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T-fuzzy Bi-ideals in Semirings

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Abstract

In this work, we initiate the study of fuzzy bi-ideals under t-norms (T-fuzzy bi-ideals) in semirings and investigate some properties of them. Also we define prime, strongly prime, semiprime, irreducible, strongly irreducible T-fuzzy bi-ideals of semirings. Next we investigate them under regular and intra-regular semirings. Finally, we characterize them under totally ordered by inclusion of T-fuzzy bi-ideals of semirings.

1. Introduction

In abstract algebra, a semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse. The term rig is also used occasionally this originated as a joke, suggesting that rigs are rings without negative elements, similar to using rng to mean a ring without a multiplicative identity. Von Neumann regular rings were introduced by von Neumann (1936) under the name of "regular rings", during his study of von Neumann algebras and continuous geometry. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. The fuzzy set theory, which is regarded as an extension the ordinary set theory, originated with Zadeh [36]. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval [0, 1]. Fuzzy sets generalize classical sets, since

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the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics [10, 11]. In [32] Shabir and Kanwal introduced prime bi-ideals, strongly prime bi-ideals and irreducible bi-ideals in semigroups. Ahsan et al. [2] initiated the study of fuzzy semirings. In [4] Bashir et al. studied fuzzy bi-ideals, prime fuzzy bi-ideals, strongly prime fuzzy bi-ideals and irreducible fuzzy bi-ideals in semirings. The triangular norm (T-norm) originated from the studies of probabilistic metric spaces [12, 31] in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [9], Alsina et al. [3], etc. introduced the T-norm into fuzzy set theory. The author by using norms, investigated some properties of fuzzy algebraic structures [14-30]. Section 2 contains some basic definitions and preliminary results which will be needed in the sequel. In Section 3, we define fuzzy bi-ideals under t-norms (T-fuzzy bi-ideals) in semiring R. We prove that $\wedge$ and sum and product of two T-fuzzy bi-ideals of semiring R is also T-fuzzy bi-ideal of semiring R. Also we investigate conditions between T-fuzzy bi-ideals and bi-ideals of semiring R. In Section 4, we introduce prime, strongly prime, semiprime, irreducible, strongly irreducible T-fuzzy bi-ideals of semiring R and we prove that every strongly prime T-fuzzy bi-ideal, is a prime T-fuzzy bi-ideal, every prime T-fuzzy bi-ideal, is a semiprime T-fuzzy bi-ideal and every strongly irreducible semiprime T-fuzzy bi-ideal, is a strongly prime T-fuzzy bi-ideal of R. Also we obtain some results about them if R be a regular and intra-regular semiring. Next, if R be totally ordered by inclusion, then we characterize irreducible T-fuzzy bi-ideals and strongly irreducible T-fuzzy bi-ideals of semiring R.

2. Preliminaries

Definition 2.1 (See [6]). A *semiring* is a set R equipped with two binary operations + and $\cdot$, called addition and multiplication, such that:

(1) $(R, +)$ is a commutative monoid with identity element 0:

$$(a+b)+c = a+(b+c),$$

$$0+a = a+0 = a,$$

$$a+b = b+a.$$

(2) $(R, \cdot)$ is a monoid with identity element 1:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$

$$1 \cdot a = a \cdot 1 = a.$$

(3) Multiplication left and right distributes over addition:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$

$$(a + b) \cdot c = (a \cdot c) + (b \cdot c).$$

(4) Multiplication by 0 annihilates *R*:

$$0 \cdot a = a \cdot 0 = 0.$$

The symbol $\cdot$ is usually omitted from the notation; that is, $a \cdot b$ is just written *ab*. Similarly, an order of operations is accepted, according to which $\cdot$ is applied before +; that is, $a + bc$ is $a + (bc)$. A commutative semiring is one whose multiplication is commutative.

Throughout this paper, *R* stands for the semiring $(R, +, \cdot)$ with an identity element 1_R and zero element 0_R.

Example 2.2. By definition, any ring is also a semiring. A motivating example of a semiring is the set of natural numbers $\mathbb{N}$ (including zero) under ordinary addition and multiplication. Likewise, the non-negative rational numbers and the non-negative real numbers form semirings. All these semirings are commutative.

Definition 2.3 (See [13])**.** A non-empty subset *B* of a semiring *R* is called a *bi-ideal* of *R* if

(1) $a + b \in B$,

(2) $ab \in B$,

(3) $arb \in B$,

for all $a, b \in B$ and $r \in R$.

Definition 2.4 (See [35, 34])**.**

(1) A semiring *R* is called *von Neumann regular* or simply *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$.

(2) A semiring R is called an *intra-regular semiring* if for each $a \in R$ there exists $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n} x_i a^2 y_i$.

Definition 2.5 (See [33]). A (non-strict) partial order is a binary relation $\leq$ over a set P satisfying particular axioms which are discussed below. When $a \leq b$, we say that a is related to b. (This does not imply that b is also related to a, because the relation need not be symmetric.)

The axioms for a non-strict partial order state that the relation $\leq$ is reflexive, antisymmetric, and transitive. That is, for all $a, b, c \in P$, it must satisfy:

(1) $a \leq a$ (reflexivity: every element is related to itself).

(2) if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry: two distinct elements cannot be related in both directions).

(3) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity: if a first element is related to a second element, and, in turn, that element is related to a third element, then the first element is related to the third element).

In other words, a partial order is an antisymmetric preorder. A set with a partial order is called a *partially ordered set* (also called a *poset*). The term ordered set is sometimes also used, as long as it is clear from the context that no other kind of order is meant. In particular, totally ordered sets can also be referred to as "ordered sets", especially in areas where these structures are more common than posets. For a, b, elements of a partially ordered set P, if $a \leq b$ or $b \leq a$, then a and b are comparable. Otherwise they are incomparable. For example, $\{x\}$ and $\{x, y, z\}$ are comparable, while $\{x\}$ and $\{y\}$ are not. A partial order under which every pair of elements is comparable is called a *total order* or *linear order*; a totally ordered set is also called a *chain* (e.g., the natural numbers with their standard order). A subset of a poset in which no two distinct elements are comparable is called an *antichain*. For example, set of singletons $\{\{x\}, \{y\}, \{z\}\}$.

Example 2.6. Standard examples of posets arising in mathematics include:

(1) The real numbers ordered by the standard less-than-or-equal relation $\leq$.

(2) The set of subsets of a given set (its power set) ordered by inclusion. Similarly, the set of sequences ordered by subsequence, and the set of strings ordered by substring.

