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Non-ergodicity of the Earth's Annual Temperature in a Precession Cycle of the Equinoxes

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Abstract

A three-parameter paleoclimate model has revealed the non-ergodicity of the Earth's annual temperature during a precession cycle. This result comes from the inverse problem of free parameter identification when the experimental input data are the mean winter and summer solstice temperatures for the period 1950-1975. The inverse problem is underdetermined and admits infinite solutions. However, it is possible to order these solutions in symmetric pairs according to the increasing level of entropy possessed by the annual half-cycle between the two solstices, up to the maximum value at which the two solutions collapse into one. Therefore, the parameter identification process admits uniqueness if, as a third constraint, one searches for the solution of the annual temperature profile that has the maximum entropy, i.e. the highest probability. The existence of a unique solution with the highest probability among the infinite other possible solutions implies the non-ergodic character of the annual temperature.

Introduction

Entropy is a measure of the degree of probability of a physical state: if a state is highly probable, its entropy is high; conversely, a state that is not highly probable has a low entropy. If the parameters that determine the Earth's temperature are the result of a physical state to which a certain level of entropy can be associated, then the value that the parameters take on will have a more or less high probability depending on the level of entropy to which they are associated. The approach of the concept of entropy to that of parameter identification warns the reader that this process, like any identification process, faces problems of non-uniqueness of solutions. In other words, the Earth's temperature profile is subject to probabilistic events, even in a modelling context that is, at least initially, strictly deterministic.

Paleoclimate Mathematical Model of Earth Annual Temperature

In a previous paper, [1], we developed a millennial energy balance equation for the Northern Hemisphere in terms of the Stefan-Boltzmann dynamic radiation equation. In this paper, we will instead analyse its annual energy balance formulation:

$$\underbrace{U_0 \frac{dT}{d\alpha}}_{Accumulation} + \underbrace{\sigma E\,(T^4 - T_{0NL}{}^4)}_{Loss} = \underbrace{F(\alpha,\beta,e,\delta)}_{Solar\ Forcing} + \underbrace{\epsilon\sigma E\,(T^4 - T_{0NL}{}^4)}_{Green\ House\ Effect}. \quad (1)$$

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In equation (1), the initial condition is $T(0) = T_i$. The annual balance is function of two independent variables, (α, β) where, (α) is the angle of revolution and (β) the angle of total precession. The model equation describes the annual temperature profile of the Earth as a function of the revolution angle (α) at each given value of the precession angle (β) considered as a parameter. The loss term $\underbrace{\sigma E\,(T^4 - {T_{0NL}}^4)}$ is the radiation equation for an opaque body. The given function ϵ describes the greenhouse effect as an additional forcing term to solar radiation, according to the Stefan-Boltzmann law. The linearisation of the radiation equation, by means of a truncated first order series development, allows an analytical solution, see Appendix. The linear model equation is then used in subsequent developments:

$$\underbrace{U_0\frac{dT}{d\alpha}}_{Accumulation} + \underbrace{U(T - T_0)}_{Loss + Green\,House\,Effect} = \underbrace{F(\alpha,\beta,e,\delta)}_{Solar\,Forcing}. \tag{2}$$

The total loss term $U(T - T_0)$ has the form of Newton's linear flux equation, assuming that the total loss transmittance has the value

$$U = 4\sigma E(1 - \epsilon)T_i^3. \tag{3}$$

The temperature T_0 at the outer boundary will assume the identified value $T_0 = -\,70°C$ corresponding to the average temperature of the tropopause at 13 km of height.

Three free parameters (U, U_0, T_0), appear in the linear energy balance. The theory of ordinary differential equations teaches how to construct a solution to linear equations of this type, given initial conditions and assigned free parameters (U, U_0, T_0). In our case, on the other hand, the experimental value of the Earth's annual temperature at a given precession angle is known, and we want to identify the free parameters, which at best determine the experimental data, through the analytical solution. The process of parameter identification is therefore an inverse problem, as described, for example, by Bellman, [2]. To determine the three free parameters, it is necessary to have (at least) a system of three equations in the three free parameters considered as unknowns. Two equations are obtained quite spontaneously by taking the experimental mean of the two annual isotherms of January and July, which roughly correspond to the temperatures at the winter and summer solstices. The third equation is missing. We usually refer to this situation as an empirical "under-determination" of the model. The inadequacy of the experimental data alone does not allow us to validate the model, and ultimately, we cannot conclude whether or not the model is adequate to represent the reality we are studying. This is a quite normal situation in science. In order to escape from indeterminacy, it is necessary to broaden the horizon by placing ourselves in a broader context. Since the energy balance equation of the model must represent a real phenomenon, we impose the constraint that the three free parameters (U, U_0, T_0) generate the most probable temperature profile. In thermodynamic terms, this constraint implies that the entropy variation of the annual half-cycle between the two winter and summer solstices is maximum. However, accepting this very natural constraint implies that the probability of the Earth's annual temperature profile is maximum when the free parameters satisfy the condition of maximum entropy variation between the two solstices. In the other cases, the temperature profile is less likely. From these considerations, we can conclude that the Earth/Sun system described by the paleoclimate model is not ergodic; in fact, a system is ergodic if, over a sufficiently long period of time, it has the same probability of being in each of the possible states. Conversely, if a system is more likely to be in a particular state than in any other possible state, then it is not ergodic.

