} \quad \text{and} \quad f(x) >_L \xi\otimes\bar{\varepsilon} \tag{2.2}$$

holds whenever $0<|x-c|<\delta$, $x\in I$*. In this case, we write* $^{\otimes}\!\lim\limits_{x\to c} f(x)=\xi$*.*

Theorem 2.14. *[12] Let* $f: I\subseteq\mathbb{R}\to\mathcal{L}^{\otimes}$, $f=(f_\mu,f_\nu)$ *and* $\xi\in\mathcal{L}^{\otimes}$*. Then,* $^{\otimes}\!\lim\limits_{x\to c} f(x)=\xi$ *if and only if* $\lim_{x\to c} f_\mu(x)=\xi_\mu$ *and* $\lim\limits_{x\to c} f_\nu(x)=\xi_\nu$*.*

Definition 2.15. *[12]* $f:[a,b]\to\mathcal{L}^{\otimes}$ *is said to be Riemann* $^{\otimes}$*integrable on* $[a,b]$ *if there exists an IFV* $\xi\in\mathcal{L}^{\otimes}$ *such that for any partition* $\mathcal{P}=\{x_0,x_1,\cdots,x_n\}$ *of* $[a,b]$ *and for any points* $c_k\in[x_k,x_{k+1}]$*, we have*

$$^{\otimes}\!\lim_{\|\mathcal{P}\|\to 0}\bigotimes_{k=0}^{n-1} f(c_k)^{\Delta x_k}=\xi.$$

In that case, we write $\xi = {}^{\otimes}\!\int_a^b f(x)^{dx}$*.*

Theorem 2.16. *[12] Let* $f:[a,b]\to\mathcal{L}^{\otimes}$ *and* $F=(f_\mu,f_\nu)$*. If* f *is* $^{\otimes}$*integrable on* $[a,b]$*, then*

$$^{\otimes}\!\int_a^b f(x)^{dx} = \left(\int_a^b f_\mu^{dx}, 1-\int_a^b (1-f_\nu)^{dx}\right).$$

$f: I\subset\mathbb{R}\to\mathcal{L}^{\otimes}$ is said to be $^{\otimes}$bounded if there exists $M\in\mathcal{L}^{\otimes}$ such that $f(x)>_L M$ for all $x\in I$. Otherwise f is said to be $^{\otimes}$unbounded.

Definition 2.17. *Let $f : [a,b) \to \mathcal{L}^{\otimes}$ be ${}^{\otimes}$integrable on interval $[a,t]$ for each $t \in [a,b)$ and ${}^{\otimes}$unbounded on the left point of b. Then, improper ${}^{\otimes}$integral of type II of the function f on $[a,b]$ is defined as*

$$\,^{\otimes}\!\!\int_a^b f(x)^{dx} = {}^{\otimes}\!\lim_{t\to b^-} {}^{\otimes}\!\!\int_a^t f(x)^{dx}.$$

In the case $f : (a,b] \to \mathcal{L}^{\otimes}$ is ${}^{\otimes}$unbounded on the right point of a, it is defined as

$$\,^{\otimes}\!\!\int_a^b f(x)^{dx} = {}^{\otimes}\!\lim_{t\to a^+} {}^{\otimes}\!\!\int_t^b f(x)^{dx}.$$

If the ${}^{\otimes}$limit exists in $\mathcal{L}^{\otimes}$, then the integral is said to be ${}^{\otimes}$convergent. Otherwise, it is said to be ${}^{\otimes}$divergent.

We define set $L_1^{\otimes}[a,b]$ by

$$L_1^{\otimes}[a,b] = \left\{ f \in \mathcal{L}^{\otimes} : f \text{ is } {}^{\otimes}\text{integrable on } [a,b] \text{ properly or improperly} \right\}.$$

Remark 2.18. *If $f \in L_1^{\otimes}[a,b]$, then $\ln(f_\mu) \in L_1[a,b]$ and $\ln(1-f_\nu) \in L_1[a,b]$.*

3 Intuitionistic Fuzzy Riemann-Liouville Fractional ${}^{\circledast}$Integrals

3.1 Intuitionistic fuzzy Riemann-Liouville fractional ${}^{\oplus}$integral

Before to define intuitionistic fuzzy Riemann-Liouville fractional ${}^{\oplus}$integral, we prove following result for n-fold ${}^{\oplus}$integrals.

