

Uniformity in d -algebras

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Abstract

In this paper, we consider a collection of d^* -ideals of a d -algebra D . We use the connotation of congruence relation regard to d^* -ideals to construct a uniformity which induces a topology on D . We debate the properties of this topology.

1. Introduction

Yoon and Kim [4] and Meng and Jun [5] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI-algebras. In [2], [3] Bourbaki and Sims introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [6] introduced the notion of d -algebras which is another generalization of BCK-algebras, and investigated relations between d -algebras and BCK-algebras. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have

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been made to study the topological structures making the star operation of d -algebra continuous. Theories of topological groups, topological rings and topological modules are well known and still investigated by many mathematicians. Even topological universal algebraic structures have been studied by some authors.

In this paper, we address the issue of attaching topologies to d -algebras in as natural a manner as possible. It turns out that we may use the class of d -ideals of a d -algebra as the underlying structure whence a certain uniformity and thence a topology is derived which provides a natural connection between the notion of a d -algebra and the notion of a topology in that we are able to conclude that in this setting a d -algebra becomes a topological d -algebra.

2. Preliminaries

Definition 2.1 [6]. A d -algebra is a non-empty set D with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms:

$$(I) \quad x * x = 0,$$

$$(II) \quad 0 * x = 0,$$

$$(III) \quad x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y$$

for all x, y in D .

A non-empty subset S of a d -algebra D is called a *sub d -algebra* of D if it is closed under the d -operation.

A non-empty subset I of a d -algebra D is called a *BCK-ideal* of D if

$$(D1) \quad o \in I$$

$$(D2) \quad x \in I \text{ and } y * x \in I \text{ imply } y \in I \text{ for all } x, y \in D.$$

And I is called a *d -ideal* of D if it satisfies (D1) and

$$(D3) \quad x \in I \text{ and } y \in D \text{ imply } x * y \in I, \text{ i.e., } I * D \in I.$$

A d -ideal I of D is called a *$d^\#$ -ideal* of D if, for arbitrary $x, y, z \in D$,

$$(D4) \quad x * z \in I \text{ whenever } x * y \in I \text{ and } y * z \in I.$$

If a $d^\#$ -ideal I of a d -algebra D satisfies

$$(D5) \quad x * y \in I \text{ and } y * x \in I \text{ imply } (x * z) * (y * z) \in I \text{ and } (z * x) * (z * y) \in I$$

for all $x, y, z \in D$, then we say that I is a d^* -ideal of D .

Theorem 2.2 [6]. *In a d -algebra D , any BCK-ideal of D is a d -subalgebra of D .*

Lemma 2.3 [6]. *In a d -algebra any d -ideal is a BCK-ideal.*

Corollary 2.4 [6]. *Any d^* -ideal of a d -algebra is a d -subalgebra.*

Definition 2.5. Let D be a d -algebra. An equivalence relation \sim on D is called a congruence if $x \sim y, u \sim v$ imply $x * u \sim y * v$, where $x, y, u, v \in D$.

Let I be a d^* -ideal of a d -algebra $(D, *, 0)$. For any x, y in D , we define $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. We claim that \sim is an equivalence relation on D . Since $0 \in I$, we have $x * x = 0 \in I$, i.e., $x \sim x$, for any $x \in D$. If $x \sim y$ and $y \sim z$, then $x * y, y * x \in I$ and $y * z, z * y \in I$. By (D4) $x * z, z * x \in I$ and hence $x \sim z$. This proves that \sim is transitive. The symmetry of \sim is trivial. By (D5) we can easily see that \sim is a congruence relation on D . We denote the congruence class containing x by $[x]_I$, i.e., $[x]_I = \{y \in X \mid x \sim y\}$. We see that $x \sim y$ if and only if $[x]_I = [y]_I$. Denote the set of all equivalence classes of D by D/I , i.e., $D/I = \{[x]_I \mid x \in D\}$ [6].