Parameters Identification: Entropic Uniqueness

The entropy of a system is the ratio between the change in heat "dQ" per unit area and the temperature T at which this change takes place. The constraint that the entropy variation of a half-cycle of revolution is maximum between the two winter and summer solstices translates into the integral of this ratio between the temperatures at the two solstices:

$$\int_{T_1}^{T_2} \frac{dQ}{T} = Max. \tag{4}$$

In the case of the paleoclimatic model, the accumulation term $U_0 \cdot dT/d\alpha$ describes the change of heat over time therefore the entropy variation between the two solstices becomes:

$$\int_{T_1}^{T_2} \frac{dQ}{T} \cong \int_{T_1}^{T_2} U_0 \frac{dT}{T} = U_0 [lnT]_{T_1}^{T_2} \cong U_0 \frac{T_2 - T_1}{T_1}. \tag{5}$$

In other words, the factor $U_0(T_2 - T_1)$ is proportional to the entropy variation of a half cycle of the Earth's revolution around the Sun between the winter and summer solstices. By replacing the expression of the analytical solution evaluated in (T_1, T_2) and defining the dimensionless extinction parameter $P = U/U_0$, we can separate the variables so that on the left we have all the known or not yet identified terms and on the right instead a function of only the extinction parameter P, which is currently unknown. Rationalisation gives us a fraction with a second degree equation in the numerator, on the right hand side.

$$\theta \equiv U_0(T_2 - T_1)\frac{2\, a_0}{\varphi F_0} = - \frac{a_1(P\, cos\beta_0 + sin\beta_0) + a_2 P}{P^2 + 1}. \tag{6}$$

The graphical representation of the two terms of the equation, shown in Figure 1, provides a very brief illustration of the problems involved in solving this equation. Graphically, the function to the left of the equal sign is a constant (made up of known terms) and describes a horizontal straight line, while the equation to the right of the equal sign has the trend shown schematically in Figure 1. Physically, the parameter P must always be positive because the two thermal transmittances (U, U_0) are both positive. By varying the constant, i.e. the ordinate at the origin of the horizontal line, four cases can occur:

1. There are two coincident and positive solutions when the horizontal line is tangent. In this case, the constant takes the maximum value (marked with a diamond in Fig. 1).
2. Two different positive solutions if the line is secant and greater than $\theta^+ > -\, a_1 sin\beta_0$.
3. One unique positive solution if the line is less than $\theta^+ < -\, a_1 sin\beta_0$.
4. No positive solution if the ordinate of the line exceeds the maximum value of the curve.

Note: The four cases are reduced to three if the precession angle coincides exactly with the perihelion or aphelion, i.e., $\beta_0 = (0, \pi)$.