Theorem 3.1. *Let f be ${}^{\oplus}$integrable on $[a,b]$. Then,*

$$(i) \quad {}^{\oplus}\!\!\int_a^x {}^{\oplus}\!\!\int_a^{t_1} \cdots {}^{\oplus}\!\!\int_a^{t_{n-2}} {}^{\oplus}\!\!\int_a^{t_{n-1}} f(t_n)dt_n dt_{n-1} \dots dt_1 = \frac{1}{(n-1)!} {}^{\oplus}\!\!\int_a^x (x-t)^{n-1} f(t)dt, \tag{3.1}$$

$$(ii) \quad {}^{\oplus}\!\!\int_x^b {}^{\oplus}\!\!\int_{t_1}^b \cdots {}^{\oplus}\!\!\int_{t_{n-2}}^b {}^{\oplus}\!\!\int_{t_{n-1}}^b f(t_n)dt_n dt_{n-1} \dots dt_1 = \frac{1}{(n-1)!} {}^{\oplus}\!\!\int_x^b (t-x)^{n-1} f(t)dt. \tag{3.2}$$

Proof. *(i)* By Theorem 2.2 and Theorem 2.10, we have

$$\begin{aligned} {}^{\oplus}\!\!\int_a^{t_{n-1}} f(t_n)dt_n &= \left(1-\exp\left(\int_a^{t_{n-1}} \ln(1-f_\mu(t_n))dt_n\right), \exp\left(\int_a^{t_{n-1}} \ln f_\nu(t_n)dt_n\right)\right) \\ {}^{\oplus}\!\!\int_a^{t_{n-2}} {}^{\oplus}\!\!\int_a^{t_{n-1}} f(t_n)dt_n dt_{n-1} &= \left(1-\exp\left(\int_a^{t_{n-2}} \int_a^{t_{n-1}} \ln(1-f_\mu(t_n))dt_n dt_{n-1}\right), \exp\left(\int_a^{t_{n-2}} \int_a^{t_{n-1}} \ln f_\nu((t_n)dt_n dt_{n-1}\right)\right) \end{aligned}$$

and proceeding in this way we get

$${}^{\oplus}\!\!\int_a^x {}^{\oplus}\!\!\int_a^{t_1} \cdots {}^{\oplus}\!\!\int_a^{t_{n-2}} {}^{\oplus}\!\!\int_a^{t_{n-1}} f(t_n)dt_n dt_{n-1} \dots dt_1$$

$$
\begin{aligned}
&= \left(1-\exp\left(\int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} \int_a^{t_{n-1}} \ln(1-f_\mu(t_n))dt_n dt_{n-1}...dt_1\right), \exp\left(\int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} \int_a^{t_{n-1}} \ln f_\nu(t_n)dt_n dt_{n-1}...dt_1\right)\right)\\
&= \left(1-\exp\left(\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\ln(1-f_\mu(t))dt\right), \exp\left(\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\ln f_\nu(t)dt\right)\right)\\
&= \frac{1}{(n-1)!}\left(1-\exp\left(\int_a^x \ln(1-f_\mu(t))^{(x-t)^{n-1}}dt\right), \exp\left(\int_a^x \ln(f_\nu(t))^{(x-t)^{n-1}}dt\right)\right)\\
&= \frac{1}{(n-1)!}{}^{\oplus}\!\!\int_a^x (x-t)^{n-1} f(t)dt
\end{aligned}
$$

by the help of classical Cauchy formula for repeated integration. This completes the proof of part *(i)*.

(ii) The proof of this part is similar to part *(i)*. □

We now define intuitionistic fuzzy Riemann-Liouville fractional $^{\oplus}$integrals.

Definition 3.2. *Let $f \in L_1^{\oplus}[a,b]$. Then, the left and right intuitionistic fuzzy Riemann-Liouville fractional $^{\oplus}$integrals of order $\alpha > 0$ are defined by*

$$
{}^{\oplus}\mathcal{I}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}{}^{\oplus}\!\!\int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x>a), \tag{3.3}
$$

$$
{}^{\oplus}\mathcal{I}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}{}^{\oplus}\!\!\int_x^b (t-x)^{\alpha-1} f(t)dt \quad (x<b), \tag{3.4}
$$

respectively.

In case of $\alpha = n$ where $n \in \mathbb{N}$, (3.3) and (3.4) coincides with $n-$fold $^{\oplus}$integrals in (3.1) and (3.2).

Theorem 3.3. *Let $f \in L_1^{\oplus}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$
\begin{aligned}
(i) \quad {}^{\oplus}\mathcal{I}_{a^+}^{\alpha} f &= \left(1-\exp\left(\mathcal{I}_{a^+}^{\alpha}\ln(1-f_\mu)\right), \exp\left(\mathcal{I}_{a^+}^{\alpha}\ln f_\nu\right)\right),\\
(ii) \quad {}^{\oplus}\mathcal{I}_{b^-}^{\alpha} f &= \left(1-\exp\left(\mathcal{I}_{b^-}^{\alpha}\ln(1-f_\mu)\right), \exp\left(\mathcal{I}_{b^-}^{\alpha}\ln f_\nu\right)\right),
\end{aligned}
$$

where $\mathcal{I}_{a^+}^{\alpha}$ and $\mathcal{I}_{b^-}^{\alpha}$ are the classical Riemann-Lioville fractional integral operators.