Lemma 2.6 [6]. *Let I be a d^* -ideal of a d -algebra $(D; *, 0)$. Then $I = [0]_I$.*

Theorem 2.7 [6]. *Let $(D, *, 0)$ be a d -algebra and I be a d^* -ideal of D . If we define $[x]_I * [y]_I = [x * y]_I$ for all $x, y \in D$, then $(D/I, *, 0)$ is a d -algebra, called the quotient d -algebra.*

3. Uniformity in d -algebras

From now on, D is a d -algebra, unless otherwise is stated. Let D be a non-empty set, and U and V be any subsets of $D \times D$. Define:

$$U \circ V = \{(x, y) \in D \times D \mid \text{for some } z \in D, (x, z) \in U \text{ and } (z, y) \in V\}.$$

$$U^{-1} = \{(x, y) \in D \times D \mid (y, x) \in U\}.$$

$$\Delta = (x, x)/x \in D.$$

Definition 3.1. By a *uniformity* on D , we mean a non-empty collection K of subsets of $D \times D$ which satisfies the following conditions:

(U1) $\nabla \subseteq U$ for any $U \in K$,

(U2) if $U \in K$, then $U^{-1} \in K$,

(U3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subseteq U$,

(U4) if $U, V \in K$, then $U \cap V \in K$,

(U5) if $U \in K$ and $U \subseteq V \subseteq D \times D$, then $V \in K$.

The pair (D, K) is called a *uniform structure*.

Theorem 3.2. Let I be a d^* -ideal of a d -algebra D . If we define

$$U^I = \{(x, y) \in D \times D / x * y \in I \text{ and } y * x \in I\}$$

and let

$$K^* = \{U^I / I \text{ is a } d^* \text{-ideal of } D\}.$$

Then K^* satisfies the conditions (U1) ~ (U4).

Proof. (U1) If $(x, x) \in \Delta$, then $(x, x) \in U^I$ since $x * x = 0 \in I$. Hence $\Delta \in U^I$ for any $U^I \in K^*$.

(U2) For any $U^I \in K^*$,

$$(x, y) \in U^I \leftrightarrow x * y \in I \text{ and } y * x \in I,$$

$$\leftrightarrow y \sim_I x,$$

$$\leftrightarrow (y, x) \in U^I,$$

$$\leftrightarrow (x, y) \in (U^I)^{-1}.$$

Hence $(U^I)^{-1} = U^I \in K^*$.

(U3) For any $U^I \in K^*$, the transitivity of \sim_I implies that $U^I \circ U^I \subseteq U^I$.

(U4) For any U^I and U^J in K^* , we claim that $U^I \cap U^J \in K^*$.

$$\begin{aligned} (x, y) \in U^I \cap U^J &\leftrightarrow (x, y) \in U^I \text{ and } (x, y) \in U^J, \\ &\leftrightarrow x * y; y * x \in I \cap J, \\ &\leftrightarrow x \sim_{I \cap J} y, \\ &\leftrightarrow (x, y) \in U^{I \cap J}. \end{aligned}$$

Since $I \cap J$ is a d^* -ideal of D , $U^I \cap U^J = U^{I \cap J} \in K^*$. This proves the theorem.

Theorem 3.3. Let $K = \{U \subseteq D \times D / U^I \subseteq U \text{ for some } U^I \in K^*\}$. Then K satisfies the conditions for a uniformity on D and hence the pair (D, K) is a uniform structure.

Proof. By Theorem 3.2, the collection K satisfies the conditions (U1) ~ (U4). It suffices to show that K satisfies (U5). Let $U \in K$ and $U \subseteq V \subseteq D \times D$. Then there exists a $U^I \subseteq U \subseteq V$, which means that $V \in K$. This proves the theorem.

Let $x \in D$ and $U \in K$. Define:

$$U[x] = \{y \in D / (x, y) \in U\}.$$

Theorem 3.4. Let D be a d -algebra. Then

$$T = \{G \subseteq D / \forall x \in G, \exists U \in K, U[x] \subseteq G\}$$

is a topology on D .

Proof. It is clear that \emptyset and the set D belong to T . Also from the definition, it is clear that T is closed under arbitrary unions. Finally to show that T is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist U and $V \in K$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$. Then $W \in K$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$. Therefore, $G \cap H \in T$. Thus, T is a topology on D .

Notion 3.5. For any x in D , $U[x]$ is a neighborhood of x .