(3) The set of natural numbers equipped with the relation of divisibility.

(4) The vertex set of a directed acyclic graph ordered by reachability.

Definition 2.7 (See [8]). A lattice is an abstract structure studied in the mathematical sub-disciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). An example is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra. Semilattices include lattices, which in turn include Heyting and Boolean algebras. These "lattice-like" structures all admit order-theoretic as well as algebraic descriptions. If $(L, \leq)$ is a partially ordered set (poset), and $S \subseteq L$ is an arbitrary subset, then an element $u \in L$ is said to be an upper bound of S if $s \leq u$ for each $s \in S$. A set may have many upper bounds, or none at all. An upper bound u of S is said to be its least upper bound, or join, or supremum, if $u \leq x$ for each upper bound x of S. A set need not have a least upper bound, but it cannot have more than one. Dually, $l \in L$ is said to be a lower bound of S if $l \leq s$ for each $s \in S$. A lower bound l of S is said to be its greatest lower bound, or meet, or infimum, if $x \leq l$ for each lower bound x of S. A set may have many lower bounds, or none at all, but can have at most one greatest lower bound. A partially ordered set $(L, \leq)$ is called a join-semilattice if each two-element subset $\{a, b\} \subseteq L$ has a join (i.e. least upper bound), and is called a meet-semilattice if each two-element subset has a meet (i.e. greatest lower bound), denoted by $a \bigvee b$ and $a \bigwedge b$, respectively. $(L, \leq)$ is called a lattice if it is both a join- and a meet-semilattice. This definition makes $\bigvee$ and $\bigwedge$ binary operations. Both operations are monotone with respect to the order: $a_1 \leq a_2$ and $b_1 \leq b_2$ implies that $a_1 \bigvee b_1 \leq a_2 \bigvee b_2$ and $a_1 \bigwedge b_1 \leq a_2 \bigwedge b_2$.

Lemma 2.8 (Zorn's lemma) (See [5]). *Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.*

Definition 2.9 (See [6]). Let X be a non-empty subset of R. Then the *characteristic function* of R denoted and defined by

$$\mu_X(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.10 (See [7])**.** A *t-norm T* is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element)

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity)

(T3) $T(x, y) = T(y, x)$ (commutativity)

(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),

for all $x, y, z \in [0, 1]$.

Example 2.11. (1) Standard intersection *T*-norm $T_m(x, y) = \min\{x, y\}$.

(2) Bounded sum *T*-norm $T_b(x, y) = \max\{0, x + y - 1\}$.

(3) Algebraic product *T*-norm $T_p(x, y) = xy$.

(4) Drastic *T*-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum *T*-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product *T*-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{xy}{x + y - xy} & \text{otherwise.} \end{cases}$$

The drastic *t*-norm is the pointwise smallest *t*-norm and the minimum is the pointwise largest *t*-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Recall that *t*-norm *T* is idempotent if for all $x \in [0, 1]$, $T(x, x) = x$.

Lemma 2.12 (See [1]). *Let T be a t-norm. Then*

$$T(T(x,\ y), T(w,\ z)) = T(T(x,\ w), T(y,\ z)),$$

for all $x, y, w, z \in [0, 1]$.

3. *T*-fuzzy Bi-ideals in Semiring

Definition 3.1. A fuzzy subset $\mu : R \to [0, 1]$ is called a *fuzzy bi-ideal* of *R* under *t*-norm *T* if

(1) $\mu(a+b) \geq T(\mu(a), \mu(b))$,

(2) $\mu(ab) \geq T(\mu(a), \mu(b))$,

(3) $\mu(abc) \geq T(\mu(a), \mu(c))$,

for all $a, b, c \in R$. We denote the set of all fuzzy bi-ideals of *R* under *t*-norm *T* by *T*-fuzzy bi-ideal of semiring *R*.

Example 3.2. Let $R = (\mathbb{N}, +, \cdot)$ be a semiring of real numbers. Define $\mu : \mathbb{N} \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.60 & \text{if } x \in \{2\mathbb{N}\}, \\ 0.80 & \text{if } x \in \{2\mathbb{N} - 1\}. \end{cases}$$

If $T_p(x,\ y) = xy$ for all $x, y \in [0, 1]$, then μ will be a *T*-fuzzy bi-ideal of semiring *R*.

Definition 3.3. Let μ and ν be two *T*-fuzzy bi-ideals of semiring *R*.

(1) The symbol $\mu \bigwedge \nu$ is a fuzzy subset $\mu \bigwedge \nu : R \to [0, 1]$ and defined by $(\mu \bigwedge \nu)(a) = T(\mu(a), \nu(a))$ for all $a \in R$.

(2) The sum of μ and ν is a fuzzy subset $\mu + \nu : R \to [0, 1]$ with $(\mu + \nu)(a) = \bigvee_{a=b+c} T(\mu(b), \nu(c))$ for all $a, b, c \in R$.

(3) The product of μ and ν is a fuzzy subset $\mu \circ \nu : R \to [0, 1]$ by $(\mu \circ \nu)(a) = \bigvee_{a=bc} T(\mu(b), \nu(c))$ for all $a, b, c \in R$.

Remark 3.4. Let μ be T-fuzzy bi-ideals of semiring R. Then for all $b, c \in R$ we have that

$$(\mu \circ \mu_R)(bc) = \bigvee_{bc} T(\mu(b), \mu_R(c)) = \bigvee_{bc} T(\mu(b), 1) = \bigvee_{bc} \mu(b) = \mu(b).$$

Proposition 3.5. (1) *A fuzzy subset* $\mu : R \to [0, 1]$ *is a T-fuzzy bi-ideal of semiring R if and only if* $\mu \geq \mu + \mu$ *and* $\mu \geq \mu \circ \mu$ *and* $\mu \geq \mu \circ \mu_R \circ \mu$.

(2) *A non-empty subset B of a semiring R is a bi-ideal of R if and only if the characteristic function* μ_B *of B is a T-fuzzy bi-ideal of R.*

(3) *Let* μ *and* ν *be two T-fuzzy bi-ideals of semiring R. Then* $\mu \wedge \nu$ *is a T-fuzzy bi-ideal of semiring R.*

(4) *Let* μ *and* ν *be two T-fuzzy bi-ideals of commutative semiring R. Then* $\mu \circ \nu$ *is a T-fuzzy bi-ideal of commutative semiring R.*

Proof. (1) Let μ is a T-fuzzy bi-ideal of semiring R.