Proof. Let $f \in L_1^{\oplus}[a,b]$ and $f = (f_\mu, f_\nu)$.

(i) From Definition 2.6, Theorem 2.10, Definition 3.2 and definitions of $\mathcal{I}_{a^+}^{\alpha}$ and $\mathcal{I}_{b^-}^{\alpha}$, we have

$$
\begin{aligned}
{}^{\oplus}\mathcal{I}_{a^+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)}{}^{\oplus}\!\!\int_a^x (x-t)^{\alpha-1} f(t)dt\\
&= \frac{1}{\Gamma(\alpha)}\left(1-\int_a^x (1-f_\mu(t))^{((x-t)^{\alpha-1})^{dt}}, \int_a^x f_\nu(t)^{((x-t)^{\alpha-1})^{dt}}\right)\\
&= \frac{1}{\Gamma(\alpha)}\left(1-\exp\left(\int_a^x (x-t)^{\alpha-1}\ln(1-f_\mu(t))dt\right), \exp\left(\int_a^x (x-t)^{\alpha-1}\ln f_\nu(t)dt\right)\right)
\end{aligned}
$$

$$= \left(1-\exp\left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\ln(1-f_\mu(t))dt\right), \exp\left(\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\ln f_\nu(t)dt\right)\right)$$
$$= \left(1-\exp\left(\mathcal{I}_{a^+}^{\alpha}\ln(1-f_\mu)(x)\right), \exp\left(\mathcal{I}_{a^+}^{\alpha}\ln f_\nu(x)\right)\right)$$

which completes the proof of part *(i)*.

(ii) The proof of this part is similar to part *(i)*, hence it is omitted. □

In the following theorem, we represent intuitionistic fuzzy Riemann-Liouville fractional $^{\oplus}$integrals via the multiplicative ones given in [3].

Theorem 3.4. *Let* $f \in L_1^{\oplus}[a,b]$ *and* $f = (f_\mu, f_\nu)$. *Then,*

$${}^{\oplus}\mathcal{I}_{a^+}^{\alpha} f = \left(1 - {}^{*}\mathcal{I}_{a^+}^{\alpha}(1-f_\mu), {}^{*}\mathcal{I}_{a^+}^{\alpha} f_\nu\right), \quad {}^{\oplus}\mathcal{I}_{b^-}^{\alpha} f = \left(1 - {}^{*}\mathcal{I}_{b^-}^{\alpha}(1-f_\mu), {}^{*}\mathcal{I}_{b^-}^{\alpha} f_\nu\right),$$

where ${}^{*}\mathcal{I}_{a^+}^{\alpha}$ *and* ${}^{*}\mathcal{I}_{b^-}^{\alpha}$ *are the multiplicative Riemann-Liouville fractional integral operators.*

Proof. The proof is direct from Theorem 3.3 and Definition 2.4, hence it is omitted. □

Theorem 3.5. *Let* $f \in L_1^{\oplus}[a,b]$ *and* $\alpha > 0$, $\beta > 0$. *Then, we have*

$$(i) \quad {}^{\oplus}\mathcal{I}_{a^+}^{\alpha}\left({}^{\oplus}\mathcal{I}_{a^+}^{\beta} f(x)\right) = {}^{\oplus}\mathcal{I}_{a^+}^{\alpha+\beta} f(x),$$
$$(ii) \quad {}^{\oplus}\mathcal{I}_{b^-}^{\alpha}\left({}^{\oplus}\mathcal{I}_{b^-}^{\beta} f(x)\right) = {}^{\oplus}\mathcal{I}_{b^-}^{\alpha+\beta} f(x).$$

Proof. Let $f \in L_1^{\oplus}[a,b]$ and $\alpha > 0$, $\beta > 0$.