Definition 3.6. Let Λ be an arbitrary family of d^* -ideals of a d -algebra D which is closed under intersection. Then the topology T comes from Theorem 3.4 is called a uniform topology on D induced by Λ .

Notation 3.7. Let Λ be a family of d^* -ideals of a d -algebra D , where Λ is closed under intersection, we denote by T_Λ the uniform topology by Λ . Especially if $\Lambda = \{I\}$, we denote it by T_I .

Example 3.8. Let $D = \{0, 1, 2, 3\}$ be a d -algebra which is not a BCK-algebra with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Then it is easy to show that $\{0\}$, $F = \{0, 3\}$, $I = \{0, 1, 2\}$ and D are the only d^* -ideals of D . We can see that $U^{\{0\}} = \Delta$, $U^F = \Delta \cup \{(0, 3), (3, 0)\}$, $U^I = \Delta \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1)\}$ and $U^D = D \times D$. Therefore, $K^* = \{U^{\{0\}}, U^F, U^I, U^D\}$ and $K = \{U \subseteq D \times D / U^I \subseteq U \text{ for some } U^I \in K^*\}$. If we take $U = U^F$, then $U[0] = U[3] = \{0, 3\}$, $U[2] = \{2\}$ and $U[1] = \{1\}$. Therefore, $T = \{G \subseteq D / \forall x \in G, \exists U \in K, U[x] \subseteq G\} = \{D, \emptyset, \{0, 3\}, \{0\}, \{1\}, \{2\}, \{0, 1, 3\}, \{0, 2, 3\}\}$. If we take $U = U^I$, then $U[0] = U[1] = U[2] = \{0, 1, 2\}$ and $U[3] = \{3\}$. Therefore, $T = \{D, \emptyset, \{3\}, \{0, 1, 2\}\}$. If we take $U = U^{\{0\}}$, then $U[x] = \{x\}, \forall x \in D$ and we obtain $T = 2^D$, the discrete topology. Moreover, if we take D as a d^* -ideal of D , then $U[x] = D, \forall x \in D$ and obtain $T = \{\emptyset, D\}$, the indiscrete topology.

4. Topological Property of Space (D, T_Λ)

Note that from Theorem 3.4 giving the Λ family of d^* -ideals of a d -algebra D which is closed under intersection. We can induce a uniform topology T_Λ on D . In this

section we study topological properties on (D, T_Δ) . Let D be a d -algebra and F, H be subsets of D . We define a set $F * H$ as follows:

$$F * H = \{x * y / x \in F, y \in H\}.$$

Definition 4.1. Let D be a d -algebra and T be a topology on the set D . Then we say that the pair (D, T) is a *topological d -algebra* if the operation “ $*$ ” is continuous with respect to T . (i.e.) If O is an open set and $a, b \in D$ such that $a * b \in O$, then there exist open sets O_1 and O_2 such that $a \in O_1, b \in O_2$ and $O_1 * O_2 \subseteq O$.

Theorem 4.2. *The pair (D, T_Δ) is a topological d -algebra.*

Proof. Let $x, y \in D$ and G be an open subset of D such that $x * y \in G$. Then there exist $U \in K, U[x * y] \subseteq G$ and a d^* -ideal I of D such that $U^I \subseteq U$. We claim that the following relation holds:

$$U^I[x] * U^I[y] \subseteq U[x * y].$$

Indeed, for any $h \in U^I[x]$ and $k \in U^I[y]$ we have that $x \sim_I h$ and $y \sim_I k$. Since \sim_I is a congruence relation, it follows that $x * y \sim_I h * k$. From that fact we have $(x * y, h * k) \in U^I \subseteq U$. Hence $h * k \in U^I[x * y] \subseteq U[x * y]$. Then $h * k \in G$.

Theorem 4.3 [3]. *Let X be a set and $S \subseteq P(X \times X)$ be a family such that for every $U \in S$ the following conditions hold:*

- (a) $\Delta \subseteq U$,
- (b) U^{-1} contains a member of S ,
- (c) *there exists a $V \in S$ such that $VoV \subseteq U$. Then there exists a unique uniformity U , for which S is a sub base.*

Theorem 4.4. *If we set $S = \{U^I / I \text{ is a } d^* \text{-ideal of a } d \text{-algebra } D\}$, then S is a sub base for a uniformity of D . We denote its associated topology by T_S .*

Proof. Since \sim_I is an equivalence relation, it is clear that S satisfies the axioms of Theorem 4.3.

