Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 15, Number 5, 2025, Pages 873-888
https://doi.org/10.34198/ejms.15525.873888

Some Interpolation Inequalities Involving Real Valued Functions of a Single Variable

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Abstract

Mathematical Inequalities play a critical role in Mathematics and especially in the field of Mathematical analysis. In this work, interpolation estimates involving Lebesgue functional norms are employed to obtain inequalities involving real-valued functions of a single variable. The inequalities involve elementary functions such as polynomials, exponentials and trigonometric functions.

1 Introduction

Mathematical inequalities play a very crucial and instrumental role in Applied Mathematical Sciences and applications to engineering problems. Inequalities are ubiquitous in the study of behavior of solutions of Partial Differential Equations as obtaining a solution in closed form most of the times it is impossible even if the PDE is linear. Then in order to study properties of the solutions, inequalities are employed. Inequalities are very powerful tools in analysis to study and explore the behaviour of functions and obtain various bounds relating different types of functions. In this work, interpolation inequalities involving Lebesgue norms are employed to derive inequalities associating elementary functions such as polynomials, exponentials and trigonometric functions. This work will further enrich the literature on mathematical inequalities. It will be a supplement of other similar works that exist in the literature [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]. These works serve as motivation for this current work. Mathematical inequalities is a contemporary topic in Analysis and still an active area of research. All the results presented in this article are rigorously proved.

Received: July 11, 2025; Accepted: August 15, 2025; Published: August 27, 2025
2020 Mathematics Subject Classification: 41A21, 26D99, 39B62.
Keywords and phrases: interpolation inequalities, real valued functions of a single variable, Pade approximants, analytical techniques.

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Theorem 1. *The following inequality holds*

$$\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^3 \le x^{\frac{3}{2}}\left(\frac{\sqrt{2}}{2}\right)^3\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{3}{2}}$$
$$\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{3}{2}},\ \forall\ x\ \in]0,1[. \quad (1)$$

Proof. First, the inequality below is proved

$$|f_1(x)(g(x)-g(0))f_2(x)(h(x)-h(0))\ f_3(x)(m(x)-m(0))| \le |f_1(x)|\ |f_2(x)\ |\ |f_3(x)\ |$$
$$\times\ x^{\frac{3}{2}}||g'||_{L_2(0,x)}\ ||h'||_{L_2(0,x)}\ ||m'||_{L_2(0,x)},$$
$$\forall x \in]0,1[.$$

The proof follows below

$$\begin{aligned}
&|f_1(x)(g(x)-g(0))f_2(x)(h(x)-h(0))\ f_3(x)(m(x)-m(0))| \\
&= |f_1(x)\ \int_0^x g'dt\ f_2(x)\ \int_0^x h'dt\ \ f_3(x)\ \int_0^x m'dt\ | \\
\textit{triangle inequality}\ &\le\ |f_1(x)|\ |f_2(x)|\ |f_3(x)|\ \int_0^x |g'|dt\ \int_0^x |h'|dt\ \int_0^x |m'|dt \\
\textit{Schwarz}-\textit{Cauchy inequality} &\le\ |f_1(x)|\ |f_2(x)|\ |f_3(x)|\ \ x^{\frac{3}{2}} \\
&\times\ ||g'||_{L_2(0,x)}\ ||h'||_{L_2(0,x)}\ ||m'||_{L_2(0,x)}, x \in]0,1[.
\end{aligned}$$

By picking $f_1(x) = f_2(x) = f_3(x) = f(x)$ and $g(x) = h(x) = m(x)$ the following estimate holds $|f^3(x)(g(x)-g(0))^3| \le |f^3(x)|\ \ x^{\frac{3}{2}}\ ||g'||^3_{L_2(0,x)}$. By letting $f(x) = b_{1,3}(x) = 3x(1-x)^2$ (Bernstein polynomial) and $g(t) = \exp(t)$, it yields

$$\begin{aligned}
|f^3(x)(g(x)-g(0))^3| &= |\left(3x(1-x)^2\right)^3(\exp(x)-1)^3| \\
&= |27x^3(1-x)^6(\exp(x)-1)^3| \\
&= 27x^3(1-x)^6(\exp(x)-1)^3 \\
&\le |\ f^3(x)|x^{\frac{3}{2}}\ ||g'||^3_{L_2(0,x)} \\
&= 27x^3(1-x)^6x^{\frac{3}{2}}\ \left(\frac{\sqrt{2}}{2}\right)^3(\exp(x)-1)^{\frac{3}{2}}\ (\exp(x)+1)^{\frac{3}{2}}.
\end{aligned}$$