(i) In view of Theorem 3.4 we obtain

$$\begin{aligned}
{}^{\oplus}\mathcal{I}_{a^+}^{\alpha}\left({}^{\oplus}\mathcal{I}_{a^+}^{\beta} f(x)\right) &= {}^{\oplus}\mathcal{I}_{a^+}^{\alpha}\left(1 - {}^{*}\mathcal{I}_{a^+}^{\beta}(1-f_\mu)(x), {}^{*}\mathcal{I}_{a^+}^{\beta} f_\nu(x)\right) \\
&= \left(1 - {}^{*}\mathcal{I}_{a^+}^{\alpha}\,{}^{*}\mathcal{I}_{a^+}^{\beta}(1-f_\mu)(x), {}^{*}\mathcal{I}_{a^+}^{\alpha}\,{}^{*}\mathcal{I}_{a^+}^{\beta} f_\nu(x)\right) \\
&= \left(1 - {}^{*}\mathcal{I}_{a^+}^{\alpha+\beta}(1-f_\mu)(x), {}^{*}\mathcal{I}_{a^+}^{\alpha+\beta} f_\nu(x)\right) \\
&= {}^{\oplus}\mathcal{I}_{a^+}^{\alpha+\beta} f(x)
\end{aligned}$$

which completes the proof of part (i).

(ii) The proof of is similar to part (i) and hence omitted. □

Example 3.6. *Here, we give a general example for intuitionistic fuzzy Riemann-Liouville fractional* $^{\oplus}$*integration in view of Definition 3.2 and Theorem 3.3.*

For $0 \le m \le n$ *and* $0 < \alpha, 0 < \beta$, *we have*

$$\begin{aligned} &{}^{\oplus}\mathcal{I}^{\alpha}_{a^+}\Big(1-\exp\big(-m(x-a)^{\beta-1}\big)\,,\exp\big(-n(x-a)^{\beta-1}\big)\Big) \\ &= \Big(1-\exp\big(\mathcal{I}^{\alpha}_{a^+}\{-m(x-a)^{\beta-1}\}\big)\,,\exp\big(\mathcal{I}^{\alpha}_{a^+}\{-n(x-a)^{\beta-1}\}\big)\Big) \\ &= \left(1-\exp\left(-\frac{m\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}\right),\exp\left(-\frac{n\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} &{}^{\oplus}\mathcal{I}^{\alpha}_{b^-}\Big(1-\exp\big(-m(b-x)^{\beta-1}\big)\,,\exp\big(-n(b-x)^{\beta-1}\big)\Big) \\ &= \left(1-\exp\left(-\frac{m\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}\right),\exp\left(-\frac{n\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}\right)\right). \end{aligned}$$

3.2 Intuitionistic fuzzy Riemann-Liouville fractional $^{\otimes}$integral

Due to the fact $(\mathcal{L}^{\oplus}, \oplus) \cong (\mathcal{L}^{\otimes}, \otimes)$ given in [12, Section 4], the proofs of theorems in this subsection is similar to those in Subsection 3.1. Hence the proofs are omitted in this subsection.

We first give following result for n-fold $^{\otimes}$integrals.

Theorem 3.7. *Let* f *be* $^{\otimes}$*integrable on* $[a,b]$. *Then, we have*

$$(i) \qquad {}^{\otimes}\!\!\int_a^x {}^{\otimes}\!\!\int_a^{t_1} \cdots {}^{\otimes}\!\!\int_a^{t_{n-2}} {}^{\otimes}\!\!\int_a^{t_{n-1}} f(t_n)^{dt_n^{dt_{n-1}^{\cdot^{\cdot^{\cdot^{dt_1}}}}}} = \left({}^{\otimes}\!\!\int_a^x f(t)^{((x-t)^{n-1})^{dt}}\right)^{\frac{1}{(n-1)!}}, \tag{3.5}$$

$$(ii) \qquad {}^{\otimes}\!\!\int_x^b {}^{\otimes}\!\!\int_{t_1}^b \cdots {}^{\otimes}\!\!\int_{t_{n-2}}^b {}^{\otimes}\!\!\int_{t_{n-1}}^b f(t_n)^{dt_n^{dt_{n-1}^{\cdot^{\cdot^{\cdot^{dt_1}}}}}} = \left({}^{\otimes}\!\!\int_x^b f(t)^{((t-x)^{n-1})^{dt}}\right)^{\frac{1}{(n-1)!}}. \tag{3.6}$$

We now define intuitionistic fuzzy Riemann-Liouville fractional $^{\otimes}$integrals.

Definition 3.8. *Let* $f \in L_1^{\otimes}[a,b]$. *Then, the left and right intuitionistic fuzzy Riemann-Liouville fractional* $^{\otimes}$*integrals of order* $\alpha > 0$ *are defined by*

$$ {}^{\otimes}\mathcal{I}^{\alpha}_{a^+} f(x) = \left({}^{\otimes}\!\!\int_a^x f(t)^{((x-t)^{\alpha-1})^{dt}}\right)^{\frac{1}{\Gamma(\alpha)}} \quad (x > a), \tag{3.7}$$

$$ {}^{\otimes}\mathcal{I}^{\alpha}_{b^-} f(x) = \left({}^{\otimes}\!\!\int_x^b f(t)^{((t-x)^{\alpha-1})^{dt}}\right)^{\frac{1}{\Gamma(\alpha)}} \quad (x < b), \tag{3.8}$$

respectively.