Example 4.5. In Example 3.7, we can see that $S = \{U^0 = \Delta, U^F = \Delta \cup \{(0, 3), (3, 0)\}, U^I = \Delta \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1)\}, U^D = D \times D\}$.

Theorem 4.6. Let Λ be a family of d^* -ideals of D which is closed under intersection. Any d^* -ideal in the collection Λ is a clopen subset of D for the topology T_Λ .

Proof. Let I be a d^* -ideal of D in Λ and $y \in I^c$. Then $y \in U^I[y]$ and we obtain that $I^c \subseteq \cup\{U^I[y]/y \in I^c\}$. We claim that $U^I[y] \subseteq I^c$ for all $y \in I^c$. Let $z \in U^I[y]$, then $y \sim_I z$. Hence $y * z \in I$. If $z \in I$, then $y \in I$, since I is a d^* -ideal of D , which is a contradiction. So $z \in I^c$ and we obtain

$$\cup\{U^I[y]/y \in I^c\} \subseteq I^c.$$

Hence $I^c = \cup\{U^I[y]/y \in I^c\}$. Since $U^I[y]$ is open for any $y \in D$, I is a closed subset of D . We show that $I = \cup\{U^I[y]/y \in I\}$. If $y \in I$, then $y \in U^I[y]$ and hence $I \subseteq \cup\{U^I[y]/y \in I\}$. Given $y \in I$, if $z \in U^I[y]$, then $y \sim_I z$ and so $z * y \in I$. Since $y \in I$ and I is a d^* -ideal of D , we have $z \in I$. Hence, we get that $\cup\{U^I[y]/y \in I\} \subseteq I$, i.e., I is also an open subset of D .

In Example 3.8, the d^* -ideals $I, F, \{0\}$ and D are clopen subsets of D , where $\Lambda = \{I, F, \{0\}, D\}$.

Theorem 4.7. $T_\Lambda = T_J$, where $J = \cap\{I/I \in \Lambda\}$.

Proof. Let K and K^* be as Theorems 3.2 and 3.3, respectively. Now consider $\Lambda_0 = \{J\}$. Define $(K_0)^* := \{U^J\}$ and $K_0 = \{U/U^J \subseteq U\}$. Let $G \in T_\Lambda$. Given an $x \in G$, there exists $U \in K$ such that $U[x] \subseteq G$. From $J \subseteq I$, we obtain that $U^J \subseteq U^I$, for any d^* -ideal I of D . Since $U \in K$, there exists $I \in \Lambda$ such that $U^I \subseteq U$. Hence $U^J[x] \subseteq U^I[x] \subseteq G$. Since $U^J \in K_0, G \in T_J$. Hence $T_\Lambda \subseteq T_J$. Conversely, if $H \in T_J$, then for any $x \in H$, there exists $U \in K_0$ such that $U[x] \subseteq H$.

So $U^J[x] \subseteq H$ and hence Λ is closed under intersection, $J \in \Lambda$. Then we get $U^J \in K$ and so $H \in T_\Lambda$. Thus $T_J \subseteq T_\Lambda$.

Corollary 4.8. *Let I and J be d^* -ideals of a d -algebra D and $I \subseteq J$. Then J is clopen in the topological space (D, T_I) .*

Proof. Consider $\Lambda = \{I, J\}$. Then by Theorem 4.7, $T_\Lambda = T_I$ and therefore J is clopen in the topological space (D, T_I) .

Theorem 4.9. *Let I and J be d^* -ideals of a d -algebra D . Then $T_I \subseteq T_J$ if $J \subseteq I$.*

Proof. Let $J \subseteq I$. Consider $\Lambda_1 = \{I\}$, $K_1^* = \{U^I\}$, $K_1 = \{U/U^I \subseteq U\}$ and $\Lambda_2 = \{J\}$, $K_2^* = \{U^J\}$, $K_2 = \{U/U^J \subseteq U\}$. Let $G \in T_I$, then for any $x \in G$, there exists $U \in K_1$ such that $U[x] \subseteq G$. Since $J \subseteq I$, we have $U^J \subseteq U^I$. $[x] \subseteq G$ implies $U^J[x] \subseteq G$. This proves that $U^J \in K_2$ and so $G \in T_J$. Thus $T_I \subseteq T_J$.