By a trivial computation, the following bound is derived

$$(\exp(x)-1)^3 \le x^{\frac{3}{2}}\ \left(\frac{\sqrt{2}}{2}\right)^3(\exp(x)-1)^{\frac{3}{2}}\ (\exp(x)+1)^{\frac{3}{2}}.$$

Using the Pade approximant for the exponential function

$$\exp(x) \approxeq \frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}},\ \forall x \in]0,1[,$$

it yields

$$\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^3 \leq x^{\frac{3}{2}}\left(\frac{\sqrt{2}}{2}\right)^3\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{3}{2}}$$
$$\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{3}{2}},\ \forall\ x\ \in]0,1[$$

identical to (1). □

Theorem 2. *The following inequality holds*

$$\begin{aligned}
&\left|\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)\left(\frac{2}{3}x^3-\frac{x^4}{2}\right)\left(2x-2x^2\right)\right| \\
&\leq x^{\frac{5}{2}}\frac{\sqrt{6}}{6}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{1}{2}} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}} \\
&\times\sqrt{\frac{4x^3}{3}-2x^4+\frac{4x^5}{5}} \\
&\times\sqrt{4x-8x^2+\frac{16x^3}{3}}, \forall x \in]0,1[. \qquad (2)
\end{aligned}$$

Proof. We start by proving the inequality

$$\left|\ (g(x)-g(0))\int_0^x m\ q\ dt\ \int_0^x m'\ q'\ dt\right| \leq \sqrt{x}\ \ ||g'||_{L_2(0,x)}\ \ ||m||_{L_2(0,x)}\ ||m'||_{L_2(0,x)}$$
$$\times\, ||q||_{L_2(0,x)}\ ||q'||_{L_2(0,x)}\ ,\ \forall x \in]0,1[.$$

The proof follows

$$\begin{aligned}
\Big|\ (g(x)-g(0)) \int_0^x m\ q\ dt \int_0^x m'\ q'\ dt\ \Big| &= \Big|\ (g(x)-g(0))\ \Big|\ \Big| \int_0^x m\ q\ dt\ \Big|\ \Big| \int_0^x m'\ q'\ dt\ \Big| \\
&= \Big| \int_0^x g' dt\ \Big|\Big| \int_0^x m\ q\ dt\ \Big|\Big| \int_0^x m'\ q'\ dt\ \Big| \\
\textit{triangle inequality}\ &\leq \left(\int_0^x |g'| dt\right) \left(\int_0^x |m\ q\ | dt\right) \left(\int_0^x |m'\ q'|\ dt\right) \\
\textit{Schwarz}-\textit{Cauchy inequality}\ &\leq \sqrt{x}\ \ ||g'||_{L_2(0,x)}\ \ ||m||_{L_2(0,x)}\ ||m'||_{L_2(0,x)} \\
&\times ||q||_{L_2(0,x)}\ ||q'||_{L_2(0,x)}\ ,\ \forall x \in]0,1[.
\end{aligned}$$

Let g be the exponential function and pick m and q to be the Bernstein polynomials $m(t) = b_{1,1}(t) = t,\ q(t) = b_{1,2}(t) = 2t - 2t^2$, then it follows

$$\begin{aligned}
\Big|\ (g(x)-g(0)) \int_0^x m\ q\ dt \int_0^x m'\ q'\ dt\ \Big| &= \Big| \int_0^x g' dt \quad \int_0^x m\ q\ dt \quad \int_0^x m'\ q'\ dt\ \Big| \\
&= \left|(\exp(x)-1)\left(\frac{2x^3}{3} - \frac{x^4}{2}\right)\left(2x - 2x^2\right)\right| \\
&\leq \sqrt{x}\ \ ||g'||_{L_2(0,x)}\ \ ||m||_{L_2(0,x)}\ ||m'||_{L_2(0,x)} \\
&\times ||q||_{L_2(0,x)}\ ||q'||_{L_2(0,x)} \\
&= x^{\frac{5}{2}} \frac{\sqrt{6}}{6}\ (\exp(x)-1)^{\frac{1}{2}}\ (\exp(x)+1)^{\frac{1}{2}} \\
&\times \sqrt{\frac{4x^3}{3} - 2x^4 + \frac{4x^5}{5}} \\
&\times \sqrt{4x - 8x^2 + \frac{16x^3}{3}}, x \in]0,1[.
\end{aligned}$$