In case of $\alpha = n$ where $n \in \mathbb{N}$, (3.7) and (3.8) coincides with $n-$fold ${}^{\otimes}$integrals in (3.5) and (3.6).

Theorem 3.9. *Let $f \in L_1^{\otimes}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$\begin{aligned}(i)\quad {}^{\otimes}\mathcal{I}_{a^+}^{\alpha} f &= \left(\exp\left(\mathcal{I}_{a^+}^{\alpha}\ln f_\mu\right), 1-\exp\left(\mathcal{I}_{a^+}^{\alpha}\ln(1-f_\nu)\right)\right),\\ (ii)\quad {}^{\otimes}\mathcal{I}_{b^-}^{\alpha} f &= \left(\exp\left(\mathcal{I}_{b^-}^{\alpha}\ln f_\mu\right), 1-\exp\left(\mathcal{I}_{b^-}^{\alpha}\ln(1-f_\nu)\right)\right),\end{aligned}$$

where $\mathcal{I}_{a^+}^{\alpha}$ and $\mathcal{I}_{b^-}^{\alpha}$ are the classical Riemann-Lioville fractional integral operators.

Theorem 3.10. *Let $f \in L_1^{\otimes}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$${}^{\otimes}\mathcal{I}_{a^+}^{\alpha} f = \left({}^{*}\mathcal{I}_{a^+}^{\alpha} f_\mu, 1 - {}^{*}\mathcal{I}_{a^+}^{\alpha}(1-f_\nu)\right),\quad {}^{\otimes}\mathcal{I}_{b^-}^{\alpha} f = \left({}^{*}\mathcal{I}_{b^-}^{\alpha} f_\mu, 1 - {}^{*}\mathcal{I}_{b^-}^{\alpha}(1-f_\nu)\right),$$

where ${}^{*}\mathcal{I}_{a^+}^{\alpha}$ and ${}^{*}\mathcal{I}_{b^-}^{\alpha}$ are the multiplicative Riemann-Liouville fractional integral operators.

Theorem 3.11. *Let $f \in L_1^{\otimes}[a,b]$ and $\alpha > 0$, $\beta > 0$. Then, we have*

$$(i)\quad {}^{\otimes}\mathcal{I}_{a^+}^{\alpha}\left({}^{\otimes}\mathcal{I}_{a^+}^{\beta} f(x)\right) = {}^{\otimes}\mathcal{I}_{a^+}^{\alpha+\beta} f(x),$$

$$(ii)\quad {}^{\otimes}\mathcal{I}_{b^-}^{\alpha}\left({}^{\otimes}\mathcal{I}_{b^-}^{\beta} f(x)\right) = {}^{\otimes}\mathcal{I}_{b^-}^{\alpha+\beta} f(x).$$

Example 3.12. *For $0 \le n \le m$ and $0 < \alpha, 0 < \beta$, we have*

$$\begin{aligned}&{}^{\otimes}\mathcal{I}_{a^+}^{\alpha}\left(\exp\left(-m(x-a)^{\beta-1}\right), 1-\exp\left(-n(x-a)^{\beta-1}\right)\right)\\ &= \left(\exp\left(\mathcal{I}_{a^+}^{\alpha}\{-m(x-a)^{\beta-1}\}\right), 1-\exp\left(\mathcal{I}_{a^+}^{\alpha}\{-n(x-a)^{\beta-1}\}\right)\right)\\ &= \left(\exp\left(-\frac{m\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}\right), 1-\exp\left(-\frac{n\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}\right)\right)\end{aligned}$$

and

$$\begin{aligned}&{}^{\otimes}\mathcal{I}_{b^-}^{\alpha}\left(\exp\left(-m(b-x)^{\beta-1}\right), 1-\exp\left(-n(b-x)^{\beta-1}\right)\right)\\ &= \left(\exp\left(-\frac{m\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}\right), 1-\exp\left(-\frac{n\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}\right)\right).\end{aligned}$$

4 Intuitionistic Fuzzy Hadamard Fractional ${}^{\circledast}$Integrals

4.1 Intuitionistic fuzzy Hadamard fractional ${}^{\oplus}$integral

Definition 4.1. *Let $f \in L_1^{\oplus}[a,b]$. Then, the left and right intuitionistic fuzzy Hadamard fractional ${}^{\oplus}$integrals of order $\alpha > 0$ are defined by*

$${}^{\oplus}\mathcal{H}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} {}^{\oplus}\!\!\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1} f(t)dt \quad (0 < a < x),$$

$$^{\oplus}\mathcal{H}^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}{}^{\oplus}\!\!\int_x^b \frac{1}{t}(\ln t - \ln x)^{\alpha-1} f(t)dt \quad (0 < x < b),$$

respectively.

