Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 6, 2025, Pages 1083-1089
https://doi.org/10.34198/ejms.15625.10831089

On a New Integral Inequality Derived from the Hardy-Hilbert Integral Inequality

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France

e-mail: christophe.chesneau@gmail.com

Abstract

This article is devoted to a new integral inequality expressed in terms of elementary integrals. The proof is notable for its use of the classical Hardy-Hilbert integral inequality, which provides an elegant and concise argument. As an illustration of its utility, we also present an application to the gamma function.

1 Introduction

Integral inequalities play a central role in analysis. They arise in areas such as harmonic analysis, functional spaces, operator theory, and special functions. See, for example, [1–5]. The Hardy-Hilbert integral inequality is one of the classics. First appearing in the early 20th century in the works of Hardy, Hilbert and their contemporaries, it has become a fundamental tool. In particular, it provides sharp bounds for integral operators and leads to elegant functional inequalities. Formally, let $p > 1$, $q = p/(p-1)$, and $f, g : [0, +\infty) \mapsto [0, +\infty)$ be two functions. Then the Hardy-Hilbert integral inequality ensures that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)dy\right]^{1/q}.$$

The constant factor $\pi/\sin(\pi/p)$ has the advantage of being optimal, which means that it cannot be improved. The inequality is sharp: for each $p > 1$, there exist functions f and g for which the equality is attained in the limit. This optimality has made the Hardy-Hilbert integral inequality a key reference point in the study of double integrals with singular kernels. The technical details can be found in [1].

The aim of this article is to present a new integral inequality involving only elementary integrals. The proof is relatively concise. It relies on suitable decompositions and the direct application of the

Received: August 3, 2025; Accepted: September 19, 2025; Published: September 22, 2025
2020 Mathematics Subject Classification: 26D15, 33E20.
Keywords and phrases: integral inequalities, double integrals, Hardy-Hilbert integral inequality, gamma function.

Hardy-Hilbert integral inequality, thus avoiding the need for lengthy arguments. The result is of intrinsic interest. It is also flexible enough to be applied to special functions. To illustrate this claim, we demonstrate how it yields a useful bound for the gamma function. This example demonstrates the relevance of the inequality for integral transforms and the theory of classical special functions.

The remainder of the paper is organized as follows: Section 2 presents the new integral inequality. Section 3 illustrates its use through an application to the gamma function. Some complements are given in Section 4. Finally, Section 5 offers concluding remarks and possible directions for further work.

2 New Integral Inequality

The theorem below presents our main integral inequality.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, $\alpha \in \mathbb{R}$, and $f : [0, +\infty) \mapsto [0, +\infty)$ be a function. Then*

$$\int_0^{+\infty} x^\alpha f(x)dx \leq \sqrt{\frac{2\pi}{\sin(\pi/p)}} \left[\int_0^{+\infty} x^{\alpha p} f^p(x)dx\right]^{1/(2p)} \left[\int_0^{+\infty} x^{(\alpha+1)q} f^q(x)dx\right]^{1/(2q)},$$

provided that the two integrals in the upper bound converge.

Proof of Theorem 2.1. For any $\alpha \in \mathbb{R}$, let us define

$$\mathcal{I}_\alpha = \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^{\alpha+1}}{x+y} f(x)f(y)dxdy.$$

By symmetry (exchanging the roles of x and y), we also have

$$\mathcal{I}_\alpha = \int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha+1} y^{\alpha}}{x+y} f(x)f(y)dxdy.$$

Combining these two expressions and using the linearity of the integral yields:

$$\begin{aligned}
\mathcal{I}_\alpha &= \frac{1}{2}(\mathcal{I}_\alpha + \mathcal{I}_\alpha) \\
&= \frac{1}{2}\left[\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha+1} y^{\alpha}}{x+y} f(x)f(y)dxdy + \int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha} y^{\alpha+1}}{x+y} f(x)f(y)dxdy\right] \\
&= \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \frac{(x+y)x^{\alpha} y^{\alpha}}{x+y} f(x)f(y)dxdy \\
&= \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} x^{\alpha} y^{\alpha} f(x)f(y)dxdy \\
&= \frac{1}{2}\left[\int_0^{+\infty} x^\alpha f(x)dx\right]\left[\int_0^{+\infty} y^\alpha f(y)dy\right] \\
&= \frac{1}{2}\left[\int_0^{+\infty} x^\alpha f(x)dx\right]^2. \qquad (1)
\end{aligned}$$