By employing Pade approximant for the exponential function meaning that

$$\exp(x) \approxeq \frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{27} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{1008} - \frac{x^5}{30240}},\ \forall x \in]0,1[$$

the inequality below follows

$$\begin{aligned}
&\Big|\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)\left(\frac{2}{3}x^3-\frac{x^4}{2}\right)(2x-2x^2)\Big| \\
&\le x^{\frac{5}{2}}\frac{\sqrt{6}}{6}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{1}{2}} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}} \\
&\times\sqrt{\frac{4x^3}{3}-2x^4+\frac{4x^5}{5}} \\
&\times\sqrt{4x-8x^2+\frac{16x^3}{3}}
\end{aligned}$$

identical to (2). □

Theorem 3. *The inequality below holds*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{72}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{72}+\frac{x^4}{1008}-\frac{x^5}{30240}}\cdot\frac{\frac{12671}{4363920}x^5-\frac{2363}{18183}x^3+x}{1+\frac{445}{12122}x^2+\frac{601}{872784}x^4+\frac{121}{16662240}x^6} \\
&-\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{72}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{72}+\frac{x^4}{1008}-\frac{x^5}{30240}}\cdot\frac{1-\frac{3665}{7788}x^2+\frac{711}{25960}x^4-\frac{2923}{7850304}x^6}{1+\frac{229}{7788}x^2+\frac{x^4}{2360}+\frac{127x^6}{39251520}}+1\Big)\left(x^2-\frac{2x^3}{3}\right)\Big| \\
&\le\frac{\sqrt{x}}{2}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{72}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{72}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{1}{2}}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{72}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{72}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}} \\
&\times\left(x-\frac{\frac{12671x^5}{4363920}-\frac{2363}{18183}x^3+x}{1+\frac{445}{12122}x^2+\frac{601}{872784}x^4+\frac{121}{16662240}x^6}\cdot\frac{1-\frac{3665x^2}{7788}+\frac{711x^4}{25960}-\frac{2923x^6}{7850304}}{1+\frac{229x^2}{7788}+\frac{x^4}{2360}+\frac{127x^6}{39251520}}\right)^{\frac{1}{2}} \\
&\times\left(\frac{4x^3}{3}-2x^4+\frac{4x^5}{5}\right)^{\frac{1}{2}}
\end{aligned}\tag{3}$$

$\forall x\in(0,\frac{\pi}{2})$.

Proof. The first step is to prove the inequality below

$$\Big|\int_0^x fgdt\int_0^x hdt\Big|\le\sqrt{x}\,||f||_{L_2(0,x)}\,||g||_{L_2(0,x)}\,||h||_{L_2(0,x)},\ \forall\ x\in(0,\frac{\pi}{2}).$$

A direct computation gives

$$\begin{aligned}
\Big|\int_0^x fgdt\int_0^x hdt\Big| &= \Big|\int_0^x fgdt\Big|\,\Big|\int_0^x hdt\Big| \\
&\le\sqrt{x}\ ||f||_{L_2(0,x)}\,||g||_{L_2(0,x)}\,||h||_{L_2(0,x)}\ ,\ \forall\ x\in(0,\frac{\pi}{2})
\end{aligned}$$

employing the Schwarz-Cauchy inequality. Pick $f(t) = \exp(t), g(t) = \sin(t), h(t) = b_{1,2}(t) = 2t - 2t^2$ where $b_{1,2}(t)$ is a Bernstein polynomial, and a straight computation gives

$$\begin{aligned}
\Big|\int_0^x \ fgdt \ \int_0^x \ hdt \ \Big| &= \Big|\int_0^x \ fgdt \ \Big| \ \Big|\int_0^x \ hdt\Big| \\
&= \Big|\frac{1}{2}(\exp(x)\sin(x) - \exp(x) \ \cos(x) + 1)\left(x^2 - \frac{2x^3}{3}\right)\Big| \\
&\le \sqrt{x} \ \ ||f||_{L_2(0,x)} \ ||g||_{L_2(0,x)} \ ||h||_{L_2(0,x)} \\
&= \frac{\sqrt{x}}{2} \ \sqrt{\exp(x) - 1} \ \sqrt{\exp(x) + 1} \ \sqrt{x - \sin(x) \ \cos(x)} \\
&\times \sqrt{\frac{4x^3}{3} - 2x^4 + \frac{4x^5}{5}}, \ \forall \ x \ \in \left(0, \ \frac{\pi}{2}\right).
\end{aligned}$$

The Pade approximations for the functions $\exp(x), \sin(x), \cos(x)$ are

$$\begin{aligned}
\exp(x) &\approxeq \frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{27} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{1008} - \frac{x^5}{30240}}, \\
\sin(x) &\approxeq \frac{\frac{12671x^5}{4363920} - \frac{2363x^3}{18183} + x}{1 + \frac{445x^2}{12122} + \frac{601x^4}{872784} + \frac{121x^6}{16662240}}, \\
cos(x) &\approxeq \frac{1 - \frac{3665x^2}{7788} + \frac{711x^4}{25960} - \frac{2923x^6}{7850304}}{1 + \frac{229x^2}{7788} + \frac{x^4}{2360} + \frac{127x^6}{39251520}}, \\
&\forall \ x \ \in \left(0, \ \frac{\pi}{2}\right).
\end{aligned}$$

