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On New General Beesack-Opial-type Integral Inequalities

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Abstract

This article presents new Beesack-Opial-type integral inequalities incorporating three functions. The upper bounds are defined using some derivatives and primitives of these functions. Detailed proofs are provided. These are accompanied by illustrative examples, including applications involving the Laplace transform.

1 Introduction

There are many different types of integral inequality, each with its own structure, assumptions, and range of applications. Beesack-Opial-type integral inequalities are of particular interest due to their foundational role in the study of differential equations, functional analysis, and mathematical physics. This article aims to further advance this topic by building on existing findings and exploring additional generalizations. Before proceeding, let us recall two classical results: the Beesack and Opial integral inequalities. The Beesack integral inequality is stated below. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \mapsto \mathbb{R}$ be a function such that f is differentiable with $f(a) = 0$ (or $f(b) = 0$). Then we have

$$\int_a^b |f(t)f'(t)|dt \leq \frac{b-a}{2} \int_a^b [f'(t)]^2 dt. \tag{1}$$

The Opial integral inequality can be viewed as a refinement of the Beesack integral inequality, under more restrictive boundary conditions on f but with a sharper constant factor. More details can be found below. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \mapsto \mathbb{R}$ be a function such that f is differentiable with $f(a) = f(b) = 0$. Then we have

$$\int_a^b |f(t)f'(t)|dt \leq \frac{b-a}{4} \int_a^b [f'(t)]^2 dt.$$

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In chronological order, the Opial integral inequality was proved before the Beesack integral inequality (see [10] and [3]). These foundational results have inspired extensive research into generalizations and refinements in various mathematical contexts. See [1–20]. In particular, reference [16] presents several important results. Of special note is the first theorem, which generalizes the Beesack integral inequality by introducing weight functions and parameters, as stated below. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a, b] \mapsto \mathbb{R}$ be a function such that f is differentiable with $f(a) = 0$. Let $p > 1$, $q = p/(p-1)$ such that $1/p + 1/q = 1$, and $m \in \mathbb{N}\backslash\{0\}$. Then we have

$$\int_a^b t^m |f(t)f'(t)|dt$$
$$\leq \frac{1}{p}\int_a^b t^m(t-a)|f'(t)|^p dt + \frac{1}{q(m+1)}\int_a^b (b^{m+1} - t^{m+1})|f'(t)|^q dt. \quad (2)$$

By choosing $m = 0$ and $p = 2$, this inequality reduces to the classical Beesack integral inequality.

This article aims to derive new Beesack-Opial-type integral inequalities based on the approach developed in [16]. This approach primarily involves suitably decomposing the integrand, and applying the Hölder integral inequality and standard integral manipulations, including changing the order of integration. Following analysis, it was found to have certain potential for generalization, an area that appears to be under-explored in the existing literature. To address this gap, we present a general theorem using three adaptable functions. The derivatives and primitives (antiderivatives) of some of these functions are considered in the upper bound. As a result, the framework in [16] is significantly generalized. Two propositions are then derived. The first proposition introduces a tunable parameter offering flexibility depending on the mathematical context or application. The second proposition optimizes this approach. It can be stated as follows:

$$\int_a^b g(t)|f(t)f'(t)|dt$$
$$\leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[g(t)h(t)|f'(t)|^p + G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}\right] dt,$$

where f, g, and h are suitably regular functions, and G denotes the primitive of $-g$. The functional structure of this inequality highlights the originality and generality of our method. All proofs are provided in full detail to ensure clarity and reproducibility. Several illustrative examples are also included to demonstrate the advantages of using three functions within this framework. Some of these examples involve the Laplace transform, which further highlights the practical relevance of our results.

The remainder of this article is organized as follows: In Section 2, we present our main theorem, along with illustrative examples. Section 3 is dedicated to the two propositions derived from this theorem. Finally, Section 4 concludes the article with a summary and possible directions for future work.

2 Main Theorem with Examples

2.1 Main theorem

Our main Beesack-Opial-type integral inequality is described in the theorem below. The proof relies on decomposing the integrand in a suitable way, introducing an auxiliary function h, using the Hölder integral inequality and applying various integral calculus techniques, such as primitive developments and changing the order of integration.

