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Strong Insertion of a Contra-α-continuous Function between Two Comparable Real-valued Functions

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Abstract

Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-α-continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [21], while the concept of a, locally dense, set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [18]. A subset A of a topological space (X, τ) is called *semi-open* [10] if

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$A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called α-*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called α-*closed* if its complement is α-open or equivalently if A is union of a closed and a nowhere dense set.

We have a set is α-open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [20]. He investigated the sets that can be represented as union of closed sets and called them V-sets. Complements of V-sets, i.e., sets that are intersection of open sets are called Λ-sets [20].

Recall that a real-valued function f defined on a topological space X is called *A-continuous* [25] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. Jafari and Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called contra-α-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 23].

Hence, a real-valued function f defined on a topological space X is called *contra-*α-*continuous* (resp. *contra-semi-continuous*, *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is α-closed (resp. semi-closed, preclosed) in X [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-α-continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *cα-property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-α-continuous function also has property P. If P_1 and P_2 are $c\alpha$-property, the following terminology is used: (i) A space X has the *weak cα-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \le f$, g has property P_1 and f has property P_2, then there exists a contra-α-continuous function h such that $g \le h \le f$. (ii) A space X has the *strong cα-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \le f$, g has property P_1 and f has property P_2, then there exists a contra-α-continuous function h such that $g \le h \le f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose α-kernel of sets are α-open, is given a sufficient condition for the weak $c\alpha$-insertion property. Also for a space with the weak $c\alpha$-insertion property, we give necessary and sufficient conditions for the space to have the strong $c\alpha$-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-α-continuous function, the necessary definitions and terminology are stated.

The abbreviations *cαc*, *cpc* and *csc* are used for contra-α-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Let (X, τ) be a topological space. Then the family of all α-open, α-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$$A^\Lambda = \bigcap\{O : O \supseteq A,\ O \in (X, \tau)\} \text{ and } A^V = \bigcup\{F : F \subseteq A,\ F^c \in (X, \tau)\}.$$

In [7, 19, 22], A^Λ is called the *kernel* of A.

We define the subsets $\alpha(A^{\Lambda})$, $\alpha(A^{V})$, $p(A^{\Lambda})$, $p(A^{V})$, $s(A^{\Lambda})$ and $s(A^{V})$ as follows:

$$\alpha(A^{\Lambda}) = \bigcap\{O : O \supseteq A,\ O \in \alpha O(X, \tau)\},$$

$$\alpha(A^{V}) = \bigcup\{F : F \subseteq A,\ F \in \alpha C(X, \tau)\},$$

$$p(A^{\Lambda}) = \bigcap\{O : O \supseteq A,\ O \in pO(X, \tau)\},$$

$$p(A^{V}) = \bigcup\{F : F \subseteq A,\ F \in pC(X, \tau)\},$$

$$s(A^{\Lambda}) = \bigcap\{O : O \supseteq A,\ O \in sO(X, \tau)\},$$

$$s(A^{V}) = \bigcup\{F : F \subseteq A,\ F \in sC(X, \tau)\}.$$

$\alpha(A^{\Lambda})$ (resp. $p(A^{\Lambda})$, $s(A^{\Lambda})$) is called the α-*kernel* (resp. *prekernel*, *semi- kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S, then $\overline{\rho}$ is defined as follows:

$x \overline{\rho}\, y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i \rho B_j$ for any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$.

(2) If $A \subseteq B$, then $A \overline{\rho} B$.

(3) If $A \rho B$, then $\alpha(A^{\Lambda}) \subseteq B$ and $A \subseteq \alpha(B^{V})$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, in which α-kernel sets are α-open, with $g \le f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\,\rho\, A(g, t_2)$, then there exists a contra-α-continuous function h defined on X such that $g \le h \le f$.*

Proof. Let g and f be real-valued functions defined on the X such that $g \le f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$, then $A(f, t_1)\,\rho\, A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\,\overline{\rho}\, F(t_2)$, $G(t_1)\,\overline{\rho}\, G(t_2)$, and $F(t_1)\,\rho\, G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\,\rho\, H(t_2)$, $H(t_1)\,\rho\, H(t_2)$ and $H(t_1)\,\rho\, G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \le h \le f$: If x is in $H(t)$, then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \le t'$, it follows that $g(x) \le t$. Hence $g \le h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \ge t$. Hence $h \le f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = \alpha(H(t_2)^V)\backslash\alpha(H(t_1)^\Lambda)$. Hence $h^{-1}(t_1, t_2)$ is α-closed in X, i.e., h is a contra-alpha-continuous function on X. □

The above proof used the technique of Theorem 1 in [14].