Employing the above Pade approximations, the final estimate is derived

$$\begin{aligned}
&\Big|\frac{1}{2}\Big(\frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{72} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{72} + \frac{x^4}{1008} - \frac{x^5}{30240}} \cdot \frac{\frac{12671}{4363920}x^5 - \frac{2363}{18183}x^3 + x}{1 + \frac{445}{12122}x^2 + \frac{601}{872784}x^4 + \frac{121}{16662240}x^6} \\
&- \frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{72} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{72} + \frac{x^4}{1008} - \frac{x^5}{30240}} \cdot \frac{1 - \frac{3665}{7788}x^2 + \frac{711}{25960}x^4 - \frac{2923}{7850304}x^6}{1 + \frac{229}{7788}x^2 + \frac{x^4}{2360} + \frac{127x^6}{39251520}} + 1)\left(x^2 - \frac{2x^3}{3}\right)\Big| \\
&\le \frac{\sqrt{x}}{2}\left(\frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{72} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{72} + \frac{x^4}{1008} - \frac{x^5}{30240}} - 1\right)^{\frac{1}{2}}\left(\frac{1 + \frac{x}{2} + \frac{x^2}{9} + \frac{x^3}{72} + \frac{x^4}{1008} + \frac{x^5}{30240}}{1 - \frac{x}{2} + \frac{x^2}{9} - \frac{x^3}{72} + \frac{x^4}{1008} - \frac{x^5}{30240}} + 1\right)^{\frac{1}{2}} \\
&\times\left(x - \frac{\frac{12671x^5}{4363920} - \frac{2363}{18183}x^3 + x}{1 + \frac{445}{12122}x^2 + \frac{601}{872784}x^4 + \frac{121}{16662240}x^6} \cdot \frac{1 - \frac{3665x^2}{7788} + \frac{711x^4}{25960} - \frac{2923x^6}{7850304}}{1 + \frac{229x^2}{7788} + \frac{x^4}{2360} + \frac{127x^6}{39251520}}\right)^{\frac{1}{2}} \\
&\times\left(\frac{4x^3}{3} - 2x^4 + \frac{4x^5}{5}\right)^{\frac{1}{2}} \quad \forall x \in (0, \frac{\pi}{2})
\end{aligned}$$

as prescribed by (3). □

Theorem 4. *The inequality below is valid*

$$\Big|\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}\left((x-1)^2+1\right)-2\Big|$$
$$\leq x^{\frac{2}{3}}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+x-1\right)^{\frac{1}{2}}$$
$$\times\left(x-\ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)+\ln(2)\right)^{\frac{1}{2}}$$
$$\times\left(\frac{x^3}{3}+x\right)^{\frac{1}{2}}(\arctan(x))^{\frac{1}{2}}$$
$$\times\ \left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{4}\right)^{\frac{2}{3}}\frac{6}{5}\left(\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}\right)^3-1\right)^{\frac{1}{3}},\ \forall\ x\ \in(0,1). \tag{4}$$

Proof. The first significant step is to prove the inequality

$$\Big|\int_0^x a(t)c(t)dt\Big|\leq\frac{1}{x^2}\left(\int_0^x(a(t)+1)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{a(t)+1}\right)dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^x(c(t)+1)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{c(t)+1}\right)dt\right)^{\frac{1}{2}}||a||_{L_p(0,x)}\ ||c||_{L_q(0,x)} \tag{5}$$

where $a(t),c(t)>0\forall t\in(0,x)$ and $\frac{1}{p}+\frac{1}{q}=1,\forall p,q\in]1,+\infty[$. To proceed with the proof we first consider the following quantity below

$$\Big|\int_0^x\frac{\sqrt{a(t)+1}}{\sqrt{b(t)+1}}\int_0^x\frac{\sqrt{c(t)+1}}{\sqrt{d(t)+1}}\int_0^x a(t)c(t)dt\Big|,\ a(t),\ b(t),\ c(t),\ d(t)>0\ \forall\ t\ \in(0,x).$$