Theorem 1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and $f, g, h : [a, b] \mapsto \mathbb{R}$ be three functions such that f is differentiable with $f(a) = 0$, g is integrable, and h is differentiable non-decreasing with $h(a) = 0$. Let us set, for any $x \in [a, b]$,*

$$G(x) = \int_x^b g(t)dt.$$

Let $p > 1$ and $q = p/(p-1)$. Then

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \left[\int_a^b g(t)h(t)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt\right]^{1/q}.$$

Proof. Using $1/p + 1/q = 1$ and the Hölder integral inequality with the parameter p, we get

$$\begin{aligned}
&\int_a^b g(t)|f(t)f'(t)|dt = \int_a^b g^{1/p+1/q}(t)h^{1/p+1/q-1}(t)|f(t)f'(t)|dt \\
&= \int_a^b \left[g^{1/p}(t)h^{1/p}(t)|f'(t)|\right] \left[g^{1/q}(t)h^{1/q-1}(t)|f(t)|\right] dt \\
&\leq A^{1/p}B^{1/q}, \qquad (3)
\end{aligned}$$

where

$$A = \int_a^b g(t)h(t)|f'(t)|^p dt$$

and

$$B = \int_a^b g(t)h^{1-q}(t)|f(t)|^q dt.$$

The term A is precisely the first integral of the desired upper bound. Let us therefore work on the term B, beginning with an intermediate inequality. Using the fact that f is differentiable with $f(a) = 0$, a suitable decomposition of the integrand with $h'(t) \geq 0$ for any $t \in [a, b]$ since h is non-decreasing, and $h(a) = 0$, we get

$$\begin{aligned}|f(t)|^q &= \left|\int_a^t f'(x)dx + f(a)\right|^q = \left|\int_a^t f'(x)dx\right|^q \\ &= \left|\int_a^t \frac{f'(x)}{[h'(x)]^{1/p}}[h'(x)]^{1/p}dx\right|^q \\ &\leq \left[\int_a^t h'(x)dx\right]^{q/p} \int_0^t \frac{|f'(x)|^q}{[h'(x)]^{q/p}}dx \\ &= [h(t)-h(a)]^{q-1} \int_0^t \frac{|f'(x)|^q}{[h'(x)]^{q/p}}dx \\ &= h^{q-1}(t) \int_a^t \frac{|f'(x)|^q}{[h'(x)]^{p-1}}dx.\end{aligned}$$

Using this into the expression of B and changing the order of integration, we find that

$$\begin{aligned}B &\leq \int_a^b g(t)h^{1-q}(t)\left[h^{q-1}(t)\int_0^t \frac{|f'(x)|^q}{[h'(x)]^{p-1}}dx\right]dt \\ &= \int_a^b \int_a^t g(t)\frac{|f'(x)|^q}{[h'(x)]^{p-1}}dxdt = \int_a^b \int_x^b g(t)\frac{|f'(x)|^q}{[h'(x)]^{p-1}}dtdx \\ &= \int_a^b \frac{|f'(x)|^q}{[h'(x)]^{p-1}}\left[\int_x^b g(t)dt\right]dx = \int_a^b \frac{|f'(x)|^q}{[h'(x)]^{p-1}}G(x)dx \\ &= \int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt. \qquad (4)\end{aligned}$$

Joining Equations (3) and (4), we obtain

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \left[\int_a^b g(t)h(t)|f'(t)|^p dt\right]^{1/p}\left[\int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt\right]^{1/q}.$$

This ends the proof of Theorem 1. □

In particular, for $p = 2$, Theorem 1 reads as

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \sqrt{\left[\int_a^b g(t)h(t)\left[f'(t)\right]^2 dt\right]\left[\int_a^b G(t)\frac{[f'(t)]^2}{h'(t)}dt\right]}.$$

The proposed theorem has one key advantage: its flexibility. This is achieved by using three functions that can be adapted to suit a wide range of mathematical problems. It is also possible to choose $b = +\infty$, dealing with improper integrals, which was not possible in the Beesack and Opial integral inequalities, and in the framework in [16].

To substantiate this, we present several illustrative examples in the subsection below. Each example is based on a different configuration of functions, demonstrating how the general structure of the theorem can accommodate various analytical scenarios.

2.2 Examples

Some direct examples of applications of Theorem 1 are described below.

