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Response of Thin Plates Subjected to Inertia Distributed Loads Moving with Variable Velocity in Opposite Directions

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Abstract

The study examined the dynamic deflection characteristic of a plate which is subjected to a pair of inertia distributed loads, moving with variable velocities and in opposite direction was investigated. The model was formulated based on the thin plate theory. The governing equation obtained for the behavior of the model was reduced from a partial differential equation to an ordinary differential equation using a series solution for the dynamic deflection in terms of the normal modes. The reduced ODE was solved using chebyshev collocation method, result obtained was presented in graphical form.

1 Introduction

In mechanics, vibration is defined as the to-and-fro movement of an object. Illustration of vibrations can be seen everywhere in nature as nearly everything vibrates. Vibrations can be felt by the sense of touch when a vehicle passes by or by our drum when it receives impulse such as sound form the environment and some vibrations may be too low or too weak to detect. The concept of vibrations is useful to man in various ways such as designing machines used for various functions in vibrators to massage the body to compact loose soil est. vibrations can cause wear and tear and even cause malfunctioning of a machine. It also cause human body to lose concentration and to fall sick.

Study of vibrations is an extremely important area owning to its wide variety of engineering applications and life applications such as aeronautical, marine, civil, mechanical and so on since the constituting members (beams, plates, column, shells etc.) form integral parts of structures; it is therefore essentials for any design engineer to have a prior knowledge of the first few modes of vibrations characteristics before finalizing the design of a given structure.

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Dynamic behaviours of flat plate under moving external loads are essential problems in structural dynamic field, which are commonly encountered in engineering, such as bridges and roads, space vehicles, submarines and mechanical engineering.

The study of dynamic behaviour of plates represents a popular trend in structural dynamics theory in the past decades. The dynamic behaviour of plates excited by concentrated moving loads has also been extensively investigated.

In Nikkhoo *et al.,* [15], the authors solved in a semi-analytical form the problem of a Kirchhoff plate vibrating under two series of moving concentrated inertial loads traversing the plate surface along parallel rectilinear trajectories with opposite directions. A Kirchhoff plate on multiple supports was studied by Marchesiello *et al.,* [12], loaded by travelling vehicles modelled as concentrated loads due to sprung masses, adopting the Rayleigh-Ritz method coupled with an iterative dynamic sub structuring method. Concentrated moving masses on a Rayleigh beam and a non-Mindlin plate taking into account rotary inertia, but not shear deformation were considered by Gbadeyan and Oni [9], providing a solution in series form via generalized finite integral transform and Struble's method.

In De Faria and Oguamanam [4], a numerical solution was found for a Mindlin plate crossed by concentrated masses, using a finite element method with adaptive meshes at low speed. In Dyniewicz *et al.,* [5], a Mindlin plate subjected to a concentrated inertial load travelling at a variable speed along an arbitrary trajectory was considered; the problem of two concentrated inertial loads travelling in opposite directions along the same trajectory was also investigated, obtaining a numerical solution using the space-time finite element method. Esen [8], presented a new FEM procedure for transverse and longitudinal vibration analysis of thin rectangular plates subjected to a variable velocity moving load along an arbitrary trajectory. Amiri *et al.,* [1], based on first-order shear deformation plate theory, studied the response of a Mindlin elastic plate under a moving mass by using direct separation of variable and eigenfunction expansion method. Eftekhari and Jafari [6], presented a mixed modal-differential quadrature method for free and forced vibration of beams in contact with fluid.

Nikkhoo *et al.,* [14] in the work titled On non-stationary response of cracked thin rectangular plates acted upon by a moving random force concluded that there are nonlinear relationships between the increasing inclined crack angles and crack lengths and the non-dimensional functions of squared mean values for both undamped and damped cracked plates, after their investigation on the dynamics of cracked thin rectangular plates under the influence of a moving non-stationary random load, characterized by a constant mean value and velocity, along with five different covariance patterns: white noise, constant, exponential, cosine wave, and exponential cosine covariance. Yao *et al.,* [21] focused on the dynamic response mechanisms of steel plates under unconfined and confined blast loads, detailing the phases of plastic hinge formation and deformation patterns in those specific conditions. The findings are centered on blast load scenarios rather than distributed load dynamics

Pi *et al.,* [16] presented a study that focuses on the vibration control of a thin rectangular plate subjected to moving masses, emphasizing the dynamic behavior of the plate over traditional beam models. The study addresses the challenges of vibration control due to multiple degrees of freedom and uncertainties in practical applications and proposed the Modal Coordinate Reconstruction (MCR) â Adaptive Sliding â Mode Control (ASMC) method for effective vibration suppression in similar scenarios. Analyzing the work of Nikkhoo *et al.,* [13] where parametric investigations on dynamics of cracked thin rectangular plates, excited by a moving mass was carried out. It was found that the presence of a crack alters the natural frequencies and mode shapes of the plate, indicating that careful consideration of crack characteristics is essential for accurate dynamic analysis and structural integrity assessments.

In the work of Song *et al.,* [17], the Ritz method with beam eigenfunctions is used to discretize the spatial partial derivatives, and the differential quadrature method and integral quadrature method were employed to analogize the resultant system of partial differential equations of rectangular thin plates of arbitrary boundary conditions under moving loads. Wu [19], investigated the dynamic analysis of a rectangular plate under a moving line load using scale beams and scaling laws. Wu [20], presented a moving distributed mass element to perform the dynamic analysis of an inclined plate under moving distributed loads using finite element method. Esen [7], presented an equivalent finite element to analyze the transverse vibration of the plate under a moving point mass. Mamandi *et al.,* [11], investigated an effects of travelling mass with variable velocity on nonlinear dynamic response of an inclined Timoshenko beam with different boundary conditions. Based on Hamilton's principle, the nonlinear governing coupled PDEs of motion are derived and solved applying Galerkin's method using the Adam-Bashforth-Moulton integration method via the MATLAB solver package to obtain the dynamic response of the plate.

Gbadeyan and Dada [10], considered the dynamics response of Mindlin elastic type of plates under the influence of a partially uniform moving load. The set of partial differential equations was transformed into its equivalent non dimensional form and the finite difference technique was used to form a new set of linear algebraic equations which was solved. A rectangular Kirchhoff plate, simply supported on two opposite edges and free on the other two edges, loaded by a partially distributed mass acting instantaneously on part of the spatial domain of the plate, travelling in parallel direction with respect to the free edges was considered by Sorrentino and Catania [18]. Their formulation include damping and it was accomplished by the Rayleigh Ritz method.

