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Study of Some New Logarithmic-Hardy-Hilbert-Type Integral Inequalities

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Abstract

Numerous logarithmic-Hardy-Hilbert-type integral inequalities have been identified in the literature, and this remains an active area of research. In this article, we introduce and analyze several new variants, thereby expanding the existing collection. Our main contributions are presented in the form of two theorems. Several secondary results are also derived. We provide detailed proofs of all our results to ensure reproducibility and highlight the underlying techniques. Finally, the appendix presents three novel integral formulas of independent interest that arose naturally during our derivations.

1 Introduction

Integral inequalities play a central role in analysis and have been extensively studied in a variety of contexts. See, for example, [2–6]. Among these, the Hardy-Hilbert integral inequality stands out as a foundational result. Its formal statement is given below. Let $p > 1$, $q = p/(p-1)$, i.e., satisfying $1/p + 1/q = 1$, and $f, g : \mathbb{R}_+ = [0, +\infty) \mapsto \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+} g^q(y)dy < +\infty$. Then the Hardy-Hilbert integral inequality states that

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_{\mathbb{R}_+} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} g^q(y)dy\right]^{1/q}.$$

The constant factor $\pi/\sin(\pi/p)$ is optimal; the inequality fails if this constant is replaced by any smaller value. The fact that this constant is optimal and that the upper bound involves the simple integral norms of f and g emphasizes the importance of the inequality. This has naturally prompted significant research aimed at extending and generalizing its form.

Over the years, numerous variants of the Hardy-Hilbert integral inequality have emerged, often introducing additional parameters, weight functions, multi-dimensional analogues, or modified kernel

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function structures. These enhancements broaden the applicability of this inequality and strengthen its links with other areas of analysis, such as harmonic analysis, operator theory, and partial differential equations. Comprehensive surveys of these developments can be found in [7–9].

Logarithmic-type integral inequalities, which incorporate logarithmic expressions into the integrands, are of particular interest. These refined versions often arise in situations where the singularity at the origin is reduced, or where the analytical framework requires more nuanced functional behaviors to be captured. They play a key role in problems involving fine regularity, asymptotic analysis, and the study of operators with slowly varying kernel functions.

A large number of logarithmic-Hardy-Hilbert-type integral inequalities have been established in the literature, and this remains an active area of research. For the purposes of this article, Table 1 provides an overview of notable inequalities of this type (specifically, those without adjustable parameters). In each case, it is assumed that the integrals in the upper bounds are finite. Note also that the constant G in the third inequality denotes the well-known Catalan constant.

This article introduces and analyzes new variants of interest that are based on different functionalities of existing logarithmic-Hardy-Hilbert-type integral inequalities. The first variant focuses on the following double integral:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{\log(x/y+y/x)}{x+y}f(x)g(y)dxdy,$$

for which we derive a sharp inequality. We then complement this result by analyzing another variant, based on the following double integral:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{\log[1/(xy)+xy]}{1+xy}f(x)g(y)dxdy.$$

A similarly structured bound is obtained in both cases. In particular, the constant factors are the same and depend explicitly on π; they are given by $\pi\log(6+4\sqrt{2})\approx 7.71542$. Based on this mathematical foundation, additional logarithmic-Hardy-Hilbert-type integral inequalities are derived as secondary results. All proofs are presented in full detail, with careful attention paid to the analytic techniques involved.

The remainder of the article is structured as follows: Section 2 presents the two main theorems, providing detailed statements and proofs. Section 3 contains further inequalities that follow as applications of these theorems. Section 4 concludes the article with a summary and possible directions for future research. The appendix covers three integral formulas related to the study that are of independent interest.