- If we take $h(t) = t - a$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we get

$$\int_a^b |f(t)f'(t)|dt \leq \left[\int_a^b (t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b (b-t)|f'(t)|^q dt\right]^{1/q}.$$

- If we take $h(t) = t - a$, $t \in [a, b]$ and $g(t) = b - t$, $t \in [a, b]$, then we find that

$$\begin{aligned}
&\int_a^b (b-t)|f(t)f'(t)|dt \\
&\leq \left[\int_a^b (b-t)(t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b \frac{(b-t)^2}{2}|f'(t)|^q dt\right]^{1/q} \\
&= \frac{1}{2^{1/q}} \left[\int_a^b (b-t)(t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b (b-t)^2|f'(t)|^q dt\right]^{1/q}.
\end{aligned}$$

- If we take $h(t) = \log(1 + t - a)$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we obtain

$$\begin{aligned}
&\int_a^b |f(t)f'(t)|dt \\
&\leq \left[\int_a^b \log(1+t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b (b-t)(1+t-a)^{p-1}|f'(t)|^q dt\right]^{1/q}.
\end{aligned}$$

- If we take $h(t) = \arctan(t - a)$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we get

$$\begin{aligned}
&\int_a^b |f(t)f'(t)|dt \\
&\leq \left[\int_a^b \arctan(t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b (b-t)[1+(t-a)^2]^{p-1}|f'(t)|^q dt\right]^{1/q}.
\end{aligned}$$

- If we take $b = +\infty$, $h(t) = t - a$, $t \in [a, b]$ and $g(t) = e^{-\lambda t}$, $t \in [a, b]$, with $\lambda > 0$, then we have

$$\begin{aligned}
&\int_a^b e^{-\lambda t}|f(t)f'(t)|dt \leq \left[\int_a^b e^{-\lambda t}(t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b \frac{1}{\lambda} e^{-\lambda t}|f'(t)|^q dt\right]^{1/q} \\
&\leq \frac{1}{\lambda^{1/q}} \left[\int_a^b e^{-\lambda t}(t-a)|f'(t)|^p dt\right]^{1/p} \left[\int_a^b e^{-\lambda t}|f'(t)|^q dt\right]^{1/q}.
\end{aligned}$$

  By considering the classical definition of the Laplace transform of a function $h : [a, b] \mapsto [0, +\infty)$, i.e.,

$$\mathcal{L}\left[h(\cdot)\right](\lambda) = \int_a^b e^{-\lambda t} h(t)dt,$$

this inequality reads as

$$\mathcal{L}\left[|f(\cdot)f'(\cdot)|\right](\lambda) \leq \frac{1}{\lambda^{1/q}}\left[\mathcal{L}\left[(\cdot - a)|f'(\cdot)|^p\right](\lambda)\right]^{1/p}\left[\mathcal{L}\left[|f'(\cdot)|^q\right](\lambda)\right]^{1/q}.$$

To the best of our knowledge, it is a new Laplace transform functional inequality. This example shows how classical transform techniques can be incorporated into our framework.

- If we take $h(t) = 1 - e^{-\lambda t}$, $t \in [a, b]$, with $\lambda > 0$, and $g(t) = 1$, $t \in [a, b]$, then we get

$$\begin{aligned} &\int_a^b |f(t)f'(t)|dt \\ &\leq \left[\int_a^b (1 - e^{-\lambda t})|f'(t)|^p dt\right]^{1/p}\left[\int_a^b (b-t)\frac{1}{\lambda^{p-1}}e^{(p-1)\lambda t}|f'(t)|^q dt\right]^{1/q} \\ &= \frac{1}{\lambda^{(p-1)^2/p}}\left[\int_a^b (1 - e^{-\lambda t})|f'(t)|^p dt\right]^{1/p}\left[\int_a^b (b-t)e^{(p-1)\lambda t}|f'(t)|^q dt\right]^{1/q}. \end{aligned}$$

- In a general setting, if we take $h = 1/g$, where g denotes a non-increasing function such that $\lim_{t\to 0} g(t) = +\infty$, then we have

$$\begin{aligned} &\int_a^b g(t)|f(t)f'(t)|dt \\ &\leq \left[\int_a^b |f'(t)|^p dt\right]^{1/p}\left[\int_a^b G(t)\frac{|f'(t)|^q}{[-g'(t)]^{p-1}}[g(t)]^{2(p-1)}dt\right]^{1/q}. \end{aligned}$$

These are just a few examples; there are many more that can be derived from Theorem 1.

Two intuitive variations of Theorem 1 are now studied.

3 Two Variations with Examples

3.1 Two variations

The first variation introduces an adjustable parameter into the upper bound of the sum. This is described in more detail in the proposition below.