If a space has the strong $c\alpha$-insertion property for (P_1, P_2), then it has the weak

$c\alpha$-insertion property for (P_1, P_2). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak $c\alpha$-insertion property to satisfy the strong $c\alpha$-insertion property.

Theorem 2.2. *Let P_1 and P_2 be $c\alpha$-property and X be a space that satisfies the weak $c\alpha$-insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g \leq f$, g has property P_1 and f has property P_2. The space X has the strong $c\alpha$-insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that* (i) *for each n, F_n and $A(f-g, 2^{-n})$ are completely separated by contra-α-continuous functions, and* (ii) $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.

Proof. Suppose that there is a sequence $(A(f-g, 2^{-n}))$ of lower cut sets for $f-g$ and suppose that there is a sequence (F_n) of subsets of X such that

$$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$$

and such that for each n, there exists a contra-α-continuous function k_n on X into $[0, 2^{-n}]$ with $k_n = 2^{-n}$ on F_n and $k_n = 0$ on $A(f-g, 2^{-n})$. The function k from X into $[0, 1/4]$ which is defined by

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x)$$

is a contra-α-continuous function by the Cauchy condition and the properties of contra-α-continuous functions, (1) $k^{-1}(0) = \{x \in X : (f-g)(x) = 0\}$ and (2) if $(f-g)(x) > 0$, then $k(x) < (f-g)(x)$: In order to verify (1), observe that if $(f-g)(x) = 0$, then $x \in A(f-g, 2^{-n})$ for each n and hence $k_n(x) = 0$ for each n. Thus $k(x) = 0$. Conversely, if $(f-g)(x) > 0$, then there exists an n such that $x \in F_n$ and hence $k_n(x) = 2^{-n}$. Thus $k(x) \neq 0$ and this verifies (1). Next, in order to establish (2), note that

$$\{x \in X : (f-g)(x) = 0\} = \bigcap_{n=1}^{\infty} A(f-g, 2^{-n})$$

and that $(A(f-g, 2^{-n}))$ is a decreasing sequence. Thus if $(f-g)(x) > 0$, then either $x \notin A(f-g, 1/2)$ or there exists a smallest n such that $x \notin A(f-g, 2^{-n})$ and $x \in A(f-g, 2^{-j})$ for $j = 1, ..., n-1$.

In the former case,

$$k(x) = 1/4 \sum_{n=1}^{\infty} k_n(x) \leq 1/4 \sum_{n=1}^{\infty} 2^{-n} < 1/2 \leq (f-g)(x),$$

and in the latter,

$$k(x) = 1/4 \sum_{j=n}^{\infty} k_j(x) \leq 1/4 \sum_{j=n}^{\infty} 2^{-j} < 2^{-n} \leq (f-g)(x).$$

Thus $0 \leq k \leq f-g$ and if $(f-g)(x) > 0$, then $(f-g)(x) > k(x) > 0$. Let $g_1 = g + (1/4)k$ and $f_1 = f - (1/4)k$. Then $g \leq g_1 \leq f_1 \leq f$ and if $g(x) < f(x)$, then

$$g(x) < g_1(x) < f_1(x) < f(x).$$

Since P_1 and P_2 are $c\alpha$-properties, g_1 has property P_1 and f_1 has property P_2. Since by hypothesis X has the weak $c\alpha$-insertion property for (P_1, P_2), there exists a contra-α-continuous function h such that $g_1 \leq h \leq f_1$. Thus $g \leq h \leq f$ and if $g(x) < f(x)$, then $g(x) < h(x) < f(x)$. Therefore X has the strong $c\alpha$-insertion property for (P_1, P_2). (The technique of this proof is by Lane [16].)

Conversely, assume that X satisfies the strong $c\alpha$-insertion for (P_1, P_2). Let g and f be functions on X satisfying P_1 and P_2, respectively such that $g \leq f$. Thus, there exists a contra-α-continuous function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. We follow an idea contained in Powderly [24]. Now consider the functions 0 and $f-h$. 0 satisfies property P_1 and $f-h$ satisfies property P_2. Thus, there exists a contra-α-continuous function h_1 such that $0 \leq h_1 \leq f - h$ and if $0 < (f - h)(x)$ for any x in X, then $0 < h_1(x) < (f - h)(x)$. We next show that