Remark 4.10. Let Λ be a family of d^* -ideals of D which is closed under intersection and $J = \bigcap \{I : I \in \Lambda\}$. We have the following statements:

(i) By Theorem 4.9, we know that $T_\Lambda = T_J$. For any $U \in K$, $x \in D$, we can get that $U^J[x] \subseteq U[x]$. Hence T_Λ is equivalent to $\{A \subseteq D : \forall x \in A, U^J[x] \subseteq A\}$. So $A \subseteq D$ is open set if and only if for all $x \in A$, $U^J[x] \subseteq A$ if and only if $A = \bigcup_{x \in A} U^J[x]$.

(ii) For all $x \in D$, by (i), we know that $U^J[x]$ is the smallest neighborhood of x .

(iii) Let $B_J = \{U^J[x] : x \in D\}$. By (i) and (ii), it is easy to check that B_J is a base of T_J .

(iv) For all $x \in D$, $\{U^J[x]\}$ is a fundamental system of neighborhoods of x .

Theorem 4.11. *If I is a d^* -ideal of D , then for all $x \in D$,*

(i) $U^I[x]$ is a clopen subset in the topological space (D, T_I) .

(ii) $U^I[x]$ is a compact set in a topological space (D, T_I) .

Proof. (i) We show that $(U^I[x])^c$ is open. If $y \in (U^I[x])^c$, then $x * y \in I^c$ or $y * x \in I^c$. We assume that $y * x \in I^c$. By applying Theorems 4.2 and 4.3, we obtain $(U^I[y] * U^I[x]) \subseteq U^I[y * x] \subseteq I^c$. We claim that $U^I[y] \subseteq (U^I[x])^c$. If $z \in U^I[y]$, then $z * x \in (U^I[z] * U^I[x])$. Hence $z * x \in I^c$, then we get $z \in (U^I[x])^c$; proving that $(U^I[x])^c$ is open. Hence $U^I[x]$ is closed. It is clear that $U^I[x]$ is open. So $U^I[x]$ is a clopen subset of D .

(ii) Let $U^I[x] \subseteq \bigcup_{(\alpha \in \Omega)} O_\alpha$, where each O_α is an open set of D . Since $x \in U^I[x]$, there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Hence $U^I[x] \subseteq O_\alpha$, proving that $U^I[x]$ is compact.

Proposition 4.12 [3]. *Let (X, T) be uniform structure. Then uniform space (X, T) is completely regular.*

Corollary 4.13. *Let Λ be a family of d^* -ideals of D which is closed under intersection. Then topological space (D, T_Λ) is completely regular.*

Theorem 4.14. *Let Λ be a family of d^* -ideals of D which is closed under intersection. Then (D, T_Λ) is a discrete space if and only if there exists $I \in \Lambda$ such that $U^I[x] = \{x\}$ for all $x \in D$.*

Proof. Let T_Λ be a discrete topology on D . If for any $I \in \Lambda$, there exists $x \in \Lambda$ such that $U^I[x] \neq \{x\}$. Let $J = \bigcap \Lambda$. Then $J \in \Lambda$ and there exists $x_0 \in D$ such that $U^J[x_0] \neq \{x_0\}$. It follows that there exists $y_0 \in U^J[x_0]$ and $x_0 \neq y_0$. By Remark 4.10(ii), $U^J[x_0]$ is the smallest neighborhood of x_0 . Hence, $\{x_0\}$ is not an open subset of D , which is a contradiction. Conversely, for any $x \in D$, there exists $I \in \Lambda$ such that $U^I[x] = \{x\}$. Hence $\{x\}$ is an open set of D . Therefore, (D, T_Λ) is a discrete space.

Theorem 4.15. *Let Λ be a family of d^* -ideals of D which is closed under intersection. Then $J = \bigcap \Lambda$ and D be a d -algebra with right identity 0. Then the following conditions are equivalent:*

(i) (D, T_J) is a discrete space;

(ii) $J = \{0\}$.