Theorem 4.2. *Let $f \in L_1^{\oplus}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$(i) \quad ^{\oplus}\mathcal{H}^{\alpha}_{a^+}f = (1 - \exp(\mathcal{H}^{\alpha}_{a^+}\ln(1-f_\mu)), \exp(\mathcal{H}^{\alpha}_{a^+}\ln f_\nu)),$$

$$(ii) \quad ^{\oplus}\mathcal{H}^{\alpha}_{b^-}f = (1 - \exp(\mathcal{H}^{\alpha}_{b^-}\ln(1-f_\mu)), \exp(\mathcal{H}^{\alpha}_{b^-}\ln f_\nu)),$$

where $\mathcal{H}^{\alpha}_{a^+}$ and $\mathcal{H}^{\alpha}_{b^-}$ are the classical Hadamard fractional integral operators.

Proof. Let $f \in L_1^{\oplus}[a,b]$ and $f = (f_\mu, f_\nu)$.

(i) From Definition 2.6, Theorem 2.10, Definition 4.1 and definitions of $\mathcal{H}^{\alpha}_{a^+}$ and $\mathcal{H}^{\alpha}_{b^-}$, we have

$$\begin{aligned}
^{\oplus}\mathcal{H}^{\alpha}_{a^+}f(x) &= \frac{1}{\Gamma(\alpha)}{}^{\oplus}\!\!\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1} f(t)dt \\
&= \frac{1}{\Gamma(\alpha)}\left(1 - \int_a^x (1-f_\mu(t))^{(\frac{(\ln x - \ln t)^{\alpha-1}}{t})^{dt}}, \int_a^x f_\nu(t)^{(\frac{(\ln x - \ln t)^{\alpha-1}}{t})^{dt}}\right) \\
&= \frac{1}{\Gamma(\alpha)}\left(1 - \exp\left(\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1}\ln(1-f_\mu(t))dt\right), \exp\left(\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1}\ln f_\nu(t)dt\right)\right) \\
&= \left(1 - \exp\left(\frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1}\ln(1-f_\mu(t))dt\right), \exp\left(\frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{t}(\ln x - \ln t)^{\alpha-1}\ln f_\nu(t)dt\right)\right) \\
&= (1 - \exp(\mathcal{H}^{\alpha}_{a^+}\ln(1-f_\mu)(x)), \exp(\mathcal{H}^{\alpha}_{a^+}\ln f_\nu(x)))
\end{aligned}$$

which completes the proof of part *(i)*.

(ii) The proof is similar to that of part *(i)*, hence it is omitted. □

In the following theorem, we represent intuitionistic fuzzy Hadamard fractional $^{\oplus}$integrals via the multiplicative ones defined in [4, 5].

Theorem 4.3. *Let $f \in L_1^{\oplus}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$^{\oplus}\mathcal{H}^{\alpha}_{a^+}f = \left(1 - {}^{*}\mathcal{H}^{\alpha}_{a^+}(1-f_\mu), {}^{*}\mathcal{H}^{\alpha}_{a^+}f_\nu\right), \quad ^{\oplus}\mathcal{H}^{\alpha}_{b^-}f = \left(1 - {}^{*}\mathcal{H}^{\alpha}_{b^-}(1-f_\mu), {}^{*}\mathcal{H}^{\alpha}_{b^-}f_\nu\right),$$

where ${}^{*}\mathcal{H}^{\alpha}_{a^+}$ and ${}^{*}\mathcal{H}^{\alpha}_{b^-}$ are the multiplicative Hadamard fractional integral operators.

Proof. The proofs are direct from Theorem 4.2 and Definition 2.5, hence omitted. □

Theorem 4.4. *Let $f \in L_1^{\oplus}[a,b]$ and $\alpha > 0$, $\beta > 0$. Then, we have*

$$(i)\quad {}^{\oplus}\mathcal{H}_{a^+}^{\alpha}\left({}^{\oplus}\mathcal{H}_{a^+}^{\beta} f(x)\right) = {}^{\oplus}\mathcal{H}_{a^+}^{\alpha+\beta} f(x),$$

$$(ii)\quad {}^{\oplus}\mathcal{H}_{b^-}^{\alpha}\left({}^{\oplus}\mathcal{H}_{b^-}^{\beta} f(x)\right) = {}^{\oplus}\mathcal{H}_{b^-}^{\alpha+\beta} f(x).$$

Proof. Let $f \in L_1^{\oplus}[a,b]$ and $\alpha > 0$, $\beta > 0$.