On the other hand, we can write

$$\mathcal{I}_\alpha = \int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^{\alpha+1}}{x+y} f(x)f(y)dxdy = \int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\dagger(x)g_\dagger(y)dxdy,$$

where

$$f_\dagger(x) = x^\alpha f(x), \quad g_\dagger(y) = y^{\alpha+1}f(y).$$

The Hardy-Hilbert integral inequality applied to the functions $f_\dagger$ and $g_\dagger$ yields

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f_\dagger(x)g_\dagger(y)dxdy \leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} f_\dagger^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} g_\dagger^q(y)dy\right]^{1/q},$$

which can be rewritten as

$$\mathcal{I}_\alpha \leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} x^{\alpha p} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(\alpha+1)q} f^q(y)dy\right]^{1/q}. \tag{2}$$

It follows from Equations (1) and (2) that

$$\frac{1}{2}\left[\int_0^{+\infty} x^\alpha f(x)dx\right]^2 \leq \frac{\pi}{\sin(\pi/p)}\left[\int_0^{+\infty} x^{\alpha p} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(\alpha+1)q} f^q(y)dy\right]^{1/q}.$$

Multiplying by 2 and taking the square root on both sides yields the final inequality, i.e.,

$$\int_0^{+\infty} x^\alpha f(x)dx \leq \sqrt{\frac{2\pi}{\sin(\pi/p)}}\left[\int_0^{+\infty} x^{\alpha p} f^p(x)dx\right]^{1/(2p)}\left[\int_0^{+\infty} x^{(\alpha+1)q} f^q(x)dx\right]^{1/(2q)}.$$

This completes the proof of Theorem 2.1. □

To the best of our knowledge, this integral inequality, which only involves elementary integrals, is a new addition to the literature. Thanks to the adjustable parameters α and p, it can be adapted to diverse mathematical scenarios.

Note that, taking $\alpha = 0$, we obtain the following simplified formulation:

$$\int_0^{+\infty} f(x)dx \leq \sqrt{\frac{2\pi}{\sin(\pi/p)}}\left[\int_0^{+\infty} f^p(x)dx\right]^{1/(2p)}\left[\int_0^{+\infty} x^q f^q(x)dx\right]^{1/(2q)}.$$

In the special case $p = 2$, it becomes

$$\int_0^{+\infty} f(x)dx \leq \sqrt{2\pi}\left[\int_0^{+\infty} f^2(x)dx\right]^{1/4}\left[\int_0^{+\infty} x^2 f^2(x)dx\right]^{1/4},$$

which corresponds to the Carlson integral inequality up to the constant factor $\sqrt{2}$. We recall that the constant factor of the Carlson integral inequality, known as the optimal constant, is $\sqrt{\pi}$. See [6]. In this specific case, therefore, our inequality is not optimal. However, it is of interest in the case of $p \in (1, +\infty)\backslash\{2\}$, for which no Carlson-type integral inequalities exist. This also suggests that our approach could be improved in some ways.

3 Application to the Gamma Function

The proposition below provides an application of Theorem 2.1 to the gamma function. For more details and properties of the gamma function, see [7].

Proposition 3.1. *For any $\beta > 0$, we consider the (standard) gamma function defined by*

$$\Gamma(\beta) = \int_0^{+\infty} x^{\beta-1} e^{-x} dx.$$

Let $p > 1$, $q = p/(p-1)$ and $\alpha > -1$. Then

$$\frac{\Gamma^2(\alpha+1)}{\Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}((\alpha+1)q+1)} \leq \frac{2\pi}{\sin(\pi/p)p^{\alpha+1/p}q^{\alpha+1+1/q}}.$$