$$\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}.$$

If x is such that $(f - g)(x) > 0$, then $g(x) < f(x)$. Therefore, $g(x) < h(x) < f(x)$. Thus, $f(x) - h(x) > 0$ or $(f - h)(x) > 0$. Hence, $h_1(x) > 0$. On the other hand, if $h_1(x) > 0$, then since $(f - h) \geq h_1$ and $f - g \geq f - h$, therefore $(f - g)(x) > 0$. For each n, let $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$, and

$$F_n = \{x \in X : h_1(x) \geq 2^{-n+1}\}$$

and

$$k_n = \sup\{\inf\{h_1, 2^{-n+1}\}, 2^{-n}\} - 2^{-n}.$$

Since $\{x \in X : (f - g)(x) > 0\} = \{x \in X : h_1(x) > 0\}$, it follows that

$$\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n.$$

We next show that k_n is a contra-α-continuous function which completely separates F_n and $A(f - g, 2^{-n})$. From its definition and by the properties of contra-α-continuous functions, it is clear that k_n is a contra-α-continuous function. Let $x \in F_n$. Then, from the definition of k_n, $k_n(x) = 2^{-n}$. If $x \in A(f - g, 2^{-n})$, then since $h_1 \leq f - h \leq f - g$, $h_1(x) \leq 2^{-n}$. Thus, $k_n(x) = 0$, according to the definition of k_n. Hence k_n completely separates F_n and $A(f - g, 2^{-n})$.

Theorem 2.3. *Let P_1 and P_2 be cα-properties and assume that the space X satisfied the weak cα-insertion property for (P_1, P_2). The space X satisfies the strong cα-insertion property for (P_1, P_2) if and only if X satisfies the strong cα-insertion property for $(P_1, c\alpha c)$ and for $(c\alpha c, P_2)$.*

Proof. Assume that X satisfies the strong cα-insertion property for $(P_1, c\alpha c)$ and for

$(c\alpha c,\ P_2)$. If g and f are functions on X such that $g \le f$, g satisfies property P_1, and f satisfies property P_2, then since X satisfies the weak $c\alpha$-insertion property for $(P_1,\ P_2)$ there is a contra-α-continuous function k such that $g \le k \le f$. Also, by hypothesis there exist contra-α-continuous functions h_1 and h_2 such that $g \le h_1 \le k$ and if $g(x) < k(x)$, then $g(x) < h_1(x) < k(x)$ and such that $k \le h_2 \le f$ and if $k(x) < f(x)$, then $k(x) < h_2(x) < f(x)$. If a function h is defined by $h(x) = (h_2(x) + h_1(x))/2$, then h is a contra-α-continuous function, $g \le h \le f$, and if $g(x) < f(x)$, then $g(x) < h(x) < f(x)$. Hence X satisfies the strong $c\alpha$-insertion property for $(P_1,\ P_2)$.

The converse is obvious since any contra-α-continuous function must satisfy both properties P_1 and P_2. (The technique of this proof is by Lane [17].)

3. Applications

Before stating the consequences of Theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space whose α-kernel sets are α-open.

Corollary 3.1. *If for each pair of disjoint preopen* (*resp. semi-open*) *sets* G_1, G_2 *of* X, *there exist* α*-closed sets* F_1 *and* F_2 *of* X *such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \emptyset$, *then* X *has the weak* $c\alpha$*-insertion property for* $(cpc,\ cpc)$ (*resp.* $(csc,\ csc)$).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \le f$. If a binary relation ρ is defined by $A\,\rho\,B$ in case $p(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $s(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,\ t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,\ t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is a preopen (resp. semi-open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. semi-closed) set, it follows that $p(A(f,\ t_1)^{\Lambda}) \subseteq p(A(g,\ t_2)^{V})$ (resp. $s(A(f,\ t_1)^{\Lambda}) \subseteq s(A(g,\ t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,\ t_1)\,\rho\,A(g,\ t_2)$. The proof follows from Theorem 2.1.

Corollary 3.2. *If for each pair of disjoint preopen (resp. semi-open) sets G_1, G_2, there exist α-closed sets F_1 and F_2 such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$, then every contra-precontinuous (resp. contra-semi-continuous) function is contra-α-continuous.*

Proof. Let f be a real-valued contra-precontinuous (resp. contra-semi-continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-α-continuous function h such that $g = h = f$. □

Corollary 3.3. *If for each pair of disjoint preopen (resp. semi-open) sets G_1, G_2 of X, there exist α-closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$, then X has the strong cα-insertion property for (cpc, cpc) (resp. (csc, csc)).*

Proof. Let g and f be real-valued functions defined on the X, such that f and g are *cpc* (resp. *csc*), and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-α-continuous functions hence h is a contra-α-continuous function. □