Logarithmic-Hardy-Hilbert-type integral inequality	Reference
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(x/y)}{x-y} f(x)g(y)dxdy \leq \left[\frac{\pi}{\sin(\pi/p)}\right]^2 \left[\int_{\mathbb{R}_+} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} g^q(y)dy\right]^{1/q}$	[2]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\lvert\log(x/y)\rvert}{\max(x,y)} f(x)g(y)dxdy \leq (p^2+q^2) \left[\int_{\mathbb{R}_+} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} g^q(y)dy\right]^{1/q}$	[10]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\lvert\log(x/y)\rvert}{x+y} f(x)g(y)dxdy \leq 8G \left[\int_{\mathbb{R}_+} f^2(x)dx\right]^{1/2} \left[\int_{\mathbb{R}_+} g^2(y)dy\right]^{1/2}$	[11]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{[\log(x/y)]^2}{x^2+xy+y^2} f(x)g(y)dxdy \leq \frac{16\sqrt{3}\pi}{243} \left[\int_{\mathbb{R}_+} x^{-1}f^2(x)dx\right]^{1/2} \left[\int_{\mathbb{R}_+} y^{-1}g^2(y)dy\right]^{1/2}$	[13]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{[\log(x/y)]^2}{x^2-xy+y^2} f(x)g(y)dxdy \leq \frac{43\sqrt{3}\pi}{36} \left[\int_{\mathbb{R}_+} x^{-1}f^2(x)dx\right]^{1/2} \left[\int_{\mathbb{R}_+} y^{-1}g^2(y)dy\right]^{1/2}$	[13]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(x/y)}{(x-y)(x^2+y^2)} f(x)g(y)dxdy \leq \frac{3\pi^2}{16} \left[\int_{\mathbb{R}_+} x^{-1}f^2(x)dx\right]^{1/2} \left[\int_{\mathbb{R}_+} y^{-3}g^2(y)dy\right]^{1/2}$	[13]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(1+x/y)}{x+y} f(x)g(y)dxdy \leq 2\pi\log(2) \left[\int_{\mathbb{R}_+} f^2(x)dx\right]^{1/2} \left[\int_{\mathbb{R}_+} g^2(y)dy\right]^{1/2}$	[14]
$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \log\left[\frac{(x+y)^2}{x^2+xy+y^2}\right] f(x)g(y)dxdy \leq \frac{\pi^2}{9} \left[\int_{\mathbb{R}_+} x^{p-1}f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q-1}g^q(y)dy\right]^{1/q}$	[15]

Table 1: Brief overview of logarithmic-Hardy-Hilbert-type integral inequalities (without adjustable parameters)

2 Main Inequalities

2.1 First logarithmic-Hardy-Hilbert-type integral inequality

The first theorem is given below, along with its proof.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy < +\infty$. Then the following inequality holds:*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(x/y+y/x)}{x+y} f(x)g(y)dxdy$$
$$\leq \pi \log(6+4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Proof of Theorem 2.1. Using the identities $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and $1/p + 1/q = 1$, and the Hölder integral inequality, we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(x/y+y/x)}{x+y} f(x)g(y)dxdy$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{1/(2q)}y^{-1/(2p)} \left[\frac{\log(x/y+y/x)}{x+y}\right]^{1/p} f(x)$$
$$\times\, x^{-1/(2q)}y^{1/(2p)} \left[\frac{\log(x/y+y/x)}{x+y}\right]^{1/q} g(y)dxdy$$
$$\leq A^{1/p}B^{1/q}, \tag{1}$$

where

$$A = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{p/(2q)}y^{-1/2} \frac{\log(x/y+y/x)}{x+y} f^p(x)dxdy$$

and

$$B = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{-1/2}y^{q/(2p)} \frac{\log(x/y+y/x)}{x+y} g^q(y)dxdy.$$

We now investigate the expressions of the double integrals A and B.