It was observed that most existing studies on plates under moving loads are limited to constant velocity, concentrated loads or uniformly distributed loads traveling in the same direction. The case of two inertia distributed loads moving in opposite directions on thin plates has not been comprehensively addressed. The novelty of this work lies in the formulation and analysis of the dynamic response of thin plates subjected to two inertia-distributed loads moving in opposite directions with variable velocities, capturing both acceleration and deceleration phases. This configuration is more representative of real world scenarios such as opposing vehicular or train movement on bridge decks and bidirectional conveyor systems.

This paper is aimed at obtaining the results of response on thin plates subjected to inertial distributed loads moving with variable velocity in opposite directions.

2 Preliminaries

Definition 2.1 (Inertia Loads [4])
Inertia load is the resisting force developed in a mass due to its resistance to a change in motion.

Definition 2.2 (Distributed Loads [8])
A distributed load is a load applied continuously along a length, area, or volume of a structural element rather than at a single point.

Definition 2.3 (Variable Velocity [16])
Variable velocity is the velocity of an object when the speed is changing with time.

Definition 2.4 (Chebyshev Collocation Method [2])
The Chebyshev Collocation Method transforms a differential equation into a system of algebraic equations by enforcing the equation to be satisfied at specific collocation points in the domain.

3 Formulation of Solution

The proposed research study is motivated by the fact that the vibration of plates subjected to a pair of distributed line loads moving in opposite directions with variable velocities in an arbitrary trajectory has not been investigated.

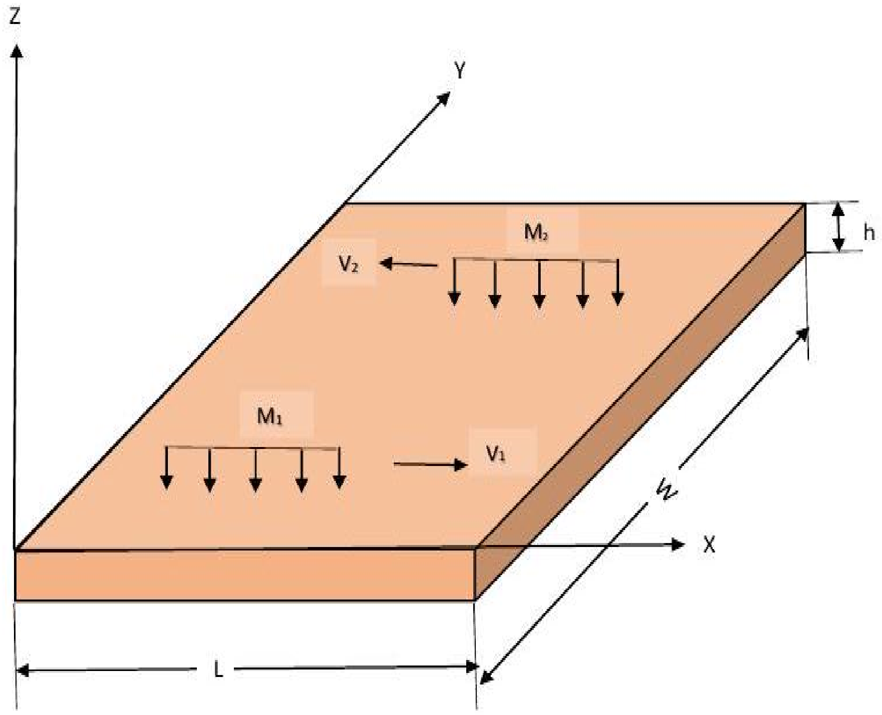

Figure 1: Thin plate under a pair of rectangular loads.

3.1 Problem Assumption

1. The plate is assumed to be a thin rectangular plate.
2. The plate is undamped and homogenous.
3. There is permanent contact condition between the moving loads and the plate surface during the whole cause of the load movement.
4. The plate is traversed by two opposite traveling loads.
5. The loads are assumed to be concentrated along x direction and distributed along y direction.

Regarding the classical plate theory, the deflection $w(x, y)$ is given by

$$D\nabla^4 w(x, y) = p(x, y), \tag{1}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}; \tag{2}$$

$$p(x, y) = \text{ load identity;}$$

$$D = \text{ flexural rigidity given by}$$

$$D = \frac{Eh^3}{12\left(1 - v^2\right)} \tag{3}$$

for which

$$E = \text{ Young's modulus;}$$

$$h = \text{ plate's thickness;}$$

$$v = \text{ Poisson's ratio.}$$

For the problem absorption above, we have

$$\begin{cases} D\nabla^4 w(x,y,t) + m\frac{\partial^2 w(x,y,t)}{\partial t^2} = \sum_{k=1}^{2} P_k \delta\left(x - x_k\right)\left(H\left(y - y_k + \frac{\mu}{2}\right) - H\left(y - y_k - \frac{\mu}{2}\right)\right) \\ P_k = \frac{1}{\mu}\left\{M_k g + M_k \frac{d^2}{dt^2} w\left(x_k(t), y_k(k), t\right)\right\} \end{cases}, \tag{4}$$

where

m = mass per unit area of the plate,

t = time,

M_k = mass per unit length of the load,

$\delta(\cdot)$ = Direc delta function,

μ = load length,

$H(\cdot)$ = Heavside function,

g = acceleration due to gravity,

P_k = contact force of the traveling load,

$(x_k(t), y_k(t))$ = location of the traveling loads at time t,

k = number of load.