Corollary 3.4. *If for each pair of disjoint subsets G_1, G_2 of X, such that G_1 is preopen and G_2 is semi-open, there exist α-closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$, then X have the weak cα-insertion property for (cpc, csc) and (csc, cpc).*

Proof. Let g and f be real-valued functions defined on X, such that g is *cpc* (resp. *csc*) and f is *csc* (resp. *cpc*), with $g \leq f$. If a binary relation ρ is defined by $A \rho B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a semi-open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. semi-closed) set, it follows that $s(A(f, t_1)^{\Lambda}) \subseteq p(A(g, t_2)^{V})$

(resp. $p(A(f, t_1)^{\Lambda}) \subseteq s(A(g, t_2)^{V}))$. Hence, $t_1 < t_2$ implies that $A(f, t_1)\,\rho\, A(g, t_2)$. The proof follows from Theorem 2.1. □

Before stating consequences of Theorems 2.2 and 2.3 we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent*:

(i) *For each pair of disjoint subsets* G_1, G_2 *of X*, *such that* G_1 *is preopen and* G_2 *is semi-open*, *there exist* α*-closed subsets* F_1, F_2 *of X such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$.

(ii) *If G is a semi-open* (*resp. preopen*) *subset of X which is contained in a preclosed* (*resp. semi-closed*) *subset F of X*, *then there exists an* α*-closed subset H of X such that* $G \subseteq H \subseteq \alpha(H^{\Lambda}) \subseteq F$.

Proof. (i) ⇒ (ii) Suppose that $G \subseteq F$, where G and F are semi-open (resp. preopen) and preclosed (resp. semi-closed) subsets of X, respectively. Hence, F^c is a preopen (resp. semi-open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint α-closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an α-open subset containing F_1, we conclude that $\alpha(F_1^{\Lambda}) \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq \alpha(F_1^{\Lambda}) \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is preopen and G_2 is semi-open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of *X*. Hence by (ii) there exists an α-closed set *H* such that $G_2 \subseteq H \subseteq \alpha(H^\Lambda) \subseteq G_1^c$.

But

$$H \subseteq \alpha(H^\Lambda) \Rightarrow H \cap \alpha((H^\Lambda)^c) = \varnothing$$

and

$$\alpha(H^\Lambda) \subseteq G_1^c \Rightarrow G_1 \subseteq \alpha((H^\Lambda)^c).$$

Furthermore, $\alpha((H^\Lambda)^c)$ is an α-closed subset of *X*. Hence $G_2 \subseteq H$, $G_1 \subseteq \alpha((H^\Lambda)^c)$ and $H \cap \alpha((H^\Lambda)^c) = \varnothing$. This means that condition (i) holds.

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets* G_1, G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, can be separated by* α*-closed subsets of X, then there exists a contra-*α*-continuous function* $h : X \to [0, 1]$ *such that* $h(G_2) = \{0\}$ *and* $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of *X*, where G_1 is preopen and G_2 is semi-open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of *X* containing the semi-open subset G_2 of *X*, by Lemma 3.1, there exists an α-closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of *X* and contains G_2, and G_1^c is a preclosed subset of *X* and contains the semi-open subset $\alpha(H_{1/2}^\Lambda)$ of *X*. Hence, by Lemma 3.1, there exists α-closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq \alpha(H_{1/4}^\Lambda) \subseteq H_{1/2} \subseteq \alpha(H_{1/2}^\Lambda) \subseteq H_{3/4} \subseteq \alpha(H_{3/4}^\Lambda) \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0, 1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain α-closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function *h* on *X* by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X$, $0 \leq h(x) \leq 1$, i.e., h maps X into $[0, 1]$. Also, we note that for any $t \in D$, $G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-α-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$, then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$, then $\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\}$, hence, they are α-closed subsets of X. Similarly, if $\alpha < 0$, then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$, then $\{x \in X : h(x) > \alpha\} = \bigcup\{\alpha((H_t^{\Lambda})^c) : t > \alpha\}$ hence, every of them is an α-closed subset. Consequently h is a contra-α-continuous function. □

Lemma 3.3. *Suppose that X is a topological space. If each pair of disjoint subsets* G_1, G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, can separate by α-closed subsets of X, and* G_1 *(resp.* G_2*) is an α-closed subsets of X, then there exists a contra-continuous function* $h : X \to [0, 1]$ *such that,* $h^{-1}(0) = G_1$ *(resp.* $h^{-1}(0) = G_2$*) and* $h(G_2) = \{1\}$ *(resp.* $h(G_1) = \{1\}$*).*