(i) In view of Theorem 4.3 and [4, Theorem 3.3]- [5, Theorem 2], we obtain

$$\begin{aligned} {}^{\oplus}\mathcal{H}_{a^+}^{\alpha}\left({}^{\oplus}\mathcal{H}_{a^+}^{\beta} f(x)\right) &= {}^{\oplus}\mathcal{H}_{a^+}^{\alpha}\left(1 - {}^{*}\mathcal{H}_{a^+}^{\beta}(1-f_\mu)(x), {}^{*}\mathcal{H}_{a^+}^{\beta} f_\nu(x)\right) \\ &= \left(1 - {}^{*}\mathcal{H}_{a^+}^{\alpha}\,{}^{*}\mathcal{H}_{a^+}^{\beta}(1-f_\mu)(x), {}^{*}\mathcal{H}_{a^+}^{\alpha}\,{}^{*}\mathcal{H}_{a^+}^{\beta} f_\nu(x)\right) \\ &= \left(1 - {}^{*}\mathcal{H}_{a^+}^{\alpha+\beta}(1-f_\mu)(x), {}^{*}\mathcal{H}_{a^+}^{\alpha+\beta} f_\nu(x)\right) \\ &= {}^{\oplus}\mathcal{H}_{a^+}^{\alpha+\beta} f(x) \end{aligned}$$

which completes the proof of part (i).

(ii) The proof of this part is similar to that of part (i), hence it is omitted. □

Example 4.5. *We give following example for intuitionistic fuzzy Hadamard fractional ${}^{\oplus}$integration in view of Definition 4.1 and Theorem 4.2.*

For $0 \le m \le n$ and $0 < \alpha, 0 < \beta$, we have

$$\begin{aligned} {}^{\oplus}\mathcal{H}_{a^+}^{\alpha}\left(1 - \left(\frac{a}{x}\right)^m, \left(\frac{a}{x}\right)^n\right) &= \left(1 - \exp\left(\mathcal{H}_{a^+}^{\alpha}\left\{-m\ln\left(\frac{x}{a}\right)\right\}\right), \exp\left(\mathcal{H}_{a^+}^{\alpha}\left\{-n\ln\left(\frac{x}{a}\right)\right\}\right)\right) \\ &= \left(1 - \exp\left(-\frac{m}{\Gamma(2+\alpha)}\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha+1}\right), \exp\left(-\frac{n}{\Gamma(2+\alpha)}\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha+1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} {}^{\oplus}\mathcal{H}_{0^+}^{\alpha}\left(1 - \exp\left(-mx^\beta\right), \exp\left(-nx^\beta\right)\right) &= \left(1 - \exp\left(-m\mathcal{H}_{0^+}^{\alpha} x^\beta\right), \exp\left(-n\mathcal{H}_{0^+}^{\alpha} x^\beta\right)\right) \\ &= \left(1 - \exp\left(-m\beta^{-\alpha} x^\beta\right), \exp\left(-n\beta^{-\alpha} x^\beta\right)\right). \end{aligned}$$

4.2 Intuitionistic fuzzy Hadamard fractional ${}^{\otimes}$integral

Due to the fact $(\mathcal{L}^{\oplus}, \oplus) \cong (\mathcal{L}^{\otimes}, \otimes)$ in [12, Section 4], the proofs of theorems in this subsection is similar to those in Subsection 4.1. Hence the proofs are omitted in this subsection.

Definition 4.6. *Let $f \in L_1^{\otimes}[a,b]$. Then, the left and right intuitionistic fuzzy Hadamard fractional $^{\otimes}$integrals of order $\alpha > 0$ are defined by*

$$\begin{aligned} {}^{\otimes}\mathcal{H}_{a^+}^{\alpha} f(x) &= \left(\int_a^{\otimes x} f(t)^{\left(\frac{(\ln x - \ln t)^{\alpha-1}}{t}\right)^{dt}}\right)^{\frac{1}{\Gamma(\alpha)}} \quad (0 < a < x), \\ {}^{\otimes}\mathcal{H}_{b^-}^{\alpha} f(x) &= \left(\int_x^{\otimes b} f(t)^{\left(\frac{(\ln x - \ln t)^{\alpha-1}}{t}\right)^{dt}}\right)^{\frac{1}{\Gamma(\alpha)}} \quad (0 < x < b), \end{aligned}$$

respectively.