Rewriting equation (4)

$$D\nabla^4 w(x,y,t) + m\frac{\partial^2 w(x,y,t)}{\partial t^2} = \sum_{k=1}^{2} \frac{1}{\mu}\left(M_k g + M_k \frac{d^2}{dt^2} w\left(x_k(t), y_k(t), t\right)\right) \tag{5}$$
$$\cdot\, \delta\left(x - x_k\right) \cdot \left(H\left(y - y_k - \frac{\mu}{2}\right) - H\left(y - y_k - \frac{\mu}{2}\right)\right).$$

Expanding the RHS of equation (5)

$$\begin{aligned} \frac{d^2}{dt^2} w\left(x_k(t), y_k(t), t\right) &= \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2}\left(\frac{dx_k}{dt}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{dy_k}{dt}\right)^2 \\ &+ \frac{2\partial^2 w}{\partial x \partial y}\left(\frac{dx_k}{dt}\right)\left(\frac{dy_k}{dt}\right) + \frac{2\partial^2 w}{\partial x \partial t}\left(\frac{dx_k}{dt}\right) + \frac{2\partial^2 w}{\partial y \partial t}\left(\frac{dy_k}{dt}\right) \\ &+ \frac{\partial w}{\partial x}\left(\frac{d^2 x_k}{dt^2}\right) + \frac{dw}{dy}\left(\frac{d^2 y_k}{dt^2}\right). \end{aligned} \tag{6}$$

Based on the problem assumption, the motion path is along x-direction and moving at a constant speed, no position change in y-direction therefore equation (6) is reduced as

$$\frac{d^2}{dt^2} w\left(x_k(t), y_k(t) t\right) = \frac{\partial^2 w}{\partial t^2} + 2U \frac{\partial^2 w}{\partial x \partial t} + U_k^2 \frac{\partial^2 w}{\partial x^2}, \tag{7}$$

where U_k is the velocity of the moving load.

Put equation (7) into (5) and expand the summation.

$$\begin{aligned} & D\nabla^4 w(x,y,t) + m\frac{\partial^2 w(x,y,t)}{\partial t^2} = \frac{1}{\mu}\left(M_1 g + M_1\left(\frac{\partial^2 w}{\partial t^2} + 2U_1\frac{\partial^2 w}{\partial x \partial t} + U_1^2\frac{\partial^2 w}{\partial x^2}\right)\right) \\ & \cdot \delta\left(x - x_1\right) \cdot \left(H\left(y - y_1 - \frac{\mu}{2}\right) - H\left(y - y_1 - \frac{\mu}{2}\right)\right) + \\ & \frac{1}{\mu}\left(M_2 g + M_2\left(\frac{\partial^2 w}{\partial t^2} + 2U_2\frac{\partial^2 w}{\partial x \partial t} + U_2^2\frac{\partial^2 w}{\partial x^2}\right)\right) \\ & \cdot \delta\left(x - x_2\right) \cdot \left(H\left(y - y_2 - \frac{\mu}{2}\right) - H\left(y - y_2 - \frac{\mu}{2}\right)\right). \end{aligned} \tag{8}$$

To solve equation (8), a semi analytical procedure is adopted which utilize the natural mode shapes of the continuous media. To extract such modes, the plate equation of free vibration is treated.

$$D\nabla^4 w(x,y,t) + m\frac{\partial^2 w(x,y,t)}{\partial t^2} = 0; \tag{9}$$

$$D\nabla^4 w(x,y,t) = -m\frac{\partial^2 w(x,y,t)}{\partial t^2}; \tag{10}$$

$$\text{Put } w(x,y,t) = \sum_{m=1}^{N}\sum_{n=1}^{N} w_m(x) w_n(y) e^{iwt} \quad \text{into (10).}$$

On simplification,

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\left\{D\left(w_m^{iv}(x)w_n(y) + 2w_m^{ii}(x)w_n^{ii}(y) + w_m(x)w_n^{iv}(y)\right)\right\} = \sum_{m=1}^{N}\sum_{n=1}^{N}\lambda_{mn}N_m(x)w_n(y)m, \tag{11}$$

where λ_{mn} = Eigen values for some specific boundary conditions.

Assuming a separation variables solution in the form of a series,

$$w(x,y,t)=\sum_{m=1}^{N}\sum_{n=1}^{N}T_{mn}(t)w_m(x)w_n(y), \tag{12}$$

where $w_m(x)$ and $w_n(y)$ are the fundamental mode shapes, m and n are the number of contributed modes and $T_{mn}(t)$ are the unknown functions of time.

Using equations (11) and (12) in (8)

$$\begin{aligned}&\sum_{m=1}^{N}\sum_{n=1}^{N}T_{mn}(t)W_m(x)W_n(y)\lambda_{mn}M+\sum_{m=1}^{N}\sum_{n=1}^{N}\ddot{T}_{mn}(t)W_m(x)W_n(y)M=\sum_{m=1}^{N}\sum_{n=1}^{N}\\&\left(\frac{1}{\mu}\left(M_1g+M_1\left(\ddot{T}_{mn}(t)W_m\left(x_1\right)W_n\left(y_1\right)+2U_1\dot{T}_{mn}(t)W_m'\left(x_1\right)W_n\left(y_1\right)\right.\right.\right.\\&\left.\left.+U_1^2T_{mn}(t)W_m''\left(x_1\right)W_n\left(y_1\right)\right)\delta\left(x-x_1\right)\cdot\left(H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)\right)\right)\\&+\frac{1}{\mu}\left(M_2g+M_2\left(\ddot{T}_{mn}(t)W_m\left(x_1\right)W_n\left(y_2\right)+2U_2\dot{T}_{mn}(t)W_m'\left(x_2\right)W_n\left(y_2\right)\right.\right.\\&\left.\left.\left.+U_2^2T_{mn}(t)W_m''\left(x_2\right)W_n\left(y_2\right)\right)\delta\left(x-x_2\right)\cdot\left(H\left(y-y_2-\frac{\mu}{2}\right)-H\left(y-y_2-\frac{\mu}{2}\right)\right)\right)\right).\end{aligned} \tag{13}$$

Multiply both sides of the equations by $w_i(x)w_j(y)$ taking the double integral of both side along the length and breadth of the plate and using the properties of orthogonal functions $w_m(x)$ and $w_n(x)$.