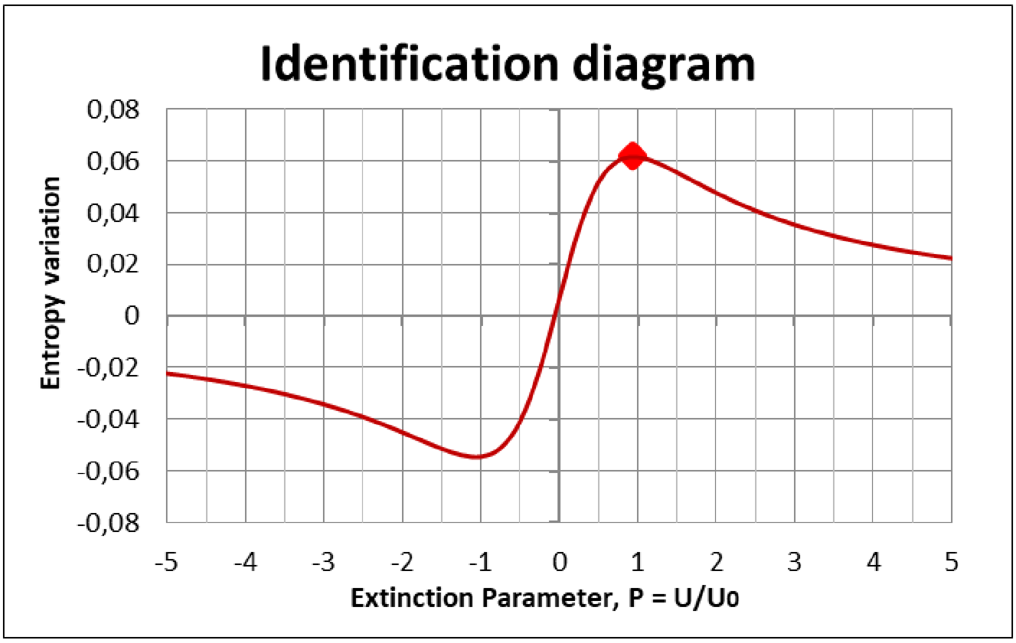

Figure 1: The maximum of the probability function, (diamond), identifies the unique extinction parameter P^*.

The curve has a single positive maximum: this dimensionless number is an astronomical invariant of the Earth/Sun system, since it depends exclusively on the eccentricity, the inclination of the axis and the angle of precession. In fact, the analytical expression of this maximum entropy value is:

$$\theta^+ = \frac{1}{2}\left(a_1 sin\beta_0 + \sqrt{(a_1 sin\beta_0)^2 + (a_1 cos\beta_0 + a_2)^2}\right). \tag{7}$$

The analytical expression for the extinction parameter is also an astronomical invariant:

$$P^* = \; -\frac{1}{2\theta^+}(a_1 cos\beta_0 + a_2). \tag{8}$$

Therefore, when the entropy variation between the two solstices is maximum, the Earth's temperature profile has the highest thermodynamic probability. The value of transmittance U_0^* identified in this way is unique and is the maximum among the possible values that identify the three free parameters (U_0^*,U^*,T_0):

$$U_0^* = \theta^+ \frac{\varphi F_0}{2\, a_0(T_2 - T_1)}\;, \tag{9}$$

$$U^* = U_0^* P^*\;, \tag{10}$$

$$T_0 = T_1 - \frac{F_0}{4a_0}\frac{\varphi}{U^*}\left\{\left\{1 + \frac{P^*}{(P^*)^2+1}\{a_1[P^* cos\beta_0 + sin\beta_0] + a_2 P^*\} + \right.\right.$$
$$+ \frac{P^*}{(P^*)^2+4}\left\{b_0\left[cos\beta_0[P^* cos\beta_0 + 2sin\beta_0] + \frac{2}{P^*}\right] + b_2\left[P^* + \frac{2}{P^*}\right] + \right.$$
$$\left.\left.\left. + \frac{b_1}{2}\left[[P^* cos\beta_0 + 2sin\beta_0] + \frac{P^2+4}{P^*}cos\beta_0\right]\right\}\right\}. \tag{11}$$

From this analysis we can conclude that the probability of the Earth's annual temperature profile is maximum when the free parameters satisfy the condition of maximum entropy variation between the two solstices, $(\theta = \theta^{+})$. In the other cases, $(\theta < \theta^{+})$, there is no longer uniqueness because there are two physically possible solutions with the same lower probability factor. This can lead to the phenomenon of bifurcation of solutions when crossing solstices, Figure 2. For each value of the entropy factor, $(\theta < \theta^{+})$, there are two temperature profiles associated with the pair of extinction parameters $P^{\pm}$:

$$P^{\pm} = \frac{-(a_1 cos\beta_0 + a_2) \pm \sqrt{\Delta}}{2\theta^{+}} . \tag{12}$$

The discriminant Δ has the value:

$$\Delta = (a_1 cos\beta_0 + a_2)^2 - 4\theta(a_1 sin\beta_0 + \theta). \tag{13}$$

Each pair of temperatures has a gradually decreasing probability. In fact, it is possible to make a descending two-by-two order of the annual temperature pairs.