If $a = b + c$, then $\mu(a) = \mu(b + c) \geq T(\mu(b), \mu(c))$ and then $\mu(a) \geq \bigvee_{a=b+c} T(\mu(b), \mu(c))$ for all $a, b, c \in R$ and so $\mu \geq \mu + \mu$.

If $a = bc$, then $\mu(a) = \mu(bc) \geq T(\mu(b), \mu(c))$ and then $\mu(a) \geq \bigvee_{a=bc} T(\mu(b), \nu(c))$ for all $a, b, c \in R$ and so $\mu \geq \mu \circ \mu$.

If $a = bcd$, then $\mu(a) = \mu(bcd) \geq T(\mu(b), \mu(d)) = T((\mu \circ \mu_R)(bc), \mu(d))$ and then $\mu(a) \geq \bigvee_{a=bcd} T((\mu \circ \mu_R)(bc), \mu(d))$ for all $a, b, c, d \in R$ and so $\mu \geq \mu \circ \mu_R \circ \mu$.

Conversely, we prove that μ is a T-fuzzy bi-ideal of semiring R.

As $\mu \geq \mu + \mu$ so

$$\mu(a + b) \geq (\mu + \mu)(a + b) = \bigvee_{a+b} T(\mu(a), \mu(b)) \geq T(\mu(a), \mu(b))$$

for all $a, b \in R$.

As $\mu \geq \mu \circ \mu$ so

$$\mu(ab) \geq (\mu \circ \mu)(ab) = \bigvee_{ab} T(\mu(a), \mu(b)) \geq T(\mu(a), \mu(b))$$

for all $a, b \in R$.

As $\mu \geq \mu \circ \mu_R \circ \mu$ so

$$\mu(abc) \geq (\mu \circ \mu_R \circ \mu)(abc) = \bigvee_{(ab)c} T((\mu \circ \mu_R)(ab), \mu(c))$$

$$= \bigvee_{ab} T(\mu(a), \mu(c)) \geq T(\mu(a), \mu(c))$$

for all $a, b, c \in R$.

Thus μ will be a T-fuzzy bi-ideal of semiring R.

(2) Let

$$\mu_B(b) = \begin{cases} 1 & \text{if } b \in B, \\ 0 & \text{otherwise} \end{cases}$$

be the characteristic function of B. Let B be a bi-ideal of semiring R such that $b_1, b_2 \in B$ and $r \in R$.

Since $b_1 + b_2 \in B$ so

$$\mu_B(b_1 + b_2) = 1 \geq 1 = T(1, 1) = T(\mu_B(b_1), \mu_B(b_2))$$

and $b_1 b_2 \in B$, then

$$\mu_B(b_1 b_2) = 1 \geq 1 = T(1, 1) = T(\mu_B(b_1), \mu_B(b_2))$$

also $b_1 r b_2 \in B$, then

$$\mu_B(b_1 r b_2) = 1 \geq 1 = T(1, 1) = T(\mu_B(b_1), \mu_B(b_2)).$$

Thus μ_B is a T-fuzzy bi-ideal of R.

Conversely, assume that μ_B is a T-fuzzy bi-ideal of R and $b_1, b_2 \in B$ and $r \in R$.

As

$$\mu_B(b_1 + b_2) \geq T(\mu_B(b_1), \mu_B(b_2)) = T(1, 1) = 1$$

so $\mu_B(b_1 + b_2) = 1$ and then $b_1 + b_2 \in B$. Since

$$\mu_B(b_1 b_2) \geq T(\mu_B(b_1), \mu_B(b_2)) = T(1, 1) = 1$$

so $\mu_B(b_1b_2) = 1$ and then $b_1b_2 \in B$. As

$$\mu_B(b_1rb_2) \geq T(\mu_B(b_1), \mu_B(b_2)) = T(1, 1) = 1$$

so $\mu_B(b_1rb_2) = 1$ and then $b_1rb_2 \in B$. Therefore B is a bi-ideal of a semiring R.

(3) Let μ and ν be two T-fuzzy bi-ideals of semiring R and $a, b, c \in R$. Then

$$\begin{aligned}(\mu \bigwedge \nu)(a+b) &= T(\mu(a+b), \nu(a+b)) \\ &\geq T(T(\mu(a), \mu(b)), T(\nu(a), \nu(b))) \\ &= T(T(\mu(a), \nu(a)), T(\mu(b), \nu(b))) \text{ (by Lemma 2.12)} \\ &= T((\mu \wedge \nu)(a), (\mu \bigwedge \nu)(b))\end{aligned}$$

and

$$\begin{aligned}(\mu \bigwedge \nu)(ab) &= T(\mu(ab), \nu(ab)) \\ &\geq T(T(\mu(a), \mu(b)), T(\nu(a), \nu(b))) \\ &= T(T(\mu(a), \nu(a)), T(\mu(b), \nu(b))) \text{ (by Lemma 2.12)} \\ &= T((\mu \bigwedge \nu)(a), (\mu \bigwedge \nu)(b))\end{aligned}$$

and

$$\begin{aligned}(\mu \bigwedge \nu)(abc) &= T(\mu(abc), \nu(abc)) \\ &\geq T(T(\mu(a), \mu(c)), T(\nu(a), \nu(c))) \\ &= T(T(\mu(a), \nu(a)), T(\mu(c), \nu(c))) \text{ (by Lemma 2.12)} \\ &= T((\mu \bigwedge \nu)(a), (\mu \bigwedge \nu)(c)).\end{aligned}$$

Then $\mu \bigwedge \nu$ will be a T-fuzzy bi-ideal of semiring R.