We bound the above quantity as follows

$$\Big|\int_0^x\frac{\sqrt{a(t)+1}}{\sqrt{b(t)+1}}\int_0^x\frac{\sqrt{c(t)+1}}{\sqrt{d(t)+1}}\int_0^x a(t)c(t)dt\Big|=\Big|\int_0^x\frac{\sqrt{a(t)+1}}{\sqrt{b(t)+1}}\,dt\Big|\ \Big|\int_0^x\frac{\sqrt{c(t)+1}}{\sqrt{d(t)+1}}\,dt\Big|$$
$$\times\Big|\int_0^x a(t)c(t)dt\ \Big|$$
$$\textit{triangle inequality}\ \leq\int_0^x\Big|\frac{\sqrt{a(t)+1}}{\sqrt{b(t)+1}}\Big|dt\ \int_0^x\Big|\frac{\sqrt{c(t)+1}}{\sqrt{d(t)+1}}\Big|dt\ \int_0^x\ \Big|a(t)c(t)\Big|dt$$
$$\textit{Schwarz}-\textit{Cauchy and Holder inequality}\ \leq\left(\int_0^x(a(t)+1)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{b(t)+1}\right)dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^x(c(t)+1)dt\right)^{\frac{1}{2}}\ \left(\int_0^x\left(\frac{1}{d(t)+1}\right)dt\right)^{\frac{1}{2}}$$
$$\times\ ||a||_{L_p(0,x)}\ ||c||_{L_q(0,x)}$$

$\frac{1}{p}+\frac{1}{q}=1, \forall p,q \in]1,+\infty[$. By picking $a(t)=b(t)$ and $c(t)=d(t)$ this yields

$$\begin{aligned} x^2\Big|\int_0^x a(t)c(t)dt\Big| \leq & \left(\int_0^x (a(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{a(t)+1}\right) dt\right)^{\frac{1}{2}} \\ & \times \left(\int_0^x (c(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{c(t)+1}\right) dt\right)^{\frac{1}{2}} \\ & \times ||a||_{L_p(0,x)}\ ||c||_{L_q(0,x)}. \end{aligned}$$

From this inequality, it follows

$$\begin{aligned} \Big|\int_0^x a(t)c(t)dt\Big| \leq & \frac{1}{x^2}\left(\int_0^x (a(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{a(t)+1}\right) dt\right)^{\frac{1}{2}} \\ \times & \left(\int_0^x (c(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{c(t)+1}\right) dt\right)^{\frac{1}{2}} ||a||_{L_p(0,x)}\ ||c||_{L_q(0,x)} \end{aligned}$$

which is the desired inequality (5). Choosing $a(t)=\exp(t)$, $c(t)=t^2$ (Bernstein), $p=3$, $q=\frac{3}{2}$, it follows that

$$\begin{aligned} \Big|\int_0^x a(t)c(t)dt\Big| &= \Big|\int_0^x t^2\exp(t)\ dt\Big| \\ &= \Big|\exp(x)\left((x-1)^2+1\right)-2\Big| \\ &\leq \frac{1}{x^2}\left(\int_0^x (a(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{a(t)+1}\right) dt\right)^{\frac{1}{2}} \\ &\times \left(\int_0^x (c(t)+1)dt\right)^{\frac{1}{2}} \left(\int_0^x \left(\frac{1}{c(t)+1}\right) dt\right)^{\frac{1}{2}} \\ &\times\ ||a||_{L_3(0,x)}\ ||c||_{L_{\frac{3}{2}}(0,x)} \\ &= x^{\frac{2}{3}}\ \sqrt{\exp(x)+x-1}\ \sqrt{x-\ln(\exp(x)+1)+\ln(2)} \\ &\times \sqrt{\frac{x^3}{3}+x}\ \sqrt{\arctan(x)} \\ &\times \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{4}\right)^{\frac{2}{3}} (\exp(3x)-1)^{\frac{1}{3}} \\ &\leq x^{\frac{2}{3}}\ \sqrt{\exp(x)+x-1}\ \sqrt{x-\ln(\exp(x)+1)+\ln(2)} \\ &\times \sqrt{\frac{x^3}{3}+x}\ \sqrt{\arctan(x)} \\ &\times \left(\frac{1}{3}\right)^{\frac{1}{3}} \left(\frac{1}{4}\right)^{\frac{2}{3}} \frac{6}{5}(\exp(3x)-1)^{\frac{1}{3}}. \end{aligned}$$

Using Pade approximant for the exponential function, meaning that

$$\exp(x) \approxeq \frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}},\ x \in]0,1[$$

it gives

$$
\begin{aligned}
&\Big|\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}\left((x-1)^2+1\right)-2\Big| \\
&\leq x^{\frac{2}{3}}\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+x-1\right)^{\frac{1}{2}} \\
&\times\left(x-\ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)+\ln(2)\right)^{\frac{1}{2}} \\
&\times\left(\frac{x^3}{3}+x\right)^{\frac{1}{2}}(\arctan(x))^{\frac{1}{2}} \\
&\times\left(\frac{1}{3}\right)^{\frac{1}{3}}\left(\frac{1}{4}\right)^{\frac{2}{3}}\frac{6}{5}\left(\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}\right)^3-1\right)^{\frac{1}{3}}, \ \forall\, x\ \in(0,1)
\end{aligned}
$$

which is identical to inequality (4).