$$\begin{aligned}&\sum_{m=1}^{N}\sum_{n=1}^{N}m\lambda_{mn}T_{mn}(t)\int_0^l\int_0^b W_m(x)W_n(y)W_i(x)W_j(y)dxdy\\&+\sum_{m=1}^{N}\sum_{n=1}^{N}m\ddot{T}_{mn}(t)\int_0^l\int_0^b W_m(x)W_n(y)W_i(x)W_j(y)dxdy=\\&\sum_{m=1}^{N}\sum_{n=1}^{N}\left(\frac{1}{\mu}\left(M_1g\int_0^l\int_0^b W_i(x)W_j(y)dxdy\right.\right.\\&+M_1\ddot{T}_{mn}(t)\int_0^l\int_0^b W_m\left(x_1\right)W_n\left(y_1\right)W_i(x)W_j(y)dxdy+\\&2M_1U_1\dot{T}_{mn}(t)\int_0^l\int_0^b W_m'\left(x_1\right)W_n\left(y_1\right)W_i(x)W_j(y)dxdy\end{aligned}$$

$$
\begin{aligned}
&+M_1U_1^2T_{mn}(t)\int_0^l\int_0^b W''_m(x_1)\,W_n(y_1)\,W_i(x)W_j(y)dxdy\Big) \\
&\delta(x-x_1)\cdot\left(H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)\right) \\
&+\frac{1}{\mu}\Bigg(M_2g\int_0^l\int_0^b W_i(x)W_j(y)dxdy+M_2\ddot{T}_{mn}(t)\int_0^l\int_0^b W_m(x_2)\,W_n(y_2)\,W_i(x)W_j(y)dxdy+ \\
&2M_2U_2\dot{T}_{mn}(t)\int_0^l\int_0^b W'_m(x_2)\,W_n(y_2)\,W_i(x)W_j(y)dxdy+ \\
&M_2U_2^2T_{mn}(t)\int_0^l\int_0^b W''_m(x_2)\,W_n(y_2)\,W_i(x)W_j(y) \\
&dxdy\cdot\delta(x-x_1)\cdot\left(H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)\right)\Bigg)\Bigg), \qquad (14)
\end{aligned}
$$

where l and b are the length and width of the plate.

Introducing the orthogonal shape functions property

$$
\begin{aligned}
&\int_A\int w_m(x)w_n(y)w_i(x)w_j(y)dA=\delta_{ij}=\begin{cases}0 & m\neq n, i\neq j\\ 1 & m=n, i=j\end{cases} \\
&\sum_{m=1}^{N}\sum_{n=1}^{N}\left(m\lambda_{mn}T_{mn}(t)+mT_{mn}(t)\right)\delta_{ij}=\sum_{m=1}^{N}\sum_{n=1}^{N}\Bigg(\frac{1}{\mu}\Bigg(M_1g\int_0^l W_i(x_1)\,\delta(x-x_1)\,dx\int_0^b W_j(y_1) \\
&\cdot H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)dy+M_1\ddot{T}_{mn}(t)\int_0^l W_m(x_1)\,W_i(x_1)\,\delta(x-x_1)\,dx \\
&\int_0^b W_n(y_1)\,W_j(y_1)\cdot H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)dy+ \\
&2M_1U_1\dot{T}_{mn}(t)\int_0^l W'_m(x_1)\,W_i(x_1)\,\delta(x-x_1)\,dx \\
&\int_0^b W_n(y_1)\,W_j(y_1)\cdot H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)dy+ \\
&M_1U_1^2T_{mn}(t)\int_0^l W''_m(x_1)\,W_i(x_1)\,\delta(x-x_1)\,dx
\end{aligned}
$$

$$\int_0^b W_n(y_1)\,W_j(y_1)\cdot H\left(y-y_1+\frac{\mu}{2}\right)-H\left(y-y_1-\frac{\mu}{2}\right)dy\Big)+$$
$$\frac{1}{\mu}\Big(M_2 g\int_0^l W_i(x_2)\,\delta(x-x_1)\,dx\int_0^b W_j(y_2)$$
$$\cdot H\left(y-y_2+\frac{\mu}{2}\right)-H\left(y-y_2-\frac{\mu}{2}\right)dy+M_2\ddot{T}_{mn}(t)\int_0^l W_m(x_2)\,W_i(x_2)\,\delta(x-x_2)\,dx$$
$$\int_0^b W_n(y_2)\,W_j(y_2)\cdot H\left(y-y_2+\frac{\mu}{2}\right)-H\left(y-y_2-\frac{\mu}{2}\right)dy+$$
$$2M_2U_2\dot{T}_{mn}(t)\int_0^l W_m'(x_2)\,W_i(x_2)\,\delta(x-x_2)\,dx$$
$$\int_0^b W_n(y_2)\,W_j(y_2)\cdot H\left(y-y_2+\frac{\mu}{2}\right)-H\left(y-y_2-\frac{\mu}{2}\right)dy+$$
$$M_2U_2^2T_{mn}(t)\int_0^l W_m''(x_2)\,W_i(x_2)\,\delta(x-x_2)\,dx$$
$$\int_0^b W_n(y_2)\,W_j(y_2)\cdot H\left(y-y_2+\frac{\mu}{2}\right)-H\left(y-y_2-\frac{\mu}{2}\right)dy\Big)\Big). \tag{16}$$

Using the Dirac delta

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\left(m\lambda_{mn}T_{mn}(t)+mT_{mn}(t)\right)\delta_{ij}=\sum_{m=1}^{N}\sum_{n=1}^{N}\Big(\frac{1}{\mu}\Big(M_1 gW_i(x_1)\int_{y_1-\frac{\mu}{2}}^{y_1+\frac{\mu}{2}}W_j(y_1)\,dy$$
$$+M_1\ddot{T}_{mn}(t)W_m(x_1)\,W_i(x_1)\int_{y_1-\frac{\mu}{2}}^{y_1+\frac{\mu}{2}}W_n(y_1)\,W_j(y_1)\,dy$$
$$+2M_1U_1\dot{T}_{mn}(t)W_m'(x_1)\,W_i(x_1)\int_{y_1-\frac{\mu}{2}}^{y_1+\frac{\mu}{2}}W_n(y_1)\,W_j(y_1)\,dy$$
$$+M_1U_1^2T_{mn}(t)W_m''(x_1)\,W_i(x_1)\int_{y_1-\frac{\mu}{2}}^{y_1+\frac{\mu}{2}}W_n(y_1)\,W_j(y_1)\,dy\Big)+$$
$$\frac{1}{\mu}\Big(M_2 gW_i(x_2)\int_{y_2-\frac{\mu}{2}}^{y_2+\frac{\mu}{2}}W_j(y_2)\,dy+M_2\ddot{T}_{mn}(t)W_m(x_2)\,W_i(x_2)\int_{y_2-\frac{\mu}{2}}^{y_2+\frac{\mu}{2}}W_n(y_2)\,W_j(y_2)\,dy$$
$$+2M_2U_2\dot{T}_{mn}(t)W_m'(x_2)\,W_i(x_2)\int_{y_2-\frac{\mu}{2}}^{y_2+\frac{\mu}{2}}W_n(y_2)\,W_j(y_2)\,dy$$
$$+M_2U_2^2T_{mn}(t)W_m''(x_2)\,W_i(x_2)\int_{y_2-\frac{\mu}{2}}^{y_2+\frac{\mu}{2}}W_n(y_2)\,W_j(y_2)\,dy\Big)\Big). \tag{17}$$