Proposition 1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and $f, g, h : [a, b] \mapsto \mathbb{R}$ be three functions such that f is differentiable with $f(a) = 0$, g is integrable, and h is differentiable non-decreasing with $h(a) = 0$. Let us set, for any $x \in [a, b]$,*

$$G(x) = \int_x^b g(t)dt.$$

Let $p > 1$, $q = p/(p-1)$ and $\theta > 0$. Then

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{\theta^p}{p}\int_a^b g(t)h(t)|f'(t)|^p dt + \frac{1}{q\theta^q}\int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt.$$

Proof. It follows from Theorem 1 that

$$\int_a^b g(t)|f(t)f'(t)|dt \leq C^{1/p}D^{1/q}, \tag{5}$$

where

$$C = \int_a^b g(t)h(t)|f'(t)|^p dt$$

and

$$D = \int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt.$$

Using $1/p + 1/q = 1$, introducing the parameter θ and applying the Young product inequality, i.e., $ab \leq a^p/p + b^q/q$ with $a, b \geq 0$, we obtain

$$\begin{aligned} C^{1/p}D^{1/q} &= \theta C^{1/p} \times \frac{1}{\theta}D^{1/q} \leq \frac{1}{p}\left[\theta C^{1/p}\right]^p + \frac{1}{q}\left[\frac{1}{\theta}D^{1/q}\right]^q \\ &= \frac{\theta^p}{p}C + \frac{1}{q\theta^q}D. \end{aligned} \tag{6}$$

Joining Equations (5) and (6), we get

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{\theta^p}{p}\int_a^b g(t)h(t)|f'(t)|^p dt + \frac{1}{q\theta^q}\int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt.$$

This ends the proof of Proposition 1. □

In particular, for $p = 2$, Proposition 1 implies that

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{1}{2}\left\{\theta^2\int_a^b g(t)h(t)[f'(t)]^2 dt + \frac{1}{\theta^2}\int_a^b G(t)\frac{[f'(t)]^2}{h'(t)}dt\right\}.$$

As an important example, in the general case, if we take $\theta = 1$, $h(t) = t - a$, $t \in [a, b]$ and $g(t) = t^m$, $t \in [a, b]$, $m \in \mathbb{N}\backslash\{0\}$, then we have

$$\begin{aligned} &\int_a^b t^m|f(t)f'(t)|dt \\ &\leq \frac{1}{p}\int_a^b t^m(t-a)|f'(t)|^p dt + \frac{1}{q(m+1)}\int_a^b (b^{m+1} - t^{m+1})|f'(t)|^q dt, \end{aligned}$$

which corresponds to [16, Theorem 1], as recalled in Equation (2).

The second variation can be considered an optimized version of Proposition 1. This optimization involves choosing a value of θ to balance the two main integral terms. This is explained in the proposition below; the manipulation of θ is only apparent in the proof.

Proposition 2. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and $f, g, h : [a, b] \mapsto \mathbb{R}$ be three functions such that f is differentiable with $f(a) = 0$, g is integrable, and h is differentiable non-decreasing with $h(a) = 0$. Let us set, for any $x \in [a, b]$,*

$$G(x) = \int_x^b g(t)dt.$$

Let $p > 1$ and $q = p/(p-1)$. Then

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{(p-1)^{1-1/p}}{p} \int_a^b \left[g(t)h(t)|f'(t)|^p + G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}} \right] dt.$$

Proof. It follows from Proposition 1 that

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{\theta^p}{p} \int_a^b g(t)h(t)|f'(t)|^p dt + \frac{1}{q\theta^q} \int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt, \tag{7}$$

with $\theta > 0$. Now, we consider a balance between the two constant factors of the main integral terms. This balance is achieved for θ such that

$$\frac{\theta^p}{p} = \frac{1}{q\theta^q} \;\Leftrightarrow\; \theta = \left(\frac{p}{q}\right)^{1/(p+q)}.$$

Using $q = p/(p-1)$, we have

$$\theta = \left(\frac{p}{q}\right)^{1/(p+q)} = \left[\frac{p}{p/(p-1)}\right]^{1/(p+p/(p-1))} = (p-1)^{1/p-1/p^2}. \tag{8}$$

Joining Equations (7) and (8), we get

$$\begin{aligned}
&\int_a^b g(t)|f(t)f'(t)|dt \\
&\leq \frac{(p-1)^{1-1/p}}{p} \int_a^b g(t)h(t)|f'(t)|^p dt + \frac{(p-1)^{1-1/p}}{p} \int_a^b G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}}dt \\
&= \frac{(p-1)^{1-1/p}}{p} \int_a^b \left[g(t)h(t)|f'(t)|^p + G(t)\frac{|f'(t)|^q}{[h'(t)]^{p-1}} \right] dt.
\end{aligned}$$

This ends the proof of Proposition 2. □

In particular, for $p = 2$, Proposition 2 implies that

$$\int_a^b g(t)|f(t)f'(t)|dt \leq \frac{1}{2}\int_a^b \left\{ g(t)h(t)[f'(t)]^2 + G(t)\frac{[f'(t)]^2}{h'(t)} \right\} dt.$$

To the best of our knowledge, the inequalities in Propositions 1 and 2 are new to the literature.