Since the integrand associated with A is non-negative, the Fubini-Tonelli theorem allows the order of integration to be exchanged. This, together with a rearrangement, gives

$$A = \int_{\mathbb{R}_+} x^{p/(2q)} f^p(x) \left\{\int_{\mathbb{R}_+} \frac{1}{\sqrt{y}} \frac{\log(x/y+y/x)}{x+y} dy\right\} dx$$
$$= \int_{\mathbb{R}_+} x^{p/(2q)-1/2} f^p(x) \left\{\int_{\mathbb{R}_+} \frac{1}{\sqrt{y/x}} \frac{\log(x/y+y/x)}{1+y/x} \frac{1}{x} dy\right\} dx. \tag{2}$$

Performing the change of variables $u = y/x$, we can express the central integral as follows:

$$\int_{\mathbb{R}_+} \frac{1}{\sqrt{y/x}} \frac{\log(x/y + y/x)}{1 + y/x} \frac{1}{x} dy = \int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log(1/u + u)}{1 + u} du. \tag{3}$$

To the best of our knowledge, this integral has not been referenced before, so it requires special treatment. We will take existing results into consideration to facilitate development. Using an appropriate logarithmic decomposition and two integral formulas from [1], i.e., [1, 4.271.7 with $n = 1$ and $b = 0$] and [1, 4.295.41 with $\mu = 1/2$ and suitable developments of the beta function denoted $\beta(x)$], we get

$$\begin{aligned}
&\int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log(1/u + u)}{1 + u} du = \int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log[(1/u)(1 + u^2)]}{1 + u} du \\
&= -\int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log(u)}{1 + u} du + \int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log(1 + u^2)}{1 + u} du \\
&= 0 + \pi \log(6 + 4\sqrt{2}) = \pi \log(6 + 4\sqrt{2}).
\end{aligned} \tag{4}$$

It follows from Equation (2), (3) and (4) and the identity $p/q = p - 1$ that

$$A = \pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx. \tag{5}$$

Similarly, for the term B but with the change of variables $v = x/y$ and details omitted for brevity, we have

$$\begin{aligned}
B &= \int_{\mathbb{R}_+} y^{q/(2p)} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{x}} \frac{\log(x/y + y/x)}{x + y} dx \right\} dy \\
&= \int_{\mathbb{R}_+} y^{q/(2p)-1/2} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{x/y}} \frac{\log(x/y + y/x)}{1 + x/y} \frac{1}{y} dx \right\} dy \\
&= \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{v}} \frac{\log(v + 1/v)}{1 + v} dv \right\} dy \\
&= \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) \pi \log(6 + 4\sqrt{2}) dy \\
&= \pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy.
\end{aligned} \tag{6}$$

It follows from Equations (1), (5) and (6), and the identity $1/p + 1/q = 1$, that

$$\begin{aligned}
&\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\log(x/y + y/x)}{x + y} f(x) g(y) dx dy \\
&\leq \left[\pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy \right]^{1/q} \\
&= \pi \log(6 + 4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy \right]^{1/q}.
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

This theorem completes Table 1 by offering a new logarithmic-Hardy-Hilbert-type integral inequality that has not been documented before.

The subsection below presents a different type of logarithmic-Hardy-Hilbert integral inequality.

2.2 Second logarithmic-Hardy-Hilbert-type integral inequality

The second theorem is given below, along with its proof.

Theorem 2.2. *Let $p > 1$, $q = p/(p-1)$, and $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy < +\infty$. Then the following inequality holds:*

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log[1/(xy)+xy]}{1+xy} f(x)g(y)dxdy \\ &\leq \pi \log(6+4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy\right]^{1/q}.\end{aligned}$$

Proof of Theorem 2.2. The proof employs mathematical techniques similar to those in the proof of Theorem 2.1, but with some important nuances. Applying the identities $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and $1/p + 1/q = 1$, and the Hölder integral inequality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log[1/(xy)+xy]}{1+xy} f(x)g(y)dxdy \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{1/(2q)}y^{-1/(2p)} \left[\frac{\log[1/(xy)+xy]}{1+xy}\right]^{1/p} f(x) \\ &\times x^{-1/(2q)}y^{1/(2p)} \left[\frac{\log[1/(xy)+xy]}{1+xy}\right]^{1/q} g(y)dxdy \\ &\leq C^{1/p}D^{1/q}, \end{aligned} \tag{7}$$

where

$$C = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{p/(2q)}y^{-1/2} \frac{\log[1/(xy)+xy]}{1+xy} f^p(x)dxdy$$

and

$$D = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} x^{-1/2}y^{q/(2p)} \frac{\log[1/(xy)+xy]}{1+xy} g^q(y)dxdy.$$

We now examine the expressions of the double integrals C and D.