Theorem 4.7. *Let $f \in L_1^{\otimes}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$\begin{aligned} (i) \quad {}^{\otimes}\mathcal{H}_{a^+}^{\alpha} f &= \left(\exp\left(\mathcal{H}_{a^+}^{\alpha} \ln f_\mu\right), 1 - \exp\left(\mathcal{H}_{a^+}^{\alpha} \ln(1 - f_\nu)\right)\right), \\ (ii) \quad {}^{\otimes}\mathcal{H}_{b^-}^{\alpha} f &= \left(\exp\left(\mathcal{H}_{b^-}^{\alpha} \ln f_\mu\right), 1 - \exp\left(\mathcal{H}_{b^-}^{\alpha} \ln(1 - f_\nu)\right)\right), \end{aligned}$$

where $\mathcal{H}_{a^+}^{\alpha}$ and $\mathcal{H}_{b^-}^{\alpha}$ are the classical Hadamard fractional integral operators.

Theorem 4.8. *Let $f \in L_1^{\otimes}[a,b]$ and $f = (f_\mu, f_\nu)$. Then,*

$$ {}^{\otimes}\mathcal{H}_{a^+}^{\alpha} f = \left({}^{*}\mathcal{H}_{a^+}^{\alpha} f_\mu, 1 - {}^{*}\mathcal{H}_{a^+}^{\alpha}(1 - f_\nu)\right), \quad {}^{\otimes}\mathcal{H}_{b^-}^{\alpha} f = \left({}^{*}\mathcal{H}_{b^-}^{\alpha} f_\mu, 1 - {}^{*}\mathcal{H}_{b^-}^{\alpha}(1 - f_\nu)\right), $$

where ${}^{*}\mathcal{H}_{a^+}^{\alpha}$ and ${}^{*}\mathcal{H}_{b^-}^{\alpha}$ are the multiplicative Hadamard fractional integral operators.

Theorem 4.9. *Let $f \in L_1^{\otimes}[a,b]$ and $\alpha > 0$, $\beta > 0$. Then, we have*

$$\begin{aligned} (i) \quad & {}^{\otimes}\mathcal{H}_{a^+}^{\alpha}\left({}^{\otimes}\mathcal{H}_{a^+}^{\beta} f(x)\right) = {}^{\otimes}\mathcal{H}_{a^+}^{\alpha+\beta} f(x), \\ (ii) \quad & {}^{\otimes}\mathcal{H}_{b^-}^{\alpha}\left({}^{\otimes}\mathcal{H}_{b^-}^{\beta} f(x)\right) = {}^{\otimes}\mathcal{H}_{b^-}^{\alpha+\beta} f(x). \end{aligned}$$

Example 4.10. *For $0 \le n \le m$ and $0 < \alpha, 0 < \beta$, we have*

$$\begin{aligned} {}^{\otimes}\mathcal{H}_{a^+}^{\alpha}\left(\left(\frac{a}{x}\right)^m, 1 - \left(\frac{a}{x}\right)^n\right) &= \left(\exp\left(\mathcal{H}_{a^+}^{\alpha}\left\{-m \ln\left(\frac{x}{a}\right)\right\}\right), 1 - \exp\left(\mathcal{H}_{a^+}^{\alpha}\left\{-n \ln\left(\frac{x}{a}\right)\right\}\right)\right) \\ &= \left(\exp\left(-\frac{m}{\Gamma(2+\alpha)}\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha+1}\right), 1 - \exp\left(-\frac{n}{\Gamma(2+\alpha)}\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha+1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} {}^{\otimes}\mathcal{H}_{0^+}^{\alpha}\left(\exp\left(-m x^\beta\right), 1 - \exp\left(-n x^\beta\right)\right) &= \left(\exp\left(-m \mathcal{H}_{0^+}^{\alpha} x^\beta\right), 1 - \exp\left(-n \mathcal{H}_{0^+}^{\alpha} x^\beta\right)\right) \\ &= \left(\exp\left(-m \beta^{-\alpha} x^\beta\right), 1 - \exp\left(-n \beta^{-\alpha} x^\beta\right)\right). \end{aligned}$$

5 Conclusion

In this paper, as a continuation of the paper [12] introducing ${}^{\circledast}$calculus(${}^{\oplus}$calculus and ${}^{\otimes}$calculus) for intuitionistic fuzzy values, we have defined intuitionistic fuzzy Riemann-Liouville and Hadamard fractional ${}^{\circledast}$integrals for intuitionistic fuzzy valued functions. Then, we have proved some theorems establishing connections with the multiplicative Riemann-Liouville and Hadamard fractional integrals and have given some examples for intuitionistic fuzzy Riemann-Liouville and Hadamard fractional ${}^{\circledast}$integrals by the help of proven theorems. Following this paper, researchers may define the other types of intuitionistic fuzzy fractional ${}^{\circledast}$integrals and obtain corresponding theoretical findings.

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