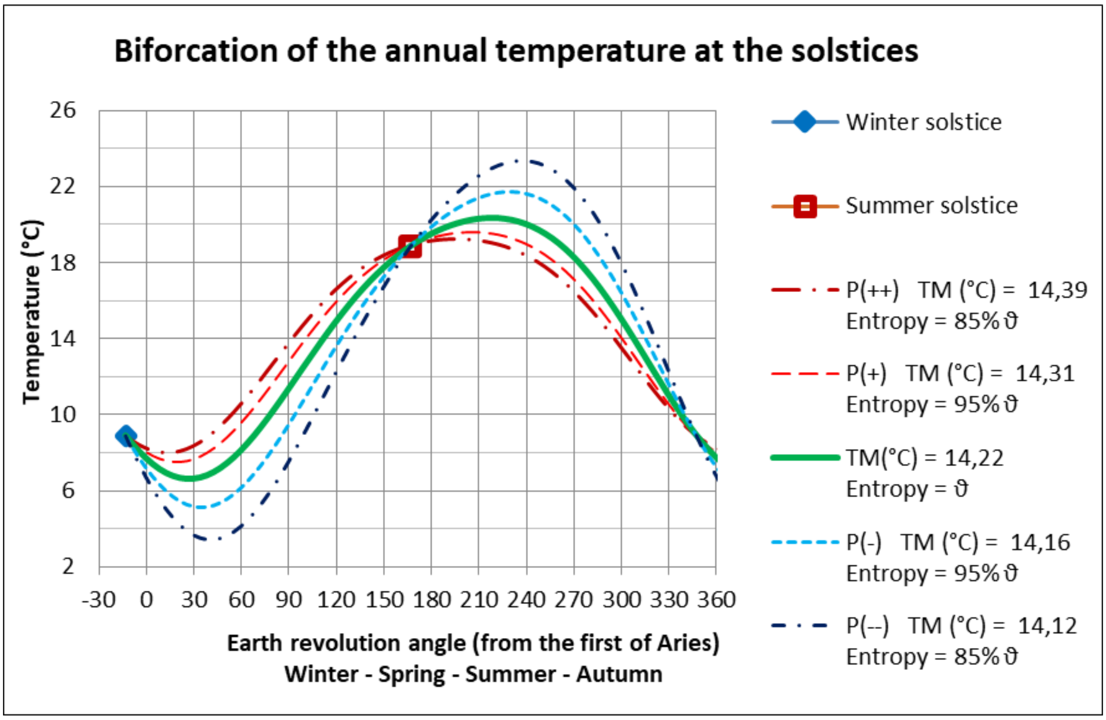

Figure 2: Two pairs of annual temperature profiles intersecting winter/summer solstice temperatures with the most likely profile (continuous line).

Figure 2 shows two pairs of annual temperature profiles of decreasing probability. All profiles intersect the winter and summer solstice temperatures. The most probable profile is in the middle (continuous line). When the extinction parameter is higher than the entropic value, $P^{++} > P^{+} > P^{*}$, there are warmer winters and colder summers than the entropic temperature profile. Conversely, if the extinction parameter is lower than the entropic value, $P^{--} < P^{-} < P^{*}$, there will be colder winters and warmer summers than the entropic temperature profile. From these considerations we can conclude that the Earth/Sun system described by the palaeoclimate model is not ergodic; in fact, a system is ergodic if, over a sufficiently long period of time, it has the same probability of being in each of the possible states. Conversely, if a system is more likely to be in a particular state than in any other possible state, then it is not ergodic.

Conclusions

We have analytically studied the solutions of a paleoclimatic mathematical model of the Earth-Sun system, consisting of a dynamic energy balance equation with three terms (storage, Stefan-Boltzmann radiation loss and solar forcing). Linearisation of the radiation equation allows the analytical solution to be constructed when the free parameters are numerical constants, independent or slightly dependent on the annual mean temperature. The paleoclimatic model has three free parameters (U, U_0, T_0). To determine their value, it is necessary to have (at least) a system of three equations in the three free parameters considered as unknowns. Two equations are obtained very spontaneously by assigning, in the expression of the stationary asymptotic solution, the value of the two isotherms of January and July (T_1, T_2), assimilated to the temperatures of the two solstices, winter and summer. The third equation is missing. This situation is usually referred to as an empirical 'under-determination' of the model. The insufficiency of experimental data alone does not allow us to validate the model, and ultimately, we cannot conclude whether or not the model is adequate to represent the reality we are studying. To get out of indeterminacy, we need to broaden our horizons and place ourselves in a broader context. Since the energy balance equation of the model must represent a real phenomenon, we impose the constraint that the three free parameters (U, U_0, T_0) take the value that makes the equation thus identified most probable. In thermodynamic terms, this constraint implies that the entropy variation of the thermal half-cycle between the two winter and summer solstices is maximum. However, accepting this very natural constraint implies that the probability of the Earth’s annual temperature profile is maximum only when the free parameters satisfy the condition of maximum entropy variation between the two solstices. In other cases, the temperature profile is less probable. From these considerations, we can conclude that the Earth/Sun system described by the paleoclimate model is not ergodic; in fact, a system is ergodic if, over a sufficiently long time, it has the same probability of being in each of the possible states. The existence of a unique solution with the highest probability among the infinite other possible solutions implies the non-ergodic character of the annual temperature.