Proof. (i) \Rightarrow (ii) By Theorem 4.14, we have $U^J\{0\} = \{0\}$. We show that $J \subseteq U^J[0]$. Let $x \in J$. Since $(x * 0) * x = 0$ and J is d^* -ideal, then $x * 0$ and $0 * x \in J$ we get that $x \in U^J[0]$. It follows that $J \subseteq U^J[0]$. Since $U^J[0] = \{0\}$ and $0 \in J$. Therefore, $J = \{0\}$.

(ii) \Rightarrow (i) Let $J = \{0\}$. Since D is d -algebra with right identity 0 , we can get that $U^J[x] = \{x\}$. It follows that (D, T_Λ) is discrete.

Corollary 4.16. *Let Λ be a family of d^* -ideal of D which is closed under intersection, $J = \bigcap \Lambda$ and D be a d -algebra with right identity 0 . Then (D, T_J) is a Hausdorff space if and only if $J = \{0\}$.*

Proof. Let (D, T_J) be a Hausdorff space. First we show that for any $x \in D$, $U^J[x] = \{x\}$. If there exists $x \neq y \in U^J[x]$, then $y \in U^J[x] \cap U^J[x]$. By Remark 4.10(ii), $U^J[x]$ and $U^J[y]$ are the smallest neighborhoods of x and y , respectively. Hence, for any neighborhood U of x and neighborhood V of y , we have that $y \in U^J[x] \cap U^J[y] \subseteq U \cap V \neq \emptyset$, which is a contradiction. Hence by Theorems 4.14 and 4.15, $J = \{0\}$. The other side of the proof directly follows from Theorem 4.15.

Definition 4.17 [3]. Recall that a uniform space (X, K) is said to be *totally bounded* if for each $U \in K$, there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n U[x_i]$.

Theorem 4.18. *Let I be a d^* -ideal of a d -algebra D . Then the following conditions are equivalent:*

- (1) the topological space (D, T_I) is compact,
- (2) the topological space (D, T_I) is totally bounded,
- (3) there exists $P = x_1, \dots, x_n \subseteq D$ such that for all $a \in D$ there exist $x_i \in P$ ($i = 1, \dots, n$) with $a * x_i \in I$ and $x_i * a \in I$.

Proof. (1) \Rightarrow (2): It is clear by [3].

(2) \Rightarrow (3): Let $U^I \in K$. Since (D, T_I) is totally bounded, there exist $x_1, \dots, x_n \in I$ such that $D = \bigcup_{i=1}^n U[x_i]$. If $a \in D$, then there exists x_i such that $a \in U_I[x_i]$, therefore $a * x_i \in I$ and $x_i * a \in I$.

(3) \Rightarrow (1): For any $a \in D$, by hypothesis, there exists $x_i \in P$ with $a * x_i \in I$ and $x_i * a \in I$. Hence $a \in U^I[x_i]$. Thus $D = \bigcup_{i=1}^n U[x_i]$. Now let $D = \bigcup_{\alpha \in \Omega} O_\alpha$ where each O_α is an open set of D . Then for any $x_i \in D$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since O_{α_i} is an open set. Hence $U^I[x_i] \subseteq O_{\alpha_i}$. Hence $D = \bigcup_{i=1}^n U[x_i] \subseteq \bigcup_{i=1}^n O_{\alpha_i}$, i.e., $D = \bigcup_{i=1}^n O_{\alpha_i}$, which means that (D, T_I) is compact.

Theorem 4.19. *If I is a d -ideal of D such that I^c is a finite set, then the topological space (D, T_I) is compact.*

Proof. Let $D = \bigcup_{\alpha \in F} O_\alpha$, where each O_α is an open subset of D . Let $I^c = \{x_1, \dots, x_n\}$. Then there exist $\alpha, \alpha_1, \dots, \alpha_n \in F$ such that $0 \in O_\alpha$, $x_1 \in O_{\alpha_1}, \dots, x_n \in O_{\alpha_n}$. Then $U^I[0] \subseteq O_\alpha$, but $U^I[0] = I$. Hence $D = \bigcup_{i=1}^n O_{\alpha_i} \cup O_\alpha$.