Proof. Suppose that G_1 (resp. G_2) is an α-closed subset of X. By Lemma 3.2, there exists a contra-α-continuous function $h : X \to [0, 1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \backslash G_1) = \{1\}$ (resp. $h(X \backslash G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \backslash G_1$ (resp. $G_1 \subseteq X \backslash G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$). □

Lemma 3.4. *Suppose that X is a topological space such that every two disjoint semi-open and preopen subsets of X can be separated by α-closed subsets of X. The following conditions are equivalent*:

(i) *For every two disjoint subsets* G_1 *and* G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, there exists a contra-α-continuous function* $h : X \to [0, 1]$ *such that,* $h^{-1}(0) = G_1$ *(resp.* $h^{-1}(0) = G_2$*) and* $h^{-1}(1) = G_2$ *(resp.* $h^{-1}(1) = G_1$*).*

(ii) *Every preopen (resp. semi-open) subset of X is an α-closed subsets of X.*

(iii) *Every preclosed (resp. semi-closed) subset of X is an α-open subsets of X.*

Proof. (i) $\Rightarrow$ (ii) Suppose that G is a preopen (resp. semi-open) subset of X. Since $\varnothing$ is a semi-open (resp. preopen) subset of X, by (i) there exists a contra-α-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G$. Set $F_n = \left\{x \in X : h(x) < \frac{1}{n}\right\}$. Then for every $n \in \mathbb{N}$, F_n is an α-closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) $\Rightarrow$ (i) Suppose that G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is semi-open. By Lemma 3.3, there exists a contra-α-continuous function $f : X \to [0, 1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \left\{x \in X : f(x) < \frac{1}{2}\right\}$, $F = \left\{x \in X : f(x) = \frac{1}{2}\right\}$, and $H = \left\{x \in X : f(x) > \frac{1}{2}\right\}$. Then $G \bigcup F$ and $H \bigcup F$ are two α-open subsets of X and $(G \bigcup F) \bigcap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-α-continuous function $g : X \to \left[\frac{1}{2}, 1\right]$ such that, $g^{-1}(1) = G_2$ and $g(G \bigcup F) = \left\{\frac{1}{2}\right\}$. Define h by $h(x) = f(x)$ for $x \in G \bigcup F$, and $h(x) = g(x)$ for $x \in H \bigcup F$. Then h is well-defined and a contra-α-continuous function, since $(G \bigcup F) \bigcap (H \bigcup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \bigcup F) \bigcup (H \bigcup F) = X$, hence h defined on X and maps to $[0, 1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every α-open subset of X is an α-closed subset of X and complement of every α-closed subset of X is an α-open subset of X, the equivalence is hold.

Corollary 3.5. *If for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. semi-open) and G_2 is semi-open (resp. preopen), there exists a contra-α-continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, then X has the strong cα-insertion property for (cpc, csc) (resp. (csc, cpc)).*

Proof. Since for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. semi-open) and G_2 is semi-open (resp. preopen), there exists a contra-α-

continuous function $h : X \to [0, 1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \left\{x \in X : h(x) < \frac{1}{2}\right\}$ and $F_2 = \left\{x \in X : h(x) > \frac{1}{2}\right\}$. Then F_1 and F_2 are two disjoint α-closed subsets of *X* that contain G_1 and G_2, respectively. Hence, by Corollary 3.4, *X* has the weak *c*α-insertion property for (cpc, csc) and (csc, cpc). Now, assume that *g* and *f* are functions on *X* such that $g \leq f$, g is *cpc* (resp. *csc*) and *f* is *c*α*c*. Since $f - g$ is *cpc* (resp. *csc*), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preopen (resp. semi-open) subset of *X*. Now setting $H_n = \{x \in X : (f - g)(x) > 2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is an α-open subset of *X* and we have $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$, H_n and $A(f - g, 2^{-n})$ are disjoint subsets of *X*. By Lemma 3.2, H_n and $A(f - g, 2^{-n})$ can be completely separated by contra-α-continuous functions. Hence by Theorem 2.2, *X* has the strong *c*α-insertion property for $(cpc, c\alpha c)$ (resp. $(csc, c\alpha c)$).

By an analogous argument, we can prove that *X* has the strong *c*α-insertion property for $(c\alpha c, csc)$ (resp. $(c\alpha c, cpc)$). Hence, by Theorem 2.3, *X* has the strong *c*α-insertion property for (cpc, csc) (resp. (csc, cpc)).

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