Since the integrand associated with C is non-negative, the Fubini-Tonelli theorem allows the order of integration to be exchanged. This, combined with the change of variables $u = xy$, Equation (4) and the identity $p/q = p - 1$, gives

$$\begin{aligned} C &= \int_{\mathbb{R}_+} x^{p/(2q)} f^p(x) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{y}} \frac{\log[1/(xy) + xy]}{1 + xy} dy \right\} dx \\ &= \int_{\mathbb{R}_+} x^{p/(2q)-1/2} f^p(x) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{xy}} \frac{\log[1/(xy) + xy]}{1 + xy} x dy \right\} dx \\ &= \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{u}} \frac{\log(1/u + u)}{1 + u} du \right\} dx \\ &= \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) \pi \log(6 + 4\sqrt{2}).dx \\ &= \pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx. \end{aligned} \tag{8}$$

Similarly, for the term D but with the change of variables $v = xy$ (omitting the details for the sake of redundancy), we have

$$\begin{aligned} D &= \int_{\mathbb{R}_+} y^{q/(2p)} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{x}} \frac{\log[1/(xy) + xy]}{1 + xy} dx \right\} dy \\ &= \int_{\mathbb{R}_+} y^{q/(2p)-1/2} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{xy}} \frac{\log[1/(xy) + xy]}{1 + xy} y dx \right\} dy \\ &= \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) \left\{ \int_{\mathbb{R}_+} \frac{1}{\sqrt{v}} \frac{\log(v + 1/v)}{1 + v} dv \right\} dy \\ &= \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) \pi \log(6 + 4\sqrt{2}) dy \\ &= \pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy. \end{aligned} \tag{9}$$

It follows from Equations (7), (8) and (9), and the identity $1/p + 1/q = 1$, that

$$\begin{aligned} &\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\log[1/(xy) + xy]}{1 + xy} f(x) g(y) dx dy \\ &\leq \left[\pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\pi \log(6 + 4\sqrt{2}) \int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy \right]^{1/q} \\ &= \pi \log(6 + 4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x) dx \right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y) dy \right]^{1/q}. \end{aligned}$$

This concludes the proof of Theorem 2.2. □

Note that despite the double integrals of interest being different, the upper bounds in Theorems 2.1 and 2.2 are the same.

The logarithmic-Hardy-Hilbert-type integral inequality in Theorem 2.2 is not documented elsewhere. Like the one in 2.1, it can therefore be legitimately added to Table 1.

The flexibility of Theorems 2.1 and 2.2 allows them to establish new logarithmic-Hardy-Hilbert integral inequalities. This aspect is developed in the subsection below.

3 Secondary Results

3.1 A double integral comparison

The proposition below shows a simple inequality comparing the double integral of the original Hardy-Hilbert integral inequality and the one in Theorem 2.1.

Proposition 3.1. *Let $p > 1$, $q = p/(p-1)$, and $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions. Then the following inequality holds:*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{1}{\log(2)} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\log(x/y + y/x)}{x+y} f(x)g(y)dxdy,$$

provided that the double integral in the upper bound converge.

Proof of Proposition 3.1. Applying the basic inequality $a^2 + b^2 \geq 2|ab|$, $a, b \in \mathbb{R}$, with $a = \sqrt{x/y}$ and $b = \sqrt{y/x}$, we get

$$\frac{x}{y} + \frac{y}{x} \geq 2\sqrt{\frac{x}{y}}\sqrt{\frac{y}{x}} = 2.$$

Therefore, we have

$$\log\left(\frac{x}{y} + \frac{y}{x}\right) \geq \log(2),$$

so that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{1}{\log(2)} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\log(x/y + y/x)}{x+y} f(x)g(y)dxdy.$$

This concludes the proof of Proposition 3.1. □

This result establishes a link between new and old inequalities. This is achieved through a simple comparison that can be used for various purposes relating to inequality.

3.2 Some new integral inequalities

The proposition below presents a new logarithmic-Hardy-Hilbert integral inequality derived from Theorem 2.1.