Theorem 4.20. *If I is a d -ideal of D , then I is a compact set in the topological space (D, T_I) .*

Proof. Let $I \subseteq \bigcup_{\alpha \in F} O_\alpha$, where each O_α is open set of D . Since $0 \in I$, there is $\alpha \in F$ such that $0 \in O_\alpha$. Then $I = U^I[0] \subseteq O_\alpha$. Hence, I is a compact set in the topological space (D, T_I) .

Definition 4.21 [8]. Let D_1 and D_2 be d -algebras. A mapping $f : D_1 \rightarrow D_2$ is called a d -morphism from D_1 to D_2 if

$$f(x * y) = f(x) * f(y)$$

for any $x, y \in D_1$. If the mapping f is bijective, then we call f a d -isomorphism and note that $f(0_1) = 0_2$ when f is a d -morphism.

Proposition 4.22. *Let $f : D_1 \rightarrow D_2$ be a d -morphism. Then the following properties hold:*

(i) *If J is a d^* -ideal of D_2 , then the set $f^{-1}(J)$ is a d^* -ideal of D_1 .*

(ii) *If f is bijective and I is a d^* -ideal of D_1 , then $f(I)$ is a d^* -ideal of D_2 .*

Proof.

(i) (1) Since $0_2 \in J$ and $f(0_1) = 0_2$, then $0_1 \in f^{-1}(J) \neq \emptyset$.

(2) Let $x \in f^{-1}(J)$ and $y \in D_1$, then $f(x) \in J$ and $f(y) \in D_2$. Since J is d^* -ideal. So $f(x) * f(y) \in J \Rightarrow f(x * y) \in J$. Thus $x * y \in f^{-1}(J)$.

(3) Let $x, y, z \in D_1$ such that $x * y, y * z \in f^{-1}(J)$. Then $f(x * y), f(y * z) \in J \Rightarrow f(x) * f(y), f(y) * f(z) \in J$. Since J is d^* -ideal, $f(x) * f(z) \in J \Rightarrow f(x * z) \in J \Rightarrow x * z \in f^{-1}(J)$. From (1), (2) and (3) we get that $f^{-1}(J)$ is d^* -ideal.

(ii) (1) Since $0_1 \in I$ and $f(0_1) = 0_2$, $0_2 \in f(I) \neq \emptyset$.

(2) Let $x \in f(I)$ and $y \in D_2$. Since f is surjective, there exist $a \in I$ and $b \in D_1$ such that $f(a) = x, f(b) = y$ and since I is d^* -ideal. So $a * b \in I \Rightarrow f(a * b) \in f(I) \Rightarrow f(a) * f(b) \in f(I)$. Thus $x * y \in f(I)$.

(3) Let $x, y, z \in D_2$ such that $x * y, y * z \in f(I)$. Then there exist $a, b, c \in D_1$ such that $f(a) = x, f(b) = y, f(c) = z \Rightarrow f(a) * f(b), f(b) * f(c) \in f(I)$. Since f is injective, $a * b = f^{-1}(f(a * b)) \subseteq f^{-1}(f(I)) = I, b * c = f^{-1}(f(b * c)) \subseteq f^{-1}(f(I)) = I$ and I is a d^* -ideal, then $a * c \in I \Rightarrow f(a * c) \in f(I) \Rightarrow x * z \in f(I)$. Thus, from (1), (2) and (3) we get that $f(I)$ is d^* -ideal.

Lemma 4.23. *Let D_1 and D_2 be d -algebras and J be a d^* -ideal of D_2 . If $f : D_1 \rightarrow D_2$ is a d -isomorphism, then $(x, y) \in U^{f^{-1}(J)} \Leftrightarrow (f(x), f(y)) \in U^J$ for every $x, y \in D_1$.*

Proof. For any $(x, y) \in U^{f^{-1}(J)} \Leftrightarrow x \sim y \in U^{f^{-1}(J)} \Leftrightarrow f(x) \sim f(y) \in J \Leftrightarrow (f(x), f(y)) \in U^J$.