3.2 Examples

Some examples of applications of Proposition 2 are given below, based on those developed in Subsection 2.2.

- If we take $h(t) = t - a$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we obtain

$$\int_a^b |f(t)f'(t)|dt \leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[(t-a)|f'(t)|^p + (b-t)|f'(t)|^q\right] dt.$$

  In particular, selecting $p = 2$ gives

$$\int_a^b |f(t)f'(t)|dt \leq \frac{b-a}{2}\int_a^b [f'(t)]^2 dt.$$

  This is the Beesack integral inequality, as recalled in Equation (1) (see [1, 2]).

- If we take $h(t) = t - a$, $t \in [a, b]$ and $g(t) = b - t$, $t \in [a, b]$, then we get

$$\int_a^b (b-t)|f(t)f'(t)|dt \leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[(t-a)|f'(t)|^p + \frac{1}{2}(b-t)^2|f'(t)|^q\right] dt.$$

- If we take $h(t) = \log(1 + t - a)$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we have

$$\int_a^b (b-t)|f(t)f'(t)|dt \leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[\log(1+t-a)|f'(t)|^p + (b-t)(1+t-a)^{p-1}|f'(t)|^q\right] dt.$$

- If we take $h(t) = \arctan(t - a)$, $t \in [a, b]$ and $g(t) = 1$, $t \in [a, b]$, then we get

$$\int_a^b |f(t)f'(t)|dt \leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[\arctan(t-a)|f'(t)|^p dt + (b-t)[1+(t-a)^2]^{p-1}|f'(t)|^q\right] dt.$$

- If we take $b = +\infty$, $h(t) = t - a$, $t \in [a, b]$ and $g(t) = e^{-\lambda t}$, $t \in [a, b]$, with $\lambda > 0$, then we find that

$$\begin{aligned} &\int_a^b e^{-\lambda t}|f(t)f'(t)|dt \\ &\leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[e^{-\lambda t}(t-a)|f'(t)|^p + \frac{1}{\lambda}e^{-\lambda t}|f'(t)|^q\right] dt \\ &= \frac{(p-1)^{1-1/p}}{p}\int_a^b e^{-\lambda t}\left[(t-a)|f'(t)|^p + \frac{1}{\lambda}|f'(t)|^q\right] dt. \end{aligned}$$

In terms of the Laplace transform, this inequality reads as

$$\mathcal{L}\left[|f(\cdot)f'(\cdot)|\right](\lambda) \leq \frac{(p-1)^{1-1/p}}{p}\mathcal{L}\left[(\cdot - a)|f'(\cdot)|^p + \frac{1}{\lambda}|f'(\cdot)|^q\right](\lambda).$$

- If we take $h(t) = 1 - e^{-\lambda t}$, $t \in [a, b]$, with $\lambda > 0$, and $g(t) = 1$, $t \in [a, b]$, then we have

$$\begin{aligned} &\int_a^b |f(t)f'(t)|dt \\ &\leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[(1-e^{-\lambda t})|f'(t)|^p + (b-t)\frac{1}{\lambda^{p-1}}e^{(p-1)\lambda t}|f'(t)|^q\right] dt. \end{aligned}$$

- In a general setting, if we take $h = 1/g$, where g denotes a non-increasing function such that $\lim_{t\to 0} g(t) = +\infty$, then we get

$$\begin{aligned} &\int_a^b g(t)|f(t)f'(t)|dt \\ &\leq \frac{(p-1)^{1-1/p}}{p}\int_a^b \left[|f'(t)|^p dt + G(t)\frac{|f'(t)|^q}{[-g'(t)]^{p-1}}[g(t)]^{2(p-1)}\right] dt. \end{aligned}$$

These are just a few examples; many more can be derived from Proposition 2.

4 Conclusion

This article introduces a new framework for Beesack-Opial-type integral inequalities. The results incorporate three functions that can be applied to a variety of contexts, thereby generalizing existing findings. Detailed proofs are provided. The theory is illustrated by examples, including applications to the Laplace transform. This methodology could potentially be extended to other integral operators, discrete analogues or higher-dimensional settings. It could also be refined through tighter parameter tuning for specific applications. We anticipate that these developments will inspire further research into more general integral inequalities and their applications in mathematics and the applied sciences.

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