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Examining New Convex Integral Inequalities

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Abstract

In this article, we introduce new convex integral inequalities based on a contemporary and adaptable analytical framework. These inequalities can handle composed functions, integral expressions, and ratio-type functionals, which make them applicable to a wide range of analysis problems. Our main result, in particular, complements a recent theorem from the literature by providing a valuable and non-trivial lower bound. The proofs are presented in full detail to ensure mathematical rigor, clarity, and reproducibility.

1 Introduction

This study is based on the fundamental concept of convexity in real-valued functions, as outlined below. Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ with $b > a$. A function $\phi : [a, b] \mapsto \mathbb{R}$ is said to be convex if, for any $x, y \in [a, b]$ and any $\eta \in [0, 1]$, the following inequality holds:

$$\phi(\eta x + (1 - \eta)y) \leq \eta\phi(x) + (1 - \eta)\phi(y).$$

If the function ϕ is twice differentiable, this inequality is equivalent to

$$\phi''(x) \geq 0$$

for any $x \in [a, b]$, which in turn implies that the first derivative ϕ' is non-decreasing. For a detailed treatment of convex functions and their properties, see [1–12].

Convex integral inequalities, in which convex functions play a central role, are a well-established subject in the fields of real and functional analysis. These inequalities are instrumental in deriving bounds for integrals, studying function spaces, and analyzing variational problems. They also feature in theories of means, probability, optimization, and partial differential equations. In this article, we contribute to the

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development of a specific convex integral inequality involving composed functions and derivatives of convex functions. This inequality was originally introduced in [11, 12], and is formally stated below.

Theorem 1.1. *[11, 12] Let $f : [0,1] \mapsto [0,1]$ be a continuous function and $\phi : [0,1] \mapsto \mathbb{R}$ be a twice differentiable convex function such that $\phi(0) = 0$. Then the following inequality holds:*

$$\phi\left(\int_0^1 f(t)dt\right) \leq \int_0^1 f(t)\phi'(t)dt.$$

This result can be viewed as an alternative to the classical Jensen integral inequality given by

$$\phi\left(\int_0^1 f(t)dt\right) \leq \int_0^1 \phi\left(f(t)\right)dt.$$

The advantage of Theorem 1.1 lies in its structural simplicity and its potential to produce sharper upper bounds in contexts where the composition $\phi(f)$ is difficult to handle or less informative. Moreover, by incorporating ϕ', the inequality can more directly exploit the local behavior of ϕ, making it particularly effective in applications where derivative-based upper bounds are preferable or more accessible than those based on the full convex image of ϕ. This inequality also opens the door to various modifications, which are discussed in Sections 2 and 3 of this article. Section 4 provides conclusions.

2 Main Result

The result below completes Theorem 1.1 by offering a lower bound of the main quantity, i.e., $\phi\left(\int_0^1 f(x)dx\right)$.

Theorem 2.1. *Let $f : [0,1] \mapsto [0,1]$ be a continuous function and $\phi : [0,1] \mapsto \mathbb{R}$ be a twice differentiable convex function such that $\phi(0) = 0$. Then the following inequality holds:*

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x}dx \leq \phi\left(\int_0^1 f(x)dx\right).$$

Proof. Since ϕ is convex with $\phi(0) = 0$, $\phi(x)/x$ is non-decreasing, as demonstrated below. Using standard differentiation rules, we have

$$\left(\frac{\phi(x)}{x}\right)' = \frac{x\phi'(x) - \phi(x)}{x^2}.$$

Since ϕ' is non-decreasing (because ϕ is twice differentiable and convex) and $\phi(0) = 0$, we have

$$\phi(x) = \int_0^x \phi'(t)dt + \phi(0) = \int_0^x \phi'(t)dt \leq \phi'(x)\int_0^x dt = x\phi'(x).$$