Proposition 3.2. *Let $p > 1$, $q = p/(p-1)$, and $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy < +\infty$. Then the following inequality holds:*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(2 + x^2/y^2 + y^2/x^2)}{x+y} f(x)g(y)dxdy$$
$$\leq 2\pi \log(6 + 4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.2. We clearly have

$$2 + \frac{x^2}{y^2} + \frac{y^2}{x^2} = \left(\frac{x}{y} + \frac{y}{x}\right)^2.$$

A basic property of the logarithmic function gives

$$\log\left(2 + \frac{x^2}{y^2} + \frac{y^2}{x^2}\right) = 2\log\left(\frac{x}{y} + \frac{y}{x}\right).$$

Using this and Theorem 2.1, we obtain

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(2 + x^2/y^2 + y^2/x^2)}{x+y} f(x)g(y)dxdy$$
$$= 2\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log(x/y + y/x)}{x+y} f(x)g(y)dxdy$$
$$\leq 2\pi \log(6 + 4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 3.2. □

On the same principle but with the use of Theorem 2.2, the proposition below offers a new logarithmic-Hardy-Hilbert integral inequality.

Proposition 3.3. *Let $p > 1$, $q = p/(p-1)$, and $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be two functions such that $\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy < +\infty$. Then the following inequality holds:*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\log[2 + 1/(xy)^2 + (xy)^2]}{1+xy} f(x)g(y)dxdy$$
$$\leq 2\pi \log(6 + 4\sqrt{2}) \left[\int_{\mathbb{R}_+} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_{\mathbb{R}_+} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

Proof of Proposition 3.3. We have

$$2+\frac{1}{(xy)^2}+(xy)^2=\left(\frac{1}{xy}+xy\right)^2.$$

A basic property of the logarithmic function gives

$$\log\left[2+\frac{1}{(xy)^2}+(xy)^2\right]=2\log\left(\frac{1}{xy}+xy\right).$$

Using this and Theorem 2.2, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{\log[2+1/(xy)^2+(xy)^2]}{1+xy}f(x)g(y)dxdy\\&=2\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{\log[1/(xy)+xy]}{1+xy}f(x)g(y)dxdy\\&\leq 2\pi\log(6+4\sqrt{2})\left[\int_{\mathbb{R}_+}x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_{\mathbb{R}_+}y^{q/2-1}g^q(y)dy\right]^{1/q}.\end{aligned}$$

This concludes the proof of Proposition 3.3. □

4 Conclusion

This article completes the collection of logarithmic-Hardy-Hilbert integral inequalities by presenting two key theorems: Theorems 2.1 and 2.2. Based on the properties of the logarithmic function, several new logarithmic Hardy-Hilbert-type integral inequalities are derived as consequences of these theorems. Detailed proofs are provided for all results. The perspectives and potential extensions of this work are outlined below.

- Examining a sharp inequality for the following double integral:

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{\log(x/y+y/x)}{1+xy}f(x)g(y)dxdy,$$

  which mixes the functionalities of the double integrals in Theorems 2.1 and 2.2.
- Investigating the optimality of the constant factors.
- Exploring analogous inequalities involving other special functions or weight conditions.
- Applying the established inequalities to problems in harmonic analysis and related fields.
- Extending the results to discrete or multidimensional settings.

Appendix

During our study, we identified several integrals that are not referenced in [1] or elsewhere in the literature. Three of these significant integrals are listed below. All of them are based on the same mathematical principles and are related by simple changes of variables.

- We have

$$\int_0^{+\infty} \frac{\log(1+x^2)}{\sqrt{x}(1+x)} dx = \pi \log(6+4\sqrt{2}).$$

- We have

$$\int_0^{+\infty} \frac{\log(1+x^4)}{1+x^2} dx = \frac{\pi}{2} \log(6+4\sqrt{2}).$$

- We have

$$\int_0^{\pi/2} \log[1+\tan^4(x)] dx = \frac{\pi}{2} \log(6+4\sqrt{2}).$$

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