□

Theorem 5. *The following inequality holds*

$$
\begin{aligned}
&\Big|\left(\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}}-\sqrt{2}\right)\left(\sqrt{x^2+1}-1\right) \\
&\times\left(\left(\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}+1\right)^{\frac{1}{2}}-\sqrt{2}\right)\Big| \\
&\leq \frac{1}{8}\frac{\sqrt{2}}{2}\sqrt{\frac{4}{3}}\frac{\sqrt{2}}{2}\,x^{\frac{3}{2}}\sqrt{\arctan(x)} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{1}{2}} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}} \\
&\times\left(ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)-\ln(2)\right)^{\frac{1}{2}} \\
&\times\left(x-ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)+\ln(2)\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times\left(1-\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}\right)^{\frac{1}{2}}$$
$$\times\left(1+\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}\right)^{\frac{1}{2}}, x \in]0,1[. \tag{6}$$

Proof. To derive the result, we firstly prove the inequality below

$$\begin{aligned}
&\left|\left(\sqrt{f(x)+1}-\sqrt{f(0)+1}\right)\left(\sqrt{g(x)+1}-\sqrt{g(0)+1}\right)\left(\sqrt{h(x)+1}-\sqrt{h(0)+1}\right)\right| \\
&\le \frac{1}{8}\,||f'||_{L_2(0,x)}||g'||_{L_2(0,x)}\,||h'||_{L_2(0,x)} \\
&\times\left(\int_0^x\left(\frac{1}{f(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{g(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{h(t)+1}\right)dt\right)^{\frac{1}{2}}
\end{aligned} \tag{7}$$

where $f, g, h > 0\ \forall t \in (0,x)$. Detailed proof follows below

$$\begin{aligned}
&\left|\left(\sqrt{f(x)+1}-\sqrt{f(0)+1}\right)\left(\sqrt{g(x)+1}-\sqrt{g(0)+1}\right)\left(\sqrt{h(x)+1}-\sqrt{h(0)+1}\right)\right| \\
&=\left|\left(\sqrt{f(x)+1}-\sqrt{f(0)+1}\right)\right|\;\left|\left(\sqrt{g(x)+1}-\sqrt{g(0)+1}\right)\right|\;\left|\left(\sqrt{h(x)+1}-\sqrt{h(0)+1}\right)\right| \\
&=\left|\int_0^x\frac{f'(t)}{2\sqrt{f(t)+1}}\,dt\right|\;\left|\int_0^x\frac{g'(t)}{2\sqrt{g(t)+1}}dt\right|\;\left|\int_0^x\frac{h'(t)}{2\sqrt{h(t)+1}}dt\right| \\
&\le\frac{1}{8}\int_0^x\left|f'(t)\right|\left|\frac{1}{\sqrt{f(t)+1}}\right|dt\int_0^x\left|g'(t)\right|\left|\frac{1}{\sqrt{g(t)+1}}\right|dt\int_0^x\left|h'(t)\right|\left|\frac{1}{\sqrt{h(t)+1}}\right|dt \\
&\le \frac{1}{8}\,||f'||_{L_2(0,x)}||g'||_{L_2(0,x)}\,||h'||_{L_2(0,x)} \\
&\times\left(\int_0^x\left(\frac{1}{f(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{g(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{h(t)+1}\right)dt\right)^{\frac{1}{2}}.
\end{aligned}$$

By picking $f(t) = \exp(t)$, $g(t) = t^2$ (Bernstein), $h(t) = \exp(-t)$ and employing the inequality (7) this yields