For simply supported rectangular plates the edges conditions can be expressed as:

$$w(0,y,t) = w(l,y,t) = \frac{\partial^2 w(0,y,t)}{\partial x^2} = \frac{\partial^2 w(l,y,t)}{\partial x^2} = 0, \tag{18}$$

$$w(x,0,t) = w(x,b,t) = \frac{\partial^2 w(x,0,t)}{\partial y^2} = \frac{\partial^2 w(x,b,t)}{\partial y^2} = 0. \tag{19}$$

With the initial conditions

$$w(x,y,t) = \frac{\partial w(x,y,t)}{\partial t} = 0. \tag{20}$$

The normalized deflection curves for simply supported boundary condition for a rectangular plate is

$$w_m(x)w_n(x) = \frac{2}{\sqrt{lb}} \sin\frac{m\pi x}{l} \sin\frac{n\pi y}{b}. \tag{21}$$

Using (21) in equation (11), we obtain

$$\lambda_{mn} = D\pi^4 \left(\frac{m^2}{l^2} + \frac{n^2}{b^2}\right)^2. \tag{22}$$

λ_{mn} is the eigenvalues for the boundary condition.

Similarly, using (21) in equation (17) and evaluating the integral, we have

$$\begin{aligned}
&\sum_{m=1}^{N}\sum_{n=1}^{N}\left(\lambda_{mn}T_{mn}(t) + T_{mn}(t)\right) = \sum_{m=1}^{N}\sum_{n=1}^{N}\left(\frac{M_1 g}{M\mu}\frac{2}{\sqrt{lb}}\sin\frac{i\pi u_1 t}{l}\cdot\frac{2}{\sqrt{j\pi}}\sin\frac{j\pi y_1}{b}\sin\frac{j\pi\mu}{2b}+\right.\\
&\frac{M_1}{M\mu}\ddot{T}_{mn}(t)\frac{4}{lb}\sin\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l}\left(\frac{b}{\pi p}\cos\frac{p\pi y_1}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_1}{b}\sin\frac{d\pi\mu}{2b}\right) + \frac{2M_1U_1}{M\mu}\dot{T}_{mn}(t)\\
&\frac{4m\pi}{l^2 b}\cos\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l}\left(\frac{b}{\pi p}\cos\frac{p\pi y_1}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_1}{b}\sin\frac{d\pi\mu}{2b}\right) + \frac{M_1U_1^2}{M\mu}T_{mn}(t)\frac{4m^2\pi^2}{l^3 b}\\
&\left.\sin\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l}\left(\frac{b}{\pi p}\cos\frac{p\pi y_1}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_1}{b}\sin\frac{d\pi\mu}{2b}\right)\right)\\
&+\sum_{m=1}^{N}\sum_{n=1}^{N}\left(\frac{M_2 g}{M\mu}\frac{2}{\sqrt{lb}}\sin\frac{i\pi\left(l-u_2 t\right)}{l}\cdot\frac{2}{\sqrt{j\pi}}\sin\frac{j\pi y_2}{b}\sin\frac{j\pi\mu}{2b}+\right.\\
&\frac{M_2}{M\mu}\ddot{T}_{mn}(t)\frac{4}{lb}\sin\frac{m\pi\left(l-u_2 t\right)}{l}\sin\frac{i\pi\left(l-u_2 t\right)}{l}\left(\frac{b}{\pi p}\cos\frac{p\pi y_2}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_2}{b}\sin\frac{d\pi\mu}{2b}\right)+\\
&\frac{2M_2U_2}{M\mu}\dot{T}_{mn}(t)\frac{4m\pi}{l^2 b}\cos\frac{m\pi\left(l-u_2 t\right)}{l}\sin\frac{i\pi\left(l-u_2 t\right)}{l}\left(\frac{b}{\pi p}\cos\frac{p\pi y_2}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_2}{b}\sin\frac{d\pi\mu}{2b}\right)\\
&+\frac{M_2U_2^2}{M\mu}T_{mn}(t)\frac{4m^2\pi^2}{l^3 b}\sin\frac{m\pi\left(l-u_2 t\right)}{l}\sin\frac{i\pi\left(l-u_2 t\right)}{l}\\
&\left.\left(\frac{b}{\pi p}\cos\frac{p\pi y_2}{b}\sin\frac{p\pi\mu}{2b} - \frac{b}{\pi d}\cos\frac{d\pi y_2}{b}\sin\frac{d\pi\mu}{2b}\right)\right)
\end{aligned} \tag{23}$$

subject to: $T_{mn}(0) = T_{mn}(\frac{l}{u}) = 0$

where $p = n - j, d = n + j$ and $n \neq j$

Also $x_1 = u_1 t$ and $x_2 = l - u_2 t$

For $n = j$

Equation (23) becomes

$$\Bigg(1 - \frac{4}{lb}\Bigg(\theta_1\psi_1 \sin\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l} + \theta_2\psi_2\Bigg(\sin(m\pi)\cos\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} - \sin(m\pi)\cos\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l} - \cos(m\pi)\sin\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} + \cos(m\pi)\sin\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l}\Bigg)\Bigg)\Bigg)\cdot\ddot{T}_{mn}(t) - \Bigg(\frac{8m\pi}{l^2 b}\Bigg(U_1\theta_1\psi_1\cos\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l} + U_2\theta_2\psi_2\Bigg(\cos(m\pi)\cos\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} - \cos(m\pi)\cos\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l} + \sin(m\pi)\sin\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} - \sin(m\pi)\sin\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l}\Bigg)\Bigg)\Bigg)\dot{T}_{mn}(t) + \Bigg(\lambda_{mn} + \frac{4m^2\pi^2}{l^3 b}\Bigg(U_1^2\theta_1\psi_1\sin\frac{m\pi u_1 t}{l}\sin\frac{i\pi u_1 t}{l} + U_2^2\theta_2\psi_2\Bigg(\sin(m\pi)\cos\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} - \sin(m\pi)\cos\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l} - \cos(m\pi)\sin\frac{m\pi u_2 t}{l}\sin(i\pi)\cos\frac{i\pi u_2 t}{l} + \cos(m\pi)\sin\frac{m\pi u_2 t}{l}\cos(i\pi)\sin\frac{i\pi u_2 t}{l}\Bigg)\Bigg)\Bigg)T_{mn}(t) = \frac{4b}{j\pi\sqrt{lb}}\Bigg(\theta_1 g\sin\frac{i\pi u_1 t}{l}\sin\frac{j\pi y_1}{b}\sin\frac{j\pi\mu}{2b} + \theta_2 g\Bigg(\sin(i\pi)\cos\frac{i\pi u_2 t}{l} - \cos(i\pi)\sin\frac{i\pi u_2 t}{l}\Bigg)\sin\frac{j\pi y_2}{b}\sin\frac{j\pi\mu}{2b}\Bigg) \tag{24}$$