This implies that

$$\left(\frac{\phi(x)}{x}\right)' = \frac{x\phi'(x) - \phi(x)}{x^2} \geq 0,$$

validating the claim that $\phi(x)/x$ is non-decreasing. As a result, we have

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x}dx \leq \int_0^{\int_0^1 f(x)dx} \frac{\phi\left(\int_0^1 f(x)dx\right)}{\int_0^1 f(x)dx}dt = \frac{\phi\left(\int_0^1 f(x)dx\right)}{\int_0^1 f(x)dx} \int_0^{\int_0^1 f(x)dx} dt$$

$$= \frac{\phi\left(\int_0^1 f(x)dx\right)}{\int_0^1 f(x)dx} \int_0^1 f(x)dx = \phi\left(\int_0^1 f(x)dx\right).$$

This ends the proof of Theorem 2.1. □

Merging the inequalities in Theorems 1.1 and 2.1, we get the following double inequality:

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x}dx \leq \phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(t)\phi'(t)dt.$$

This result thus completes the theory in [11, 12]. Its innovation lies in its ability to handle composed functions, integral expressions, and ratio-type functionals. In particular, the presence of the integral of f in the upper limit is a characteristic similar to the Steffensen integral inequality (see [13–16]). However, the results are intrinsically of a different nature.

For an example of illustration, if we set $\phi(x) = \tan((\pi/4)x)$, $x \in [0, 1]$ which is twice differentiable and convex such that $\phi(0) = 0$, then Theorem 2.1 gives

$$\int_0^{\int_0^1 f(x)dx} \frac{\tan((\pi/4)x)}{x}dx = \int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x}dx \leq \phi\left(\int_0^1 f(x)dx\right)$$

$$= \tan\left(\frac{\pi}{4}\left(\int_0^1 f(x)dx\right)\right).$$

In this case, we obtain a manageable upper bound of a complex integral.

3 Secondary Results

The proposition below is about a new convex integral inequality in the vein of Theorem 2.1, but with a more sophisticated functional structure.

Proposition 3.1. *Let $f : [0, 1] \mapsto [0, 1]$ be a continuous function and $\phi : [0, 1] \mapsto \mathbb{R}$ be a twice differentiable convex function such that $\phi(0) = 0$. Then the following inequality holds:*

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x}dx \leq \int_0^1 f(t)\frac{\phi(t)}{t}dt.$$

Proof. For any $t \in [0,1]$, let us set

$$T(t) = \int_0^{\int_0^t f(x)dx} \frac{\phi(x)}{x} dx,$$

noticing that the main term of interest satisfies

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = T(1).$$

Using standard differentiation rules, we get

$$T'(t) = \left(\int_0^{\int_0^t f(x)dx} \frac{\phi(x)}{x} dx \right)' = \left(\int_0^t f(x)dx \right)' \left[\left(\int_0^y \frac{\phi(x)}{x} dx \right)'(y) \bigg|_{y=\int_0^t f(x)dx} \right]$$
$$= f(t) \frac{\phi\left(\int_0^t f(x)dx \right)}{\int_0^t f(x)dx}. \tag{1}$$

Noticing that

$$T(0) = \int_0^{\int_0^0 f(x)dx} \frac{\phi(x)}{x} dx = \int_0^0 \frac{\phi(x)}{x} dx = 0,$$

and using Equation (1), we obtain

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = T(1) = \int_0^1 T'(t)dt + T(0) = \int_0^1 T'(t)dt$$
$$= \int_0^1 f(t) \frac{\phi\left(\int_0^t f(x)dx \right)}{\int_0^t f(x)dx} dt. \tag{2}$$

Since ϕ is convex with $\phi(0) = 0$, $\phi(x)/x$ is non-decreasing (see the first part of the proof of Theorem 2.1), and, since $f(t) \in [0,1]$ for any $t \in [0,1]$, we have $\int_0^t f(x)dx \geq 0$ and $\int_0^t f(x)dx \leq \int_0^t dx = t$. These results combined with Equation (2) give

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = \int_0^1 f(t) \frac{\phi\left(\int_0^t f(x)dx \right)}{\int_0^t f(x)dx} dt \leq \int_0^1 f(t) \frac{\phi(t)}{t} dt.$$

This ends the proof of Proposition 3.1. □

The inequality presented in Proposition 3.1 thus offers an elegant convex integral inequality. It involves the ratio function $\phi(x)/x$, which naturally arises in various analytical contexts, particularly when studying logarithmic convexity or dealing with normalized forms of convex functions.