Proof of Proposition 3.1. Applying Theorem 2.1 to the function $f(x) = e^{-x}$, $x \geq 0$, we obtain

$$\int_0^{+\infty} x^{\alpha} e^{-x} dx \leq \sqrt{\frac{2\pi}{\sin(\pi/p)}} \left[\int_0^{+\infty} x^{\alpha p} e^{-px} dx\right]^{1/(2p)} \left[\int_0^{+\infty} x^{(\alpha+1)q} e^{-qx} dx\right]^{1/(2q)}. \quad (3)$$

Let us now express each integral of this inequality in terms of the gamma function. First, we have

$$\int_0^{+\infty} x^{\alpha} e^{-x} dx = \Gamma(\alpha+1).$$

Making the change of variables $y = px$, we obtain

$$\int_0^{+\infty} x^{\alpha p} e^{-px} dx = \frac{1}{p^{\alpha p+1}} \int_0^{+\infty} y^{\alpha p} e^{-y} dy = \frac{1}{p^{\alpha p+1}} \Gamma(\alpha p+1).$$

In a similar way, making the change of variables $z = qx$, we have

$$\int_0^{+\infty} x^{(\alpha+1)q} e^{-qx} dx = \frac{1}{q^{(\alpha+1)q+1}} \int_0^{+\infty} z^{(\alpha+1)q} e^{-z} dz = \frac{1}{q^{(\alpha+1)q+1}} \Gamma((\alpha+1)q+1).$$

Combining these gamma function expressions into Equation (3), we get

$$\Gamma(\alpha+1) \leq \sqrt{\frac{2\pi}{\sin(\pi/p)}} \left[\frac{1}{p^{\alpha p+1}} \Gamma(\alpha p+1)\right]^{1/(2p)} \left[\frac{1}{q^{(\alpha+1)q+1}} \Gamma((\alpha+1)q+1)\right]^{1/(2q)}.$$

Taking the square of both sides and grouping the gamma function terms on the left yields the final inequality, i.e.,

$$\frac{\Gamma^2(\alpha+1)}{\Gamma^{1/p}(\alpha p+1)\Gamma^{1/q}((\alpha+1)q+1)} \leq \frac{2\pi}{\sin(\pi/p)p^{\alpha+1/p}q^{\alpha+1+1/q}}.$$

This completes the proof of Proposition 3.1. □

To the best of our knowledge, this is a new inequality involving the gamma function. It is also flexible due to the presence of the adjustable parameters α and p.

4 Complements

The theory developed in Theorem 2.1 for the case $p = 2$ can be adapted for integration over a finite interval. In this case, a key tool is [8, Theorem 1]. The theorem below provides the technical details.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$, $\gamma > 0$, and $f : [0, \gamma] \mapsto [0, +\infty)$ be a function. Then*

$$\int_0^{\gamma} x^{\alpha} f(x)dx$$
$$\leq \sqrt{2\pi} \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{x}{\gamma}} \right] x^{2\alpha} f^2(x)dx \right\}^{1/4} \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{x}{\gamma}} \right] x^{2(\alpha+1)} f^2(x)dx \right\}^{1/4},$$

provided that the two integrals in the upper bound converge.

Proof of Theorem 4.1. For any $\alpha \in \mathbb{R}$, let us define

$$\mathcal{J}_{\alpha} = \int_0^{\gamma} \int_0^{\gamma} \frac{x^{\alpha} y^{\alpha+1}}{x+y} f(x)f(y)dxdy.$$

Proceeding exactly as in the first steps of the proof of Theorem 2.1, replacing $+\infty$ by γ, we obtain

$$\mathcal{J}_{\alpha} = \frac{1}{2} \left[\int_0^{\gamma} x^{\alpha} f(x)dx \right]^2. \tag{4}$$