(4) Let μ and ν be two T-fuzzy bi-ideals of semiring R. Let $a_1, a_2, b_1, b_2, c_1, c_2 \in R$ such that $a_1 = b_1c_1, a_2 = b_2c_2, a_3 = b_3c_3, b_1c_2 = b_2c_1 = 0$. Then

$$(\mu \circ \nu)(a_1 + a_2) = \bigvee_{(a_1+a_2)=(b_1+b_2)(c_1+c_2)} T(\mu(b_1 + b_2), \nu(c_1 + c_2))$$

$$= \bigvee_{(a_1+a_2)=b_1c_1+b_2c_2} T(\mu(b_1+b_2), \nu(c_1+c_2))$$

$$= \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(\mu(b_1+b_2), \nu(c_1+c_2))$$

$$\geq \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(T(\mu(b_1), \mu(b_2)), T(\nu(c_1), \nu(c_2)))$$

$$= \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(T(\mu(b_1), \nu(c_1)), T(\mu(b_2), \nu(c_2))) \text{ (by Lemma 2.12)}$$

$$= T(\bigvee_{a_1=b_1c_1} T(\mu(b_1), \nu(c_1)), \bigvee_{a_2=b_2c_2} T(\mu(b_2), \nu(c_2)))$$

$$= T((\mu \circ \nu)(a_1), (\mu \circ \nu)(a_2))$$

and

$$(\mu \circ \nu)(a_1a_2) = \bigvee_{(a_1a_2)=(b_1b_2)(c_1c_2)} T(\mu(b_1b_2), \nu(c_1c_2))$$

$$= \bigvee_{(a_1a_2)=b_1c_1b_2c_2} T(\mu(b_1b_2), \nu(c_1c_2))$$

$$= \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(\mu(b_1b_2), \nu(c_1c_2))$$

$$\geq \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(T(\mu(b_1), \mu(b_2)), T(\nu(c_1), \nu(c_2)))$$

$$= \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2} T(T(\mu(b_1), \nu(c_1)), T(\mu(b_2), \nu(c_2))) \text{ (by Lemma 2.12)}$$

$$= T(\bigvee_{a_1=b_1c_1} T(\mu(b_1), \nu(c_1)), \bigvee_{a_2=b_2c_2} T(\mu(b_2), \nu(c_2)))$$

$$= T((\mu \circ \nu)(a_1), (\mu \circ \nu)(a_2))$$

and

$$(\mu \circ \nu)(a_1a_2a_3) = \bigvee_{(a_1a_2a_3)=(b_1b_2b_3)(c_1c_2c_3)} T(\mu(b_1b_2b_3), \nu(c_1c_2c_3))$$

$$= \bigvee_{(a_1a_2a_3)=b_1c_1b_2c_2b_3c_3} T(\mu(b_1b_2b_3), \nu(c_1c_2c_3))$$

$$= \bigvee_{a_1=b_1c_1,\ a_2=b_2c_2,\ a_3=b_3c_3} T(\mu(b_1b_2b_3), \nu(c_1c_2c_3))$$

$$\geq \bigvee_{a_1=b_1c_1,\ a_3=b_3c_3} T(T(\mu(b_1), \mu(b_3)), T(\nu(c_1), \nu(c_3)))$$

$$= \bigvee_{a_1=b_1c_1,\ a_3=b_3c_3} T(T(\mu(b_1), \nu(c_1)), T(\mu(b_3), \nu(c_3))) \text{ (by Lemma 2.12)}$$

$$= T(\bigvee_{a_1=b_1c_1} T(\mu(b_1), \nu(c_1)), \bigvee_{a_3=b_3c_3} T(\mu(b_3), \nu(c_3)))$$

$$= T((\mu \circ \nu)(a_1), (\mu \circ \nu)(a_3)).$$

Then $\mu \circ \nu$ is a *T*-fuzzy bi-ideal of commutative semiring *R*. □

Proposition 3.6. *Let* μ, ν, β *be T-fuzzy bi-ideals of semiring R and* X, Y *be two non-empty subsets of semiring R.*

(1) *If* $\mu \leq \nu$, *then* $\mu \circ \beta \leq \nu \circ \beta$ *and* $\beta \circ \mu \leq \beta \circ \nu$.

(2) $\mu_X \circ \mu_Y = \mu_{XY}$.

(3) $\mu_X \wedge \mu_Y = \mu_{X \cap Y}$.

(4) $\mu_X + \mu_Y = \mu_{X+Y}$.

Proof. Let $x, y, z \in R$.

(1) If $\mu \leq \nu$, then $\mu(y) \leq \nu(y)$. Now

$$(\mu \circ \beta)(x) = \bigvee_{x=yz} T(\mu(y), \beta(z)) \leq \bigvee_{x=yz} T(v(y), \beta(z)) = (\nu \circ \beta)(x)$$

and then $\mu \circ \beta \leq \nu \circ \beta$. Also as $\mu \leq \nu$, then $\mu(z) \leq \nu(z)$ and

$$(\beta \circ \mu)(x) = \bigvee_{x=yz} T(\beta(y), \mu(z)) \leq \bigvee_{x=yz} T(\beta(y), \nu(z)) = (\beta \circ \nu)(x)$$

and thus $\beta \circ \mu \leq \beta \circ \nu$.

(2) We know that

$$\mu_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_Y(y) = \begin{cases} 1 & \text{if } y \in Y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu_{XY}(z) = \begin{cases} 1 & \text{if } z \in XY \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } z = xy \in XY \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in X,\ y \in Y \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in X$ and $y \in Y$, then $xy \in XY$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=xy} T(\mu_X(x), \mu_Y(y)) = \bigvee T(1, 1) = 1 = \mu_{XY}(z = xy).$$

If $x \in X$ and $y \notin Y$, then $xy \notin XY$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=xy} T(\mu_X(x), \mu_Y(y)) = \bigvee T(1, 0) = 0 = \mu_{XY}(z = xy).$$

If $x \notin X$ and $y \notin Y$, then $xy \notin XY$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=xy} T(\mu_X(x), \mu_Y(y)) = \bigvee T(0, 1) = 0 = \mu_{XY}(z = xy).$$

If $x \notin X$ and $y \notin Y$, then $xy \notin XY$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=xy} T(\mu_X(x), \mu_Y(y)) = \bigvee T(0, 0) = 0 = \mu_{XY}(z = xy).$$

Thus $\mu_X \circ \mu_Y = \mu_{XY}$.

(3) We have that

$$\mu_{X\cap Y}(z)=\begin{cases}1 & \text{if } z\in X\cap Y\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}1 & \text{if } z\in X,\ z\in Y\\ 0 & \text{otherwise.}\end{cases}$$

If $z\in X$ and $z\in Y$, then $z\in X\cap Y$ and then

$$(\mu_X \bigwedge \mu_Y)(z)=T(\mu_X(z),\mu_Y(z))=T(1,1)=1=\mu_{X\cap Y}(z).$$

If $z\in X$ and $z\notin Y$, then $z\notin X\cap Y$ and then

$$(\mu_X \bigwedge \mu_Y)(z)=T(\mu_X(z),\mu_Y(z))=T(1,0)=0=\mu_{X\cap Y}(z).$$

If $z\notin X$ and $z\in Y$, then $z\notin X\cap Y$ and then

$$(\mu_X \bigwedge \mu_Y)(z)=T(\mu_X(z),\mu_Y(z))=T(0,1)=0=\mu_{X\cap Y}(z).$$

If $z\notin X$ and $z\notin Y$, then $z\notin X\cap Y$ and then

$$(\mu_X \bigwedge \mu_Y)(z)=T(\mu_X(z),\mu_Y(z))=T(0,0)=0=\mu_{X\cap Y}(z).$$

Then $\mu_X \bigwedge \mu_Y=\mu_{X\cap Y}$.