subject to: $T_{mn}(0) = T_{mn}(\frac{l}{u}) = 0$
where:
$\theta_1 = \frac{M_1}{M\mu}$, $\theta_2 = \frac{M_2}{M\mu}$
$\psi_1 = \frac{\mu}{2} - \frac{b}{2n\pi}\cos\frac{2n\pi Y_1}{b}\sin\frac{n\pi\mu}{b}$, $\psi_2 = \frac{\mu}{2} - \frac{b}{2n\pi}\cos\frac{2n\pi Y_2}{b}\sin\frac{n\pi\mu}{b}$

3.2 Numerical Illustration

In this work, the Chebyshev collocation method was considered to solve equations (23) and (24) as it is not possible to obtain a closed form solution. The Chebyshev collocation method is a numerical method for solving differential equations that is based on the spectral method, in which the solution is represented as a series expansion in terms of a set of basis functions (Boyd, [2]). In the case of Chebyshev collocation, the basis functions are Chebyshev polynomials, which are defined in terms of the roots of the polynomial.

One advantage of the Chebyshev collocation method is its ability to achieve high accuracy with relatively few collocation points (Canuto *et al.,* [3]). This makes it an efficient method for solving differential equations, particularly those that are stiff or have singularities (Boyd, [2]). Additionally, the Chebyshev collocation method is well-suited for problems with periodic boundary conditions, as the Chebyshev polynomials are periodic functions.

To use the Chebyshev collocation method, the domain of the differential equation is transformed to the interval $[-1, 1]$, and the solution is expressed as a series expansion in terms of Chebyshev polynomials (Canuto *et al.,* [3]):

$$u(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) + \dots$$

where $T_n(x)$ is the nth Chebyshev polynomial and the coefficients an are determined by collocating the differential equation at a set of points in the interval $[-1, 1]$ (Boyd, [2]). These points, known as collocation points, are chosen to be the roots of the Chebyshev polynomial of the second kind, which are closely spaced near the ends of the interval. The motion of the load is considered under the following

- Case One: The loads are considered to be moving in a rectilinear trajectory and with a constant velocity, hence the load displacement is defined as:

$$\begin{cases} X_1(t) = v_1t, \quad Y_1(t) = 0.25b \\ X_2(t) = l - v_2t, \quad Y_2(t) = 0.75b \end{cases}. \tag{25}$$

- Case two: The loads are considered to be moving in a rectilinear trajectory and with a variable velocity, hence the load displacement is defined as:

$$\begin{cases} X_1(t) = v_1t + \frac{a_1t^2}{2}, \quad Y_1(t) = 0.25b \\ X_2(t) = l - \left(v_2t + \frac{a_2t^2}{2}\right), \quad Y_2(t) = 0.75b \end{cases}. \tag{26}$$

Table 1: Parameters of the model of the plate

Parameter	Notation	Value
Length of the plate	l	10 m
Width of the plate	b	5 m
Poisson's ratio	ν	0.2
Plate thickness	h	0.2 m
Young's modulus	E	2.109×10^7
Mass of the plate	M	1000kg

4 Results and Discussion

In this section, the deflection response of a simply supported thin rectangular plate subjected to a pair of moving line load with constant and variable velocities was presented in graphical forms. The response investigated focuses on the effect of the load lengths, mass ratio of the loads to the plate, velocities, acceleration and deceleration of the moving line loads.

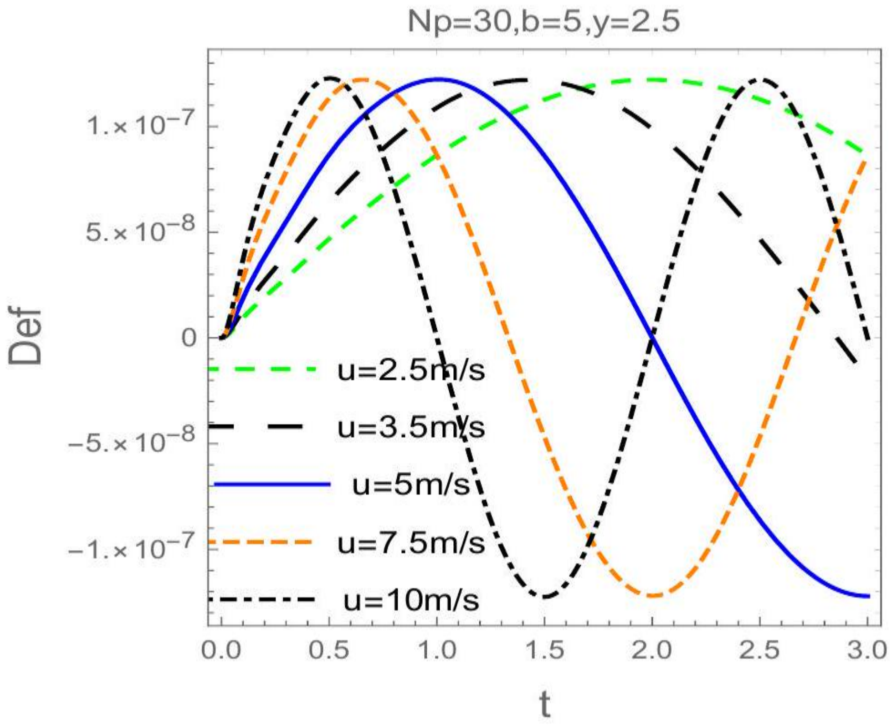

Figure 2: Shape of the deflection along the mid-plane of the plate with same velocities.