For an example of illustration, if we set $\phi(x) = \tan((\pi/4)x)$, $x \in [0,1]$, which is twice differentiable and convex such that $\phi(0) = 0$, then Proposition 3.1 gives

$$\int_0^{\int_0^1 f(x)dx} \frac{\tan((\pi/4)x)}{x} dx = \int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx \leq \int_0^1 f(t)\frac{\phi(t)}{t} dt$$
$$= \int_0^1 f(t)\frac{\tan((\pi/4)t)}{t} dt.$$

The proposition below presents an alternative of Proposition 3.1 under a monotonicity assumption on f. A lower bound is also established.

Proposition 3.2. *Let $f : [0,1] \mapsto [0,1]$ be a non-decreasing function and $\phi : [0,1] \mapsto \mathbb{R}$ be a twice differentiable convex function such that $\phi(0) = 0$. Then the following inequality holds:*

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx \leq \int_0^1 \frac{\phi(tf(t))}{t} dt.$$

Furthermore, if $f(0) > 0$, we have

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx \geq \frac{1}{f(0)} \int_0^1 f(t)\frac{\phi(tf(0))}{t} dt.$$

Proof. We can reuse Equation (2), that is

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = \int_0^1 f(t)\frac{\phi\left(\int_0^t f(x)dx\right)}{\int_0^t f(x)dx} dt. \tag{3}$$

Since ϕ is convex with $\phi(0) = 0$, $\phi(x)/x$ is non-decreasing (see the first part of the proof of Theorem 2.1), and, since $f(t) \in [0,1]$ for any $t \in [0,1]$ with f non-decreasing, we have $\int_0^t f(x)dx \geq 0$ and $\int_0^t f(x)dx \leq f(t)\int_0^t dx = tf(t)$. These results combined with Equation (3) and the fact that f is non-negative give

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = \int_0^1 f(t)\frac{\phi\left(\int_0^t f(x)dx\right)}{\int_0^t f(x)dx} dt \leq \int_0^1 f(t)\frac{\phi(tf(t))}{tf(t)} dt$$
$$= \int_0^1 \frac{\phi(tf(t))}{t} dt. \tag{4}$$

This is the first stated inequality.

Furthermore, if $f(0) > 0$, then we have $\int_0^t f(x)dx \geq f(0)\int_0^t dx = tf(0)$. With the same arguments than above, we get

$$\int_0^{\int_0^1 f(x)dx} \frac{\phi(x)}{x} dx = \int_0^1 f(t)\frac{\phi\left(\int_0^t f(x)dx\right)}{\int_0^t f(x)dx} d \geq \int_0^1 f(t)\frac{\phi(tf(0))}{tf(0)} dt$$
$$= \frac{1}{f(0)} \int_0^1 f(t)\frac{\phi(tf(0))}{t} dt.$$

This ends the proof of Proposition 3.2. □

Another way of summarizing this proposition is as follows:

$$\frac{1}{f(0)}\int_0^1 f(t)\frac{\phi(tf(0))}{t}dt \leq \int_0^{\int_0^1 f(x)dx}\frac{\phi(x)}{x}dx \leq \int_0^1 \frac{\phi(tf(t))}{t}dt.$$

We end this section with some important comments.

In Proposition 3.2, if f is assumed to be non-increasing instead of non-decreasing, the two inequalities are reversed; we are able to prove that

$$\int_0^1 \frac{\phi(tf(t))}{t}dt \leq \int_0^{\int_0^1 f(x)dx}\frac{\phi(x)}{x}dx \leq \frac{1}{f(0)}\int_0^1 f(t)\frac{\phi(tf(0))}{t}dt.$$

In Theorem 2.1 and Propositions 3.1 and 3.2, if ϕ is supposed to be concave instead of convex, then the corresponding final inequalities are reversed. The details are omitted for the sake of brevity.

4 Conclusion

In this article, we have presented new convex integral inequalities. In particular, we have determined a lower bound that complements and strengthens the main theorem in [11, 12]. The other inequalities are characterized by the use of composed functions, integral expressions and ratio-type functionals. The detailed proofs provided ensure that the results are rigorous and reproducible, providing a solid foundation for future research and applications.

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