(4) We get that

$$\mu_{X+Y}(z)=\begin{cases}1 & \text{if } z\in X+Y\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}1 & \text{if } z=x+y\in X+Y\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}1 & \text{if } x\in X,\ y\in Y\\ 0 & \text{otherwise.}\end{cases}$$

If $x\in X$ and $y\in Y$, then $x+y\in X+Y$ and then

$$(\mu_X \circ \mu_Y)(z)=\bigvee_{z=x+y} T(\mu_X(x),\mu_Y(y))=\bigvee T(1,1)=1=\mu_{X+Y}(z=x+y).$$

If $x \in X$ and $y \notin Y$, then $x + y \notin X + Y$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=x+y} T(\mu_X(x), \mu_Y(y)) = \bigvee T(1, 0) = 0 = \mu_{X+Y}(z = x + y).$$

If $x \notin X$ and $y \in Y$, then $x + y \notin X + Y$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=x+y} T(\mu_X(x), \mu_Y(y)) = \bigvee T(0, 1) = 0 = \mu_{X+Y}(z = x + y).$$

If $x \notin X$ and $y \notin Y$, then $x + y \notin X + Y$ and then

$$(\mu_X \circ \mu_Y)(z) = \bigvee_{z=x+y} T(\mu_X(x), \mu_Y(y)) = \bigvee T(0, 0) = 0 = \mu_{X+Y}(z = x + y)$$

Therefore $\mu_X + \mu_Y = \mu_{X+Y}$. □

4. Prime, Strongly Prime, Semiprime, Irreducible and Strongly Irreducible *T*-fuzzy Bi-ideals in Semirings

Definition 4.1. Let μ be *T*-fuzzy bi-ideal of semiring *R*.

(1) μ is called a *prime T-fuzzy bi-ideal* if for any *T*-fuzzy bi-ideals α, β of *R*, if $\alpha \circ \beta \leq \mu$, then $\alpha \leq \mu$ or $\beta \leq \mu$.

(2) μ is called a *strongly prime T-fuzzy bi-ideal* if for any *T*-fuzzy bi-ideals α, β of *R*, if $(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$, then $\alpha \leq \mu$ or $\beta \leq \mu$.

(3) μ is called *idempotent* if $\mu = \mu \circ \mu = \mu^2$.

(4) μ is said to be a *semiprime T-fuzzy bi-ideal* if $\alpha \circ \alpha = \alpha^2 \leq \mu$ implies $\alpha \leq \mu$ for every *T*-fuzzy bi-ideal α of *R*.

(5) μ is said to be an *irreducible T-fuzzy bi-ideal* if for any *T*-fuzzy bi-ideals α, β of *R*, if $\alpha \wedge \beta = \mu$, then $\alpha = \mu$ or $\beta = \mu$.

(6) μ is said to be a *strongly irreducible T-fuzzy bi-ideal* if for any *T*-fuzzy bi-ideals α, β of *R*, if $\alpha \wedge \beta \leq \mu$, then $\alpha \leq \mu$ or $\beta \leq \mu$.

Proposition 4.2. *We have the following assertions.*

(1) *Every strongly prime T-fuzzy bi-ideal of a semiring R is a prime T-fuzzy bi-ideal of R.*

(2) *Every prime T-fuzzy bi-ideal of a semiring R is a semiprime T-fuzzy bi-ideal of R.*

(3) *The intersection of any family of prime T-fuzzy bi-ideals of a semiring R is a semiprime T-fuzzy bi-ideal of R.*

(4) *Every strongly irreducible semiprime T-fuzzy bi-ideal of a semiring R is a strongly prime T-fuzzy bi-ideal of R.*

Proof. Let μ, α, β be T-fuzzy bi-ideals of semiring R.

(1) Let μ be strongly prime T-fuzzy bi-ideal and $\alpha \circ \beta \leq \mu$, then $(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$, and so $\alpha \leq \mu$ or $\beta \leq \mu$. This implies that μ is prime T-fuzzy bi-ideal.

(2) Let μ be prime T-fuzzy bi-ideal and $\alpha \circ \alpha \leq \mu$. Then $\alpha \leq \mu$ and then μ is semiprime T-fuzzy bi-ideal.

(3) Let μ_1 and μ_2 be two prime T-fuzzy bi-ideals of semiring R and $\alpha \circ \beta \leq \mu_1 \cap \mu_2$. Then $\alpha \circ \beta \leq \mu_1$ and $\alpha \circ \beta \leq \mu_2$. This implies that $\alpha \leq \mu_1$ or $\beta \leq \mu_1$ and $\alpha \leq \mu_2$ or $\beta \leq \mu_2$. Thus $\alpha \leq \mu_1 \cap \mu_2$ and $\beta \leq \mu_1 \cap \mu_2$. Therefore, $\mu_1 \cap \mu_2$ will be prime T-fuzzy bi-ideal of a semiring R.

(4) Let μ be strongly irreducible semiprime T-fuzzy bi-ideal such that $(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$. As $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$ so $(\alpha \wedge \beta) \circ (\alpha \wedge \beta) = (\alpha \wedge \beta)^2 \leq \alpha \circ \beta$. Also $\alpha \wedge \beta \leq \beta$ and $\alpha \wedge \beta \leq \alpha$ so $(\alpha \wedge \beta) \circ (\alpha \wedge \beta) = (\alpha \wedge \beta)^2 \leq \beta \circ \alpha$. Thus $(\alpha \wedge \beta)^2 \leq (\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$. Since μ is a semiprime T-fuzzy bi-ideal so $\alpha \wedge \beta \leq \mu$. Now since μ is a strongly irreducible T-fuzzy bi-ideal, $\alpha \leq \mu$ or $\beta \leq \mu$. Hence, μ is a strongly prime T-fuzzy. □

Proposition 4.3. *Let* μ *be T-fuzzy bi-ideal of a semiring R with* $\mu(a) = \varepsilon > 0$ *for all* $a \in R$ *and* $\varepsilon \in (0, 1]$. *Then there exists an irreducible T-fuzzy bi-ideal* β *of R such that* $\mu \leq \beta$ *and* $\beta(a) = \varepsilon$ *for all* $a \in R$ *and* $\varepsilon \in (0, 1]$.