Figure 2 above represents the shape of the deflection at the center of the plate when the two line loads moves with same velocities in opposite direction for various velocities. It will be observed that the deflection increases as the velocity increases. At the lowest velocity, the deflection profile exhibits a gradual increase, reaching a relatively small peak before commencing oscillatory recovery. The response is quasi-static in nature, as the load traverses the plate slowly, allowing the structure to deform without significant dynamic amplification. For higher velocities, the deflection profiles show more frequent oscillations within the same time interval, and the time to reach maximum deflection is reduced. This behavior occurs because the moving load traverses the plate more rapidly, exciting higher vibration modes and introducing phase shifts in the response.

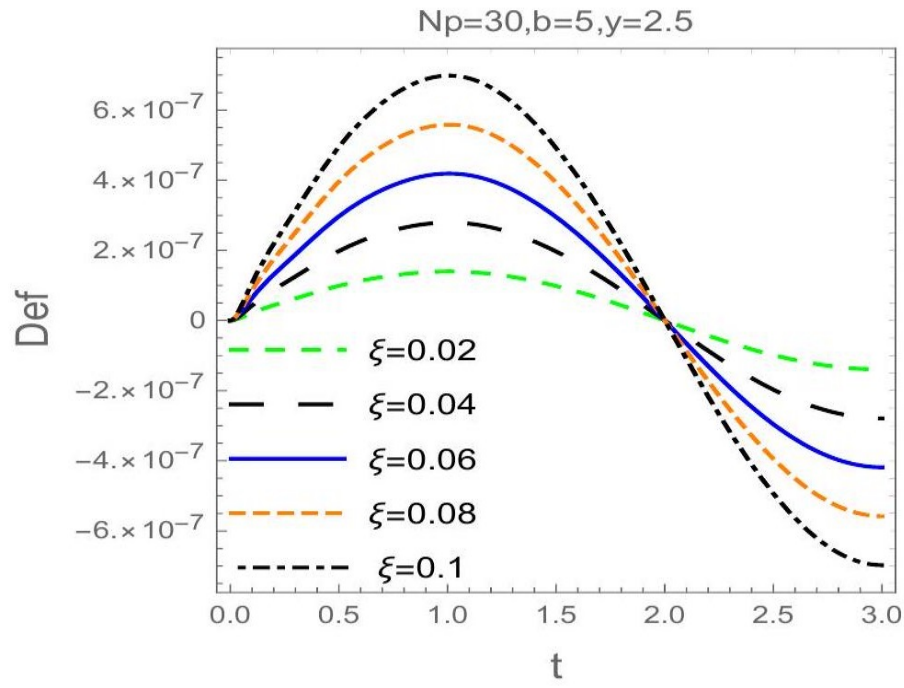

Figure 3: Shape of the deflection along the mid-plane of the plate with different mass ratio.

The effect the mass ratio on the deflection of the plate is illustrated in Figure 3. The mass ratio here represents the ratio of the moving load's mass to the equivalent modal mass of the plate. For the smallest mass ratio the plate experiences the lowest peak deflection, indicating that the moving load imparts relatively small inertial effects on the structure. As the mass ratio increases, the peak deflection amplitude rises significantly. This demonstrates the strong influence of load mass on the dynamic response. Physically, a higher mass ratio corresponds to a heavier moving load relative to the plate's stiffness and inertia. Such a load introduces greater inertial forces into the system, resulting in increased bending and larger vibratory motion. This can have significant implications for structural fatigue, serviceability, and safety in engineering systems where moving loads of varying masses are present. Figure 4 shows the effect of the length of the moving loads on the plate's deflection. At short load length, the plate experiences the highest peak deflection amplitude, indicating that the concentrated nature of the load produces stronger localized bending effects. As μ increases, the peak deflection gradually decreases. This trend is due to the distribution of the load over a larger surface area, which reduces the peak stress and deflection at any single point. For the longest load length, the deflection amplitude is the smallest among the cases considered, reflecting the load's broad footprint and reduced localized impact. Physically, shorter load lengths behave similarly to point loads, producing more intense local deformation and higher vibration amplitudes. In contrast, longer loads behave more like uniformly distributed forces, which spread their effect and thus induce smaller deflections. The oscillation patterns for all load lengths remain symmetric about the zero-deflection axis, indicating that the plate's response is primarily elastic and not permanently deformed under the given loading conditions.

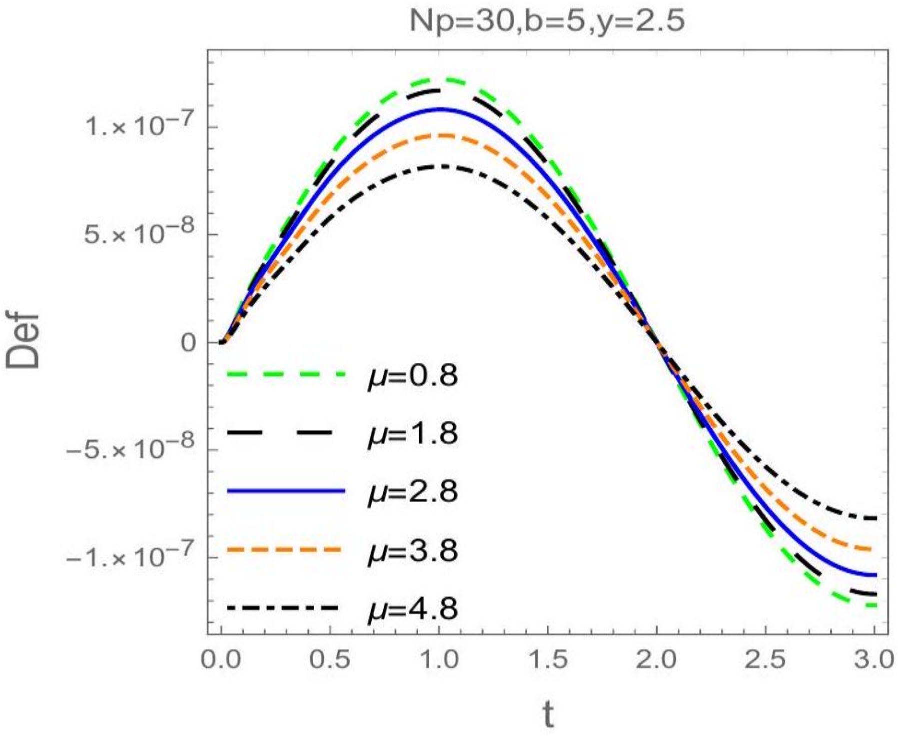

Figure 4: Shape of the deflection along the mid-plane of the plate with different load length.