$$
\begin{aligned}
&\left|\left(\sqrt{f(x)+1}-\sqrt{f(0)+1}\right)\left(\sqrt{g(x)+1}-\sqrt{g(0)+1}\right)\ \left(\sqrt{h(x)+1}-\sqrt{h(0)+1}\right)\right| \\
&=\left|\left(\sqrt{f(x)+1}-\sqrt{f(0)+1}\right)\right|\ \left|\left(\sqrt{g(x)+1}-\sqrt{g(0)+1}\right)\right|\ \left|\left(\sqrt{h(x)+1}-\sqrt{h(0)+1}\right)\right| \\
&=\left|\int_0^x \frac{f'(t)}{2\ \sqrt{f(t)+1}}\ dt\right|\ \left|\int_0^x \frac{g'(t)}{2\sqrt{g(t)+1}}dt\ \right|\ \left|\int_0^x\ \frac{h'(t)}{2\ \sqrt{h(t)+1}}dt\right| \\
&=\left|\left(\sqrt{\exp(x)+1}-\sqrt{2}\right)\left(\sqrt{x^2+1}-1\right)\left(\sqrt{\exp(-x)+1}-\sqrt{2}\right)\right| \\
&\frac{1}{8}\ ||f'||_{L_2(0,x)}||g'||_{L_2(0,x)}\ ||h'||_{L_2(0,x)} \\
&\times\left(\int_0^x\left(\frac{1}{f(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{g(t)+1}\right)dt\right)^{\frac{1}{2}}\left(\int_0^x\left(\frac{1}{h(t)+1}\right)dt\right)^{\frac{1}{2}} \\
&=\frac{1}{8}\ \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\ \sqrt{\frac{4}{3}}x^{\frac{3}{2}}\ \sqrt{\exp(x)-1}\ \sqrt{\exp(x)+1} \\
&\times\sqrt{1-\exp(-x)}\ \sqrt{1+\exp(-x)} \\
&\times(x-\ln(\exp(x)+1)+\ln(2))^{\frac{1}{2}} \\
&\times\sqrt{\arctan(x)}\ \ (\ln(\exp(x)+1)-\ln(2))^{\frac{1}{2}}.
\end{aligned}
$$

Using Pade approximation for the exponential function gives the final inequality

$$
\begin{aligned}
&\left|\left(\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}}-\sqrt{2}\right)\ \left(\sqrt{x^2+1}-1\right)\right. \\
&\times\ \left.\left(\left(\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}+1\right)^{\frac{1}{2}}-\sqrt{2}\right)\right| \\
&\leq\frac{1}{8}\frac{\sqrt{2}}{2}\ \sqrt{\frac{4}{3}}\ \frac{\sqrt{2}}{2}\ x^{\frac{3}{2}}\ \sqrt{\arctan(x)} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}-1\right)^{\frac{1}{2}} \\
&\times\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\times\left(ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)-\ln(2)\right)^{\frac{1}{2}}$$
$$\times\left(x-ln\left(\frac{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}+1\right)+\ln(2)\right)^{\frac{1}{2}}$$
$$\times\left(1-\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}\right)^{\frac{1}{2}}$$
$$\times\left(1+\frac{1-\frac{x}{2}+\frac{x^2}{9}-\frac{x^3}{27}+\frac{x^4}{1008}-\frac{x^5}{30240}}{1+\frac{x}{2}+\frac{x^2}{9}+\frac{x^3}{27}+\frac{x^4}{1008}+\frac{x^5}{30240}}\right)^{\frac{1}{2}}$$

identical to (6) and the proof is completed. □

2 Conclusion

In this article, inequalities have been derived involving polynomial, trigonometric and exponential functions relying on purely analytical techniques such as functional interpolation inequalities. To derive all the inequalities in this work Pade approximations have been used for the exponential and trigonometric functions. All the results presented in this work are obtained using rigorous analysis.

3 Graphical Illustration of the Results

In this section, we include evidence that all the inequalities presented in this work are valid. More precisely, there are five graphs in total extended to three dimensions for better visualization. These five graphs represent the inequalities (1), (2), (3), (4), (6) where the red curve corresponds to the left hand side of each inequality and the blue curve to the right hand side respectively. All the graphs below verify the rigorous results obtained in this research article.

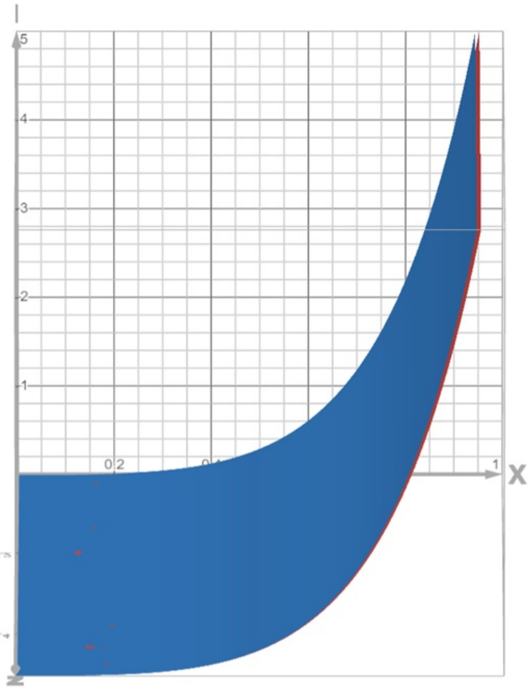

Figure 1: Graphical illustration of inequality (1).