On the other hand, we can write

$$\mathcal{J}_{\alpha} = \int_0^{\gamma} \int_0^{\gamma} \frac{x^{\alpha} y^{\alpha+1}}{x+y} f(x)f(y)dxdy = \int_0^{\gamma} \int_0^{\gamma} \frac{1}{x+y} f_{\dagger}(x) g_{\dagger}(y)dxdy,$$

where

$$f_{\dagger}(x) = x^{\alpha} f(x), \quad g_{\dagger}(y) = y^{\alpha+1} f(y).$$

The result in [8, Theorem 1, Equation (4) with $b = \gamma$ and $t = 1$] applied to the functions $f_{\dagger}$ and $g_{\dagger}$ yields

$$\int_0^{\gamma} \int_0^{\gamma} \frac{1}{x+y} f_{\dagger}(x) g_{\dagger}(y)dxdy$$
$$\leq \pi \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{x}{\gamma}} \right] f_{\dagger}^2(x)dx \right\}^{1/2} \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{y}{\gamma}} \right] g_{\dagger}^2(y)dy \right\}^{1/2},$$

which can be rewritten as

$$\mathcal{J}_{\alpha} \leq \pi \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{x}{\gamma}} \right] x^{2\alpha} f^2(x)dx \right\}^{1/2} \left\{ \int_0^{\gamma} \left[1 - \frac{1}{2}\sqrt{\frac{y}{\gamma}} \right] y^{2(\alpha+1)} f^2(y)dy \right]^{1/2}. \tag{5}$$

It follows from Equations (4) and (5) that

$$\frac{1}{2}\left[\int_0^\gamma x^\alpha f(x)dx\right]^2$$
$$\leq \pi\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{x}{\gamma}}\right]x^{2\alpha}f^2(x)dx\right\}^{1/2}\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{y}{\gamma}}\right]y^{2(\alpha+1)}f^2(y)dy\right]^{1/2}.$$

We then derive the desired inequality, i.e.,

$$\int_0^\gamma x^\alpha f(x)dx$$
$$\leq \sqrt{2\pi}\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{x}{\gamma}}\right]x^{2\alpha}f^2(x)dx\right\}^{1/4}\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{x}{\gamma}}\right]x^{2(\alpha+1)}f^2(x)dx\right\}^{1/4}.$$

This ends the proof of Theorem 4.1. □

Applying $\gamma \to +\infty$, Theorem 4.1 reduces to Theorem 2.1 with $p = 2$.

In particular, setting $\alpha = 0$ yields

$$\int_0^\gamma f(x)dx \leq \sqrt{2\pi}\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{x}{\gamma}}\right]f^2(x)dx\right\}^{1/4}\left\{\int_0^\gamma\left[1-\frac{1}{2}\sqrt{\frac{x}{\gamma}}\right]x^2f^2(x)dx\right\}^{1/4}.$$

To the best of our knowledge, this is a new addition to the literature extending the scope of the Carlson integral inequality.

5 Conclusion

In this article, we establish a new integral inequality using only elementary integrals. The proof relies on suitable decompositions and the classical Hardy-Hilbert integral inequality. Applying the new inequality to the gamma function demonstrates its potential for deriving bounds for other special functions. Future work could involve extending this to weighted versions, multidimensional settings or connections with fractional integral operators and related inequalities.

References

[1] Hardy, G. H., Littlewood, J. E., & Pólya, G. (1934). *Inequalities*. Cambridge: Cambridge University Press.

[2] Beckenbach, E. F., & Bellman, R. (1961). *Inequalities*. Berlin: Springer. https://doi.org/10.1007/978-3-642-64971-4

[3] Walter, W. (1970). *Differential and integral inequalities*. Berlin: Springer. https://doi.org/10.1007/978-3-642-86405-6

[4] Bainov, D., & Simeonov, P. (1992). *Integral inequalities and applications*. Mathematics and Its Applications (Vol. 57). Dordrecht: Kluwer Academic.

[5] Cvetkovski, Z. (2012). *Inequalities: Theorems, techniques and selected problems*. Berlin: Springer. https://doi.org/10.1007/978-3-642-23792-8

[6] Carlson, F. (1934). Une inégalité. *Arkiv för Matematik, Astronomi och Fysik*, *25B*, 1–5.

[7] Gradshteyn, I. S., & Ryzhik, I. M. (2007). *Table of integrals, series, and products* (7th ed.). San Diego, CA: Academic Press.

[8] Yang, B. C. (1998). On Hilbert's integral inequality. *Journal of Mathematical Analysis and Applications*, *220*, 778–785. https://doi.org/10.1006/jmaa.1997.5877