Proof. Let $P = \{\alpha : \alpha$ is T-fuzzy bi-ideal of a semiring $R : \mu \leq \alpha : \alpha(a) = \varepsilon > 0\}$.

As $\mu \in P$ so $P \neq \varnothing$. Let $H = \{h_i : h_i$ is T-fuzzy bi-ideal of a semiring $R : h_i(a) = \varepsilon :$ $\mu \leq h_i : \forall i \in I\}$ be any totally ordered subset of P. Now we prove that $\bigvee_{i \in I} h_i$ is a T-fuzzy bi-ideal of R such that $\mu \leq \vee_{i \in I} h_i$. Assume that $a, b, c \in R$ and as h_i is T-fuzzy bi-ideal of a semiring R, then

(1) $(\bigvee_{i \in I})h_i(a+b) = \bigvee_{i \in I}(h_i(a+b))$

$$\geq \bigvee_{i \in I} T(h_i(a), h_i(b)) = T(\bigvee_{i \in I} h_i(a), \bigvee_{i \in I} h_i(b)).$$

(2) $(\bigvee_{i \in I})h_i(ab) = \bigvee_{i \in I}(h_i(ab))$

$$\geq \bigvee_{i \in I} T(h_i(a), h_i(b)) = T(\bigvee_{i \in I} h_i(a), \bigvee_{i \in I} h_i(b)).$$

(3) $(\bigvee_{i \in I})h_i(abc) = \bigvee_{i \in I}(h_i(abc))$

$$\geq \bigvee_{i \in I} T(h_i(a), h_i(c)) = T(\bigvee_{i \in I} h_i(a), \bigvee_{i \in I} h_i(c)).$$

Thus $\bigvee_{i \in I} h_i$ is a T-fuzzy bi-ideal of R. Since $\mu \leq h_i$ for all $i \in I$, $\mu \leq \bigvee_{i \in I} h_i$. Also $(\bigvee_{i \in I} h_i)(a) = \bigvee_{i \in I}(h_i)(a) = \varepsilon$ with $\varepsilon \in (0, 1]$. Therefore, $\bigvee_{i \in I}(h_i) \in P$ and $\bigvee_{i \in I}(h_i)$ is an upper bound of H. Now by Zorn's lemma, there exists a T-fuzzy bi-ideal β of R which is maximal with respect to the property $\mu \leq \beta$ and $\beta(a) = \varepsilon$. Now we show that β is an irreducible T-fuzzy bi-ideal of R. Let β_1, β_2 be T-fuzzy bi-ideals of R such that $\beta_1 \wedge \beta_2 = \beta$, then $\beta \leq \beta_1$ and $\beta \leq \beta_2$. We claim that $\beta = \beta_1$ or $\beta = \beta_2$. By the contrary, assume that $\beta \neq \beta_1$ and $\beta \neq \beta_2$. This implies $\beta < \beta_1$ and $\beta < \beta_2$. So $\beta_1(a) \neq \beta(a)$ and $\beta_2(a) \neq \beta(a)$. Hence $(\beta_1 \wedge \beta_2)(a) = \beta(a) \neq \varepsilon$ which is a contradiction to the fact that $(\beta_1 \wedge \beta_2)(a) = \beta(a) = \varepsilon$. Hence either $\beta = \beta_1$ or $\beta = \beta_2$.

□

In the following proposition and corollaries we investigate T-fuzzy bi-ideals of regular and intra-regular semiring R.

Proposition 4.4. *Let T be a idempotent t-norm. Then for a semiring R the following assertions hold*:

(1) *R is both regular and intra-regular* $\Rightarrow$ (2) $\mu \circ \mu = \mu$ *for every T-fuzzy bi-ideal* μ *of R* $\Rightarrow$ (3) $\alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha)$ *for all T-fuzzy bi-ideals* α *and* β *of R*.

Proof. (1) $\Rightarrow$ (2) Let R be both regular and intra-regular and $a \in R$. Then there exist elements $x, y_i, z_i \in R$ such that $a = axa$ and $a = \sum_{i=1}^{n} y_i a^2 z_i$ and then

$$a = axa = axaxa = ax\left(\sum_{i=1}^{n} y_i a^2 z_i\right)xa = \sum_{i=1}^{n} (axy_i a)(az_i xa).$$

Now

$$(\mu \circ \mu)(a) = (\mu \circ \mu)\left(\sum_{i=1}^{n} (axy_i a)(az_i xa)\right)$$

$$= \sum_{i=1}^{n} (\mu \circ \mu)(axy_i a)(az_i xa) = \sum_{i=1}^{n} \bigvee T(\mu(axy_i a), \mu(az_i xa))$$

$$\geq \sum_{i=1}^{n} T(T(\mu(a), \mu(a)), T(\mu(a), \mu(a))) \text{ (by definition of } T\text{-fuzzy bi-ideal)}$$

$$= \sum_{i=1}^{n} T(\mu(a), \mu(a)) = \sum_{i=1}^{n} \mu(a) = \mu(a). \text{ (Since } T \text{ is idempotent)}$$

Thus $\mu \circ \mu \geq \mu$. Also as μ is T-fuzzy bi-ideal μ of R so by Proposition 3.5(part 1) we get that $\mu \circ \mu \leq \mu$. Thus $\mu \circ \mu = \mu$.

(2) $\Rightarrow$ (3) Let α and β be two T-fuzzy bi-ideals of R. Then by Proposition 3.5(part 3) we have that $\alpha \wedge \beta$ is also T-fuzzy bi-ideal of R. Thus by hypothesis, we have $\alpha \wedge \beta = (\alpha \wedge \beta) \circ (\alpha \wedge \beta)$. As $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$ so $\alpha \wedge \beta = (\alpha \wedge \beta) \circ (\alpha \wedge \beta) \leq \alpha \circ \beta$. Similarly, $\alpha \wedge \beta \leq \beta \circ \alpha$. Then

$$\alpha \wedge \beta \leq (\alpha \circ \beta) \wedge (\beta \circ \alpha). \tag{a}$$