Theorem 4.24. Let D_1 and D_2 be d -algebras and J be a d^* -ideal of D_2 . If $f : D_1 \rightarrow D_2$ is a d -isomorphism, then the following properties hold:

(i) for any $x \in D_1$, $f(U^{f^{-1}(J)}[x]) = U^J[f(x)]$,

(ii) for any $y \in D_2$, $f^{-1}(U^J[y]) = U^{f^{-1}(J)}[f^{-1}(y)]$.

Proof. (i) Let $y \in f(U^{f^{-1}(J)}[x])$. Then there exists $x_0 \in U^{f^{-1}(J)}[x]$ such that $y = f(x_0)$. It follows that $x \sim x_0 \in f^{-1}(J) \Rightarrow f(x) \sim f(x_0) \Rightarrow f(x) \sim y \in J \Rightarrow b \in U^J[f(x)]$.

Conversely, let $y \in U^J[f(x)] \Rightarrow f(x) \sim y \in J \Rightarrow f^{-1}(f(x) \sim y) \in f^{-1}(J) \Rightarrow x \sim f^{-1}(y) \in f^{-1}(J) \Rightarrow f^{-1}(y) \in U^{f^{-1}(J)}[x] \Rightarrow b \in f(U^{f^{-1}(J)}[x])$.

(ii) $x \in f^{-1}(U^J[y]) \Leftrightarrow f(x) \in U^J[y] \Leftrightarrow f(x) \sim y \in J \Leftrightarrow f^{-1}(f(x) \sim y) \in f^{-1}(J) \Leftrightarrow x \sim f^{-1}(y) \in f^{-1}(J) \Leftrightarrow x \in U^{f^{-1}(J)}[f^{-1}(y)]$.

Theorem 4.25. Let D_1 and D_2 be d -algebras and J be a d^* -ideal of D_2 . If $f : D_1 \rightarrow D_2$ is a d -isomorphism, then f is a homeomorphism map from $(D_1, \tau_{f^{-1}(J)})$ to (D_2, τ_J) .

Proof. First we prove that f is continuous. Let $A \in \tau$. By Remark 4.10, we can get that $A = \bigcup_{\alpha \in A} U^J[a]$. It follows that $f^{-1}(A) = f^{-1}(\bigcup_{\alpha \in A} U^J[a]) = \bigcup_{\alpha \in A} f^{-1}(U^J[a])$. We claim that if $b \in f^{-1}(U^J[a])$, then $U^{f^{-1}(J)}[b] \subseteq f^{-1}(U^J[a])$. Indeed, let $c \in U^{f^{-1}(J)}[b]$, we get that $c \sim b \in f^{-1}(J)$, so $f(c) \sim f(b) \in J$ by Lemma 4.23. Since $f(b) \in U^J[a]$, we get that $f(b) \sim a \in J$. It follows that $f(c) \sim a \in J$. Thus, we have that $f(c) \in U^J[a]$. So $c \in f^{-1}(U^J[a])$. Hence $f^{-1}(U^J[a]) =$

$\bigcup_{b \in f^{-1}(U^J[a])} U^{f^{-1}(J)}[b] \in \tau_{f^{-1}(J)}$. Therefore, $f^{-1}(A) = \bigcup_{\alpha \in A} f^{-1}(U^J[a]) \in \tau_{f^{-1}(J)}$. So f is a continuous map. Finally we show that f is an open map. Let A be an open set of $(D_1, \tau_{f^{-1}(J)})$. We claim that $f(A)$ is an open set of (D_2, τ_J) . Let $a \in f(A)$. We shall show that $U^J[a] \subseteq f(A)$. Indeed, for any $b \in U^J[a]$, we get that $b \sim a \in J$. By Lemma 4.23, we have $f^{-1}(a) \sim f^{-1}(b) \in f^{-1}(J)$. Hence $f^{-1}(b) \in U^{f^{-1}(J)}[f^{-1}(a)]$. Since $a \in f(A)$ and f is injective we get that $f^{-1}(a) \in A$. By Remark 4.10(i), it follows that $U^{f^{-1}(J)}[f^{-1}(a)] \subseteq A$. So $f^{-1}(b) \in A$, we get that $b \in f(A)$. Therefore $U^J[a] \subseteq f(A)$. So f is open map.

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