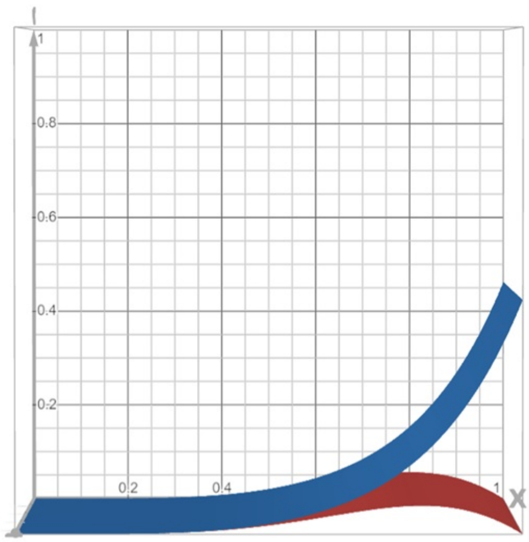

Figure 2: Graphical illustration of inequality (2).

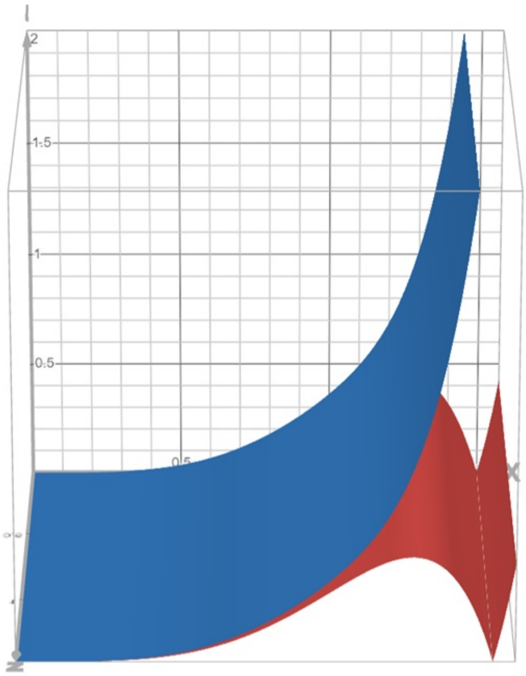

Figure 3: Graphical illustration of inequality (3).

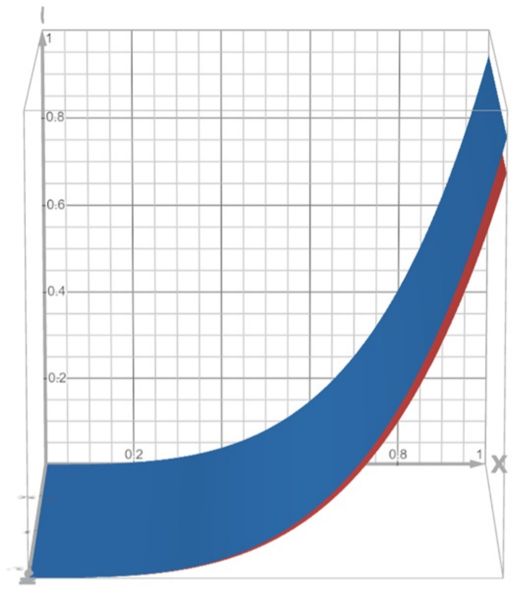

Figure 4: Graphical illustration of inequality (4).

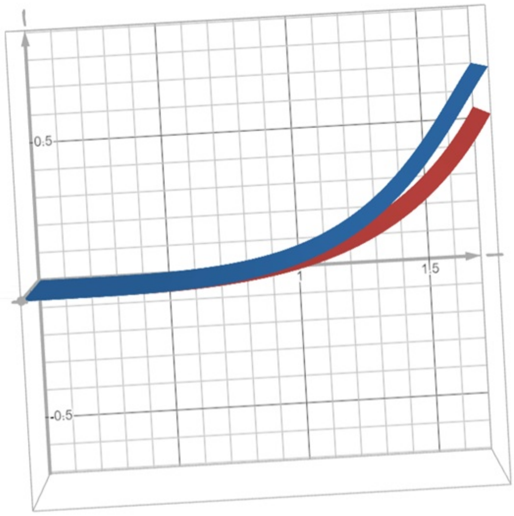

Figure 5: Graphical illustration of inequality (6).

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