Figure 5 shows the deflection behaviour of the mid-plane of the plate when subjected to a pair of moving line load in opposite directions with variable velocity. The deflection with time for a fixed acceleration and different load velocities. It can be clearly seen that maximum magnitude of the dynamic deflection increases with increasing initial load velocity, showing the same trend as in Figure 2.

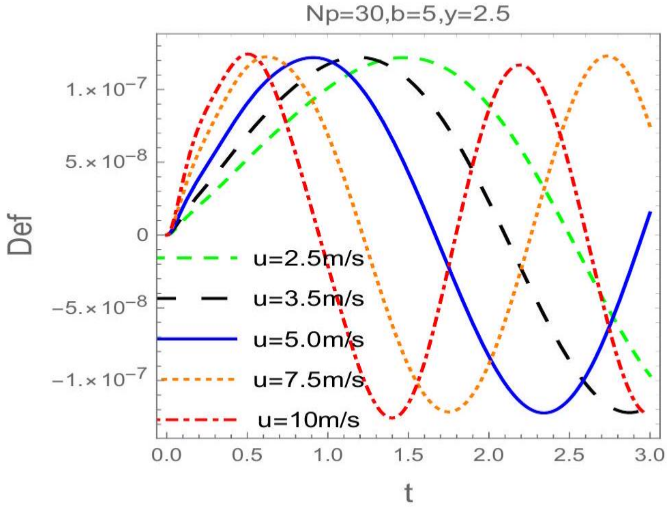

Figure 5: Deflection of plate with different velocities moving with variable velocity.

Figure 6 and Figure 7 show the effect of acceleration and deceleration on the dynamic deflection of the thin plate. The effect of load acceleration on the deflection is significant. It can be observed that several peaks can be observed and the larger the acceleration the wider the deflection are spread out. The deceleration effect can be clearly seen in Figure 7, the dynamic deflection increases for some times and later decreases. The magnitude of the dynamic deflection decreases with increase in deceleration.

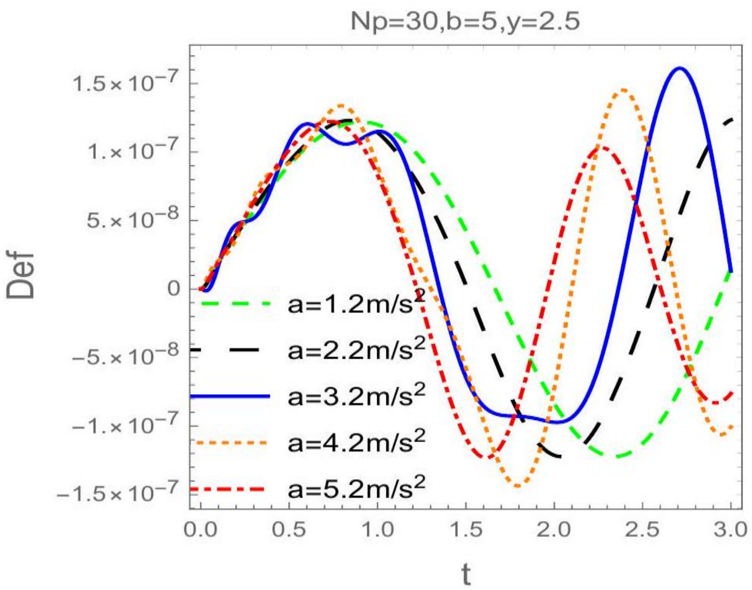

Figure 6: Deflection of plate with different acceleration moving with variable velocity.

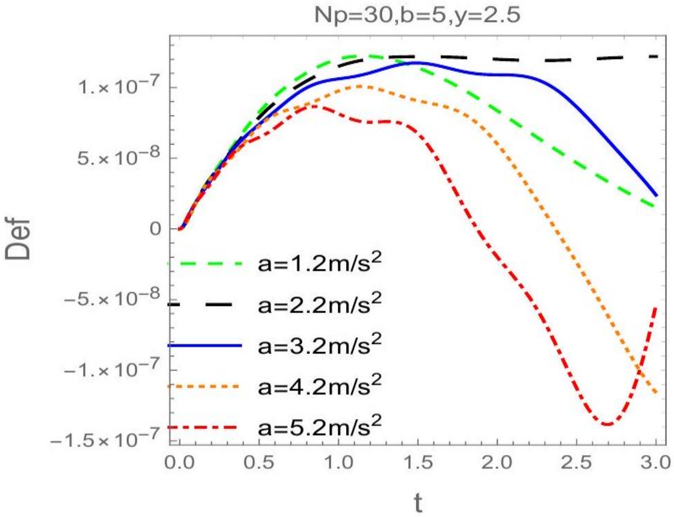

Figure 7: Deflection of plate with different deceleration moving with variable velocity.

5 Conclusion

In this work, the dynamic deflection characteristic of a plate which is subjected to a pair of inertia distributed loads, moving with variable velocities and in opposite direction was investigated. The model was formulated based on the thin plate theory. The governing equation obtained for the behavior of the model was reduced from a partial differential equation to an ordinary differential equation using a series solution for the dynamic deflection in terms of the normal modes. The reduced ODE was solved using chebyshev collocation method. The effects of acceleration, deceleration, initial velocity, mass ratio and load length on the dynamic deflection response have been investigated. The main conclusions of this work is as follows:

- The deflection amplitude of the thin plate increases with increasing velocity of the distributed moving loads in both directions. This trend is more pronounced at higher velocities, suggesting that inertial effects become dominant as the load motion approaches the critical velocity region.
- The interaction of inertia loads moving in opposite directions results in more complex deflection patterns compared to single-direction motion. The presence of counter-moving loads alters the dynamic amplification, producing regions of constructive and destructive interference in the plate's vibration response.
- An increase in the load mass ratio produces a proportional increase in the maximum deflection of the plate. This is due to the enhanced inertial force imparted to the plate by heavier moving loads,

thereby increasing the dynamic response amplitude.

- The combined effects of varying velocities and opposing motion directions amplify the dynamic response nonlinearly. The results indicate that both the magnitude and the location of peak deflections are sensitive to these factors, which is crucial for predicting structural performance under real-life loading scenarios.

- The findings are particularly relevant to engineering applications involving bidirectional moving loads such as vehicular-pedestrian interactions on bridges, automated material handling systems, and high-speed transport over plate-like structures where understanding the coupled influence of velocity, mass, and motion direction is essential for design safety and serviceability.

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