Now from Proposition 3.5(part 4) we have that $\alpha \circ \beta$ and $\beta \circ \alpha$ are T-fuzzy bi-ideals of R and by Proposition 3.5(part 3) we get that $(\alpha \circ \beta) \wedge (\beta \circ \alpha)$ will be T-fuzzy bi-ideal of R. Thus by hypothesis, we have

$$(\alpha \circ \beta) \wedge (\beta \circ \alpha) = [(\alpha \circ \beta) \wedge (\beta \circ \alpha)] \circ [(\alpha \circ \beta) \wedge (\beta \circ \alpha)]$$

$$\leq (\alpha \circ \beta) \circ (\beta \circ \alpha) = \alpha \circ \underbrace{\beta \circ \beta}_{\beta} \circ \alpha = \alpha \circ \underbrace{\beta}_{\leq \mu_R} \circ \alpha$$

$$\leq \alpha \circ \mu_R \circ \alpha \leq \alpha. \text{ (by Proposition 3.5 (part 1))}$$

Similarly, $(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \beta$ and then

$$(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \alpha \wedge \beta. \tag{b}$$

Then from (a) and (b) we get that $(\alpha \circ \beta) \wedge (\beta \circ \alpha) = \alpha \wedge \beta.$ □

Corollary 4.5. *Let R be a regular and intra-regular semiring. Then* μ *is a strongly prime T-fuzzy bi-ideal of R if and only if* μ *is a strongly irreducible T-fuzzy bi-ideal of R.*

Proof. Let R be a regular and intra-regular semiring and α and β be two T-fuzzy bi-ideals of R. Then from Proposition 4.4 we get that $\alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha)$. Now if μ is a strongly prime T-fuzzy bi-ideal of R, then $\alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$ and then $\alpha \leq \mu$ or $\beta \leq \mu$ and so μ will be a strongly irreducible T-fuzzy bi-ideal of R. Also if μ is a strongly irreducible T-fuzzy bi-ideal of R, then $(\alpha \circ \beta) \wedge (\beta \circ \alpha) = \alpha \wedge \beta \leq \mu$ which means that $\alpha \leq \mu$ or $\beta \leq \mu$ and then μ is a strongly prime T-fuzzy bi-ideal of R. □

Corollary 4.6. *Let R be a regular and intra-regular semiring. If the set of T-fuzzy bi-ideals of R is totally ordered by inclusion. Then each T-fuzzy bi-ideal of a semiring R is a strongly prime T-fuzzy bi-ideal of R.*

Proof. Let α, β, μ be three T-fuzzy bi-ideals of R. We prove that μ will be a strongly prime T-fuzzy bi-ideal of R. Since the set of T-fuzzy bi-ideals of R is totally ordered by inclusion, $\alpha \leq \beta$ or $\beta \leq \alpha$. This gets that $\alpha \wedge \beta = \alpha$ or $\alpha \wedge \beta = \beta$. As R is a regular and intra-regular semiring, so from Proposition 4.4 we get that $\alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha)$. Now let $(\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$. Then $\alpha = \alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$ or $\beta = \alpha \wedge \beta = (\alpha \circ \beta) \wedge (\beta \circ \alpha) \leq \mu$. This implies that μ will be a strongly prime T-fuzzy bi-ideal of R. □

Corollary 4.7. *Let R be a regular and intra-regular semiring. If the set of T-fuzzy bi-ideals of R is totally ordered by inclusion, then each T-fuzzy bi-ideal of a semiring R is a prime T-fuzzy bi-ideal of R.*

Proof. Let α, β, μ be three T-fuzzy bi-ideals of R. We prove that μ will be a prime T-

fuzzy bi-ideal of R. As R is a regular and intra-regular semiring so from Proposition 4.4 we get that $\alpha \circ \alpha = \alpha$ and $\beta \circ \beta = \beta$. Now let $\alpha \circ \beta \leq \mu$ and as the set of T-fuzzy bi-ideals of R is totally ordered by inclusion so $\alpha \leq \beta$ or $\beta \leq \alpha$. If $\alpha \leq \beta$, then $\alpha = \alpha \circ \alpha \leq \alpha \circ \beta \leq \mu$ and if $\beta \leq \alpha$, then $\beta = \beta \circ \beta \leq \alpha \circ \beta \leq \mu$. Thus $\alpha \leq \mu$ or $\beta \leq \mu$ and then μ will be a prime T-fuzzy bi-ideal of R. □

In the following proposition we characterize strongly irreducible T-fuzzy bi-ideals and irreducible T-fuzzy bi-ideals of R.

Proposition 4.8. *Let R be a semiring. Then the following assertions are equivalent*:

(1) *Set of T-fuzzy bi-ideals of R is totally ordered by inclusion.*

(2) *Each T-fuzzy bi-ideal of R is strongly irreducible T-fuzzy bi-ideal of R.*

(3) *Each T-fuzzy bi-ideal of R is an irreducible T-fuzzy bi-ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let α, β, μ be three T-fuzzy bi-ideals of R. We prove that μ will be a strongly irreducible T-fuzzy bi-ideal of R. Since the set of T-fuzzy bi-ideals of R is totally ordered by inclusion, $\alpha \leq \beta$ or $\beta \leq \alpha$. Thus $\alpha \wedge \beta = \alpha$ or $\alpha \wedge \beta = \beta$. Now let $\alpha \wedge \beta \leq \mu$. Then $\alpha = \alpha \wedge \beta \leq \mu$ or $\beta = \alpha \wedge \beta \leq \mu$. Thus $\alpha \leq \mu$ and $\beta \leq \mu$ and then μ will be a strongly irreducible T-fuzzy bi-ideal of R.

(2) $\Rightarrow$ (3) Let α, β, μ be three T-fuzzy bi-ideals of R. Let μ be a strongly irreducible T-fuzzy bi-ideal of R. We prove that μ will be an irreducible T-fuzzy bi-ideal of R. Let $\alpha \wedge \beta = \mu$. Since $\alpha \wedge \beta \leq \alpha$ and $\alpha \wedge \beta \leq \beta$, $\mu \leq \alpha$ and $\mu \leq \beta$. Also since μ is a strongly irreducible T-fuzzy bi-ideal of R, $\alpha \wedge \beta = \mu \leq \mu$. Thus $\alpha \leq \mu$ and $\beta \leq \mu$. Therefore, we obtain that $\alpha = \mu$ or $\beta = \mu$ and this implies that μ is irreducible T-fuzzy bi-ideal of R.

(3) $\Rightarrow$ (1) Let α, β be two T-fuzzy bi-ideals of R. We must prove that $\alpha \leq \beta$ or $\beta \leq \alpha$. By Proposition 3.5(part 3) we have that $\alpha \wedge \beta$ is also T-fuzzy bi-ideal of R and then will be an irreducible T-fuzzy bi-ideal of R. As $\alpha \wedge \beta = \beta \wedge \alpha$, then $\alpha = \alpha \wedge \beta$ or $\beta = \alpha \wedge \beta$ and so $\alpha \leq \beta$ or $\beta \leq \alpha$. This means that the set of T-fuzzy bi-ideals of R is totally ordered by inclusion. □

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References

[1] M. T. Abu Osman, On some products of fuzzy subgroups, *Fuzzy Sets and Systems* 24 (1987), 79-86. https://doi.org/10.1016/0165-0114(87)90115-1

[2] J. Ahsan, Semirings characterized by their fuzzy ideals, *The Journal of Fuzzy Mathematics* 6 (1998), 181-192.

[3] C. Alsina, E. Trillas and L. Valverde, On some logical connectives for fuzzy sets theory, *J. Math. Anal. Appl.* 93 (1983), 15-26. https://doi.org/10.1016/0022-247X(83)90216-0

[4] S. Bashir, J. Mehmood and M. Shabir, Prime bi-ideals and prime fuzzy bi-ideals in semirings, *World Applied Sciences Journal* 22 (2013), 106-121.

[5] G. M. Bergman, *An Invitation to General Algebra and Universal Constructions*, Universitext, 2nd ed., Springer-Verlag, 2015. https://doi.org/10.1007/978-3-319-11478-1

[6] J. Berstel and D. Perrin, *Theory of Codes*, Academic Press, Inc., 1985.

[7] J. J. Buckley and E. Eslami, *An Introduction to Fuzzy Logic and Fuzzy Sets*, Springer-Verlag, Berlin, Heidelberg, GmbH, 2002. https://doi.org/10.1007/978-3-7908-1799-7

[8] G. Gratzer, *Lattice Theory: First Concepts and Distributive Lattices*, W. H. Freeman, 1971.

[9] U. Höhle, Probabilistie uniformization of fuzzy topologies, *Fuzzy Sets and Systems* 1 (1978), 311-332. https://doi.org/10.1016/0165-0114(78)90021-0

[10] N. Kehayopulu, Ordered semi-groups whose elements are separated by prime ideals, *Math. Slovaca* 62 (2012), 417-424. https://doi.org/10.2478/s12175-012-0018-9

[11] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum* 65 (2002), 128-132. https://doi.org/10.1007/s002330010079

[12] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. U.S.A.* 28 (1942), 535-537. https://doi.org/10.1073/pnas.28.12.535

[13] M. Munir, Bi-ideals in semirings, Department of Mathematics, Quaid-e-Azam University, Islamabad, 2010.

[14] R. Rasuli, Fuzzy ideals of subtraction semigroups with respect to a t-norm and a t-conorm, *The Journal of Fuzzy Mathematics Los Angeles* 24(4) (2016), 881-892.

[15] R. Rasuli, Fuzzy modules over a t-norm, *Int. J. Open Problems Compt. Math.* 9(3) (2016), 12-18. https://doi.org/10.12816/0033740

[16] R. Rasuli, Fuzzy subrings over a t-norm , *The Journal of Fuzzy Mathematics Los Angeles* 24(4) (2016), 995-1000.

[17] R. Rasuli, Norms over intuitionistic fuzzy subrings and ideals of a ring, *Notes on Intuitionistic Fuzzy Sets* 22(5) (2016), 72-83.

[18] R. Rasuli, Norms over fuzzy Lie algebra, *Journal of New Theory* 15 (2017), 32-38.

[19] R. Rasuli, Fuzzy subgroups on direct product of groups over a t-norm, *Journal of Fuzzy Set Valued Analysis* 3 (2017), 96-101. https://doi.org/10.5899/2017/jfsva-00339

[20] R. Rasuli, Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms, *Journal of New Theory* 18 (2017), 39-52.

[21] R. Rasuli, *Intuitionistic Fuzzy Subrings and Ideals of a Ring Under Norms*, LAP LAMBERT Academic Publishing, 2017.

[22] R. Rasuli, Characterization of Q-fuzzy subrings (anti Q-fuzzy subrings) with respect to a t-norm (t-conorms), *J. Inf. Optim. Sci.* 39 (2018), 827-837. https://doi.org/10.1080/02522667.2016.1228316

[23] R. Rasuli, T-fuzzy submodules of $R \times M$, *Journal of New Theory* 22 (2018), 92-102.

[24] R. Rasuli, Fuzzy subgroups over a t-norm, *J. Inf. Optim. Sci.* 39 (2018), 1757-1765. https://doi.org/10.1080/02522667.2018.1427028

[25] R. Rasuli, Fuzzy sub-vector spaces and sub-bivector spaces under t-norms, *General Letters in Mathematics* 5 (2018), 47-57. https://doi.org/10.31559/glm2018.5.1.6

[26] R. Rasuli, Anti fuzzy submodules over a t-conorm and some of their properties, *The Journal of Fuzzy Mathematics Los Angles* 27(2019), 229-236.

[27] R. Rasuli, Artinian and Noetherian fuzzy rings, *Int. J. Open Problems Compt. Math*. 12 (2019), 1-7.

[28] R. Rasuli and H. Narghi, T-norms over Q-fuzzy subgroups of group, *Jordan Journal of Mathematics and Statistics (JJMS)* 12 (2019), 1-13.

[29] R. Rasuli, Fuzzy equivalence relation, fuzzy congrunce relation and fuzzy normal subgroups on group G over t-norms, *Asian Journal of Fuzzy and Applied Mathematics* 7 (2019), 14-28.

[30] R. Rasuli, Norms over anti fuzzy *G*-submodules, *MathLAB Journal* 2 (2019), 56-64.

[31] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.

[32] M. Shabir and N. Kanwal, Prime bi-ideals of semigroups, *Southeast Asian Bulletin of Mathematics* 31 (2007), 757-764.

[33] D. A. Simovici and C. Djeraba, Partially ordered sets, in: *Mathematical Tools for Data Mining: Set Theory, Partial Orders, Combinatorics*, Springer, 2008.

[34] L. A. Skornyakov, *Regular Ring* (*In the Sense of von Neumann*), Springer Science + Business Media B. V. / Kluwer Academic Publishers, 2001.

[35] J. von Neumann, On regular rings, *Proc. Natl. Acad. Sci. USA* 22 (1936), 707-712. https://doi.org/10.1073/pnas.22.12.707

[36] L. A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X