Appendix

Any astronomical theory of paleoclimate aims to establish a link between the flow of solar energy and the Earth's climate, on a global scale. M. Milankovitch [4] developed such an approach in the last century: he was the first to calculate the solar radiation received by the Earth (insolation) as a function of the geographical coordinates of the Earth and the astronomical parameters, eccentricity and inclination, both variable on a millennial scale. For a critical review of paleoclimatic models, see Berger, [3] and Loutre, [4]. In the following, we summarise the main features of the paleoclimate model discussed in this paper. Previously [1], we developed a millennial energy balance equation for the Northern Hemisphere in terms of the Stefan-Boltzmann dynamic radiation equation. In this paper, we have now analysed its annual energy balance formulation. In order to construct this model, we have adopted the planetary perspective of a distant observer who sees the Earth as a small sphere, essentially blue due to the predominance of water that covers it, and surrounded by a tenuous and thin gaseous atmosphere. The paleoclimate model calculates the solar radiation in terms of the Earth’s alt-azimuth coordinates with respect to the ecliptic plane, i.e. independently of the geographical coordinates. The planetary model is based on the following three constitutive hypotheses

1- The planets describe elliptical orbits around the Sun (Kepler).

2- The planets are material points (Newton).

3- Each planet is associated with an energy balance, in the same spirit as Newton.

In this sense, the paleoclimatic model applies to all the planets of the Solar System. The daily energy balance takes into account the spherical geometry of the Earth. The solar energy incident on the outer surface of the atmosphere is a function of the position on the spherical surface through a Lambert extinction function, specifically modified by the author for the horizontal coordinates of this model. The time scales taken into account by the model are the daily scale of the Earth's rotation, the annual scale of revolution, the millennial scale of variation of eccentricity [5], the inclination of the axis of rotation [6], and the precession of the equinoxes. The angles of revolution and precession are related to perihelion and not to the fixed stars. Therefore, the millennial rotation of the apsidal line connecting perihelion and aphelion has no effect. This model therefore uses a mobile reference of the Eulerian type. This choice is not unimportant: the combination of the anticlockwise precession velocity of - 50.256"/year and the clockwise rotation of the apsis line of + 11.077"/year gives an equivalent velocity of - 61.333"/year, so that the combined cycle of the two movements takes place in about 21,200 years, whereas the precession alone takes place in 25,800 years. The model consists of an energy balance ordinary differential equation for each assigned daily value of the rotation angle ($\alpha = 2\pi t/\tau; \tau = 360\ t_0$). The equation has three terms, a source term (radiative forcing), a loss term and an accumulation term. The forcing term takes into account the tilt of the Earth's axis, (δ), and the eccentricity of the orbit, (e), both of which are variable on a millennial scale, and the angle of precession, ($\beta = 2\pi t/\vartheta$; $\vartheta = 360\ t_1$). We can give the model equation the character of a definition and consider it, in fact, as the constitutive equation of the Earth model considered in this work:

$$\underbrace{U_0\frac{dT}{d\alpha}}_{Accumulation} + \underbrace{U(T-T_0)}_{Loss + Green\ House\ Effect} = \underbrace{F(\alpha,\beta,e,\delta)}_{Solar\ Forcing}. \tag{A1}$$

To obtain an analytical solution of the paleoclimatic model, we linearised the "exact" relationship of the solar energy flux $F(\alpha,\beta,e,\delta)$ by means of a series development of circular functions truncated to the second order [1], thus obtaining the forcing term in the form:

$$F(\alpha,\beta) = \frac{\varphi F_0}{4a_0}\{1 + a_1 cos\alpha + a_2 cos(\alpha-\beta) + b_0 cos^2\alpha + $$

$$+ b_1 cos\alpha \cdot \cos(\alpha-\beta) + b_2 cos^2(\alpha-\beta)\}. \tag{A2}$$

The solution of the linearised energy balance equation (A1) is the sum of two components: an infinitesimal component and a finite component, both of which have a periodic character. The first tends asymptotically to zero as the angle of revolution (α) increases and is a function of the initial condition. The second is independent of the initial condition and describes the stable, periodic behaviour of the solution. This finite component of the solution is the asymptotic solution, i.e. the limit cycle. The analytical solution of the paleoclimate model implicitly takes into account the effect of the atmosphere and the different distribution of Antarctic ice, land and oceans through the accumulation transmittance, U_0 The periodic solution has the following expression, where $P = U/U_0$ is the extinction parameter:

$$T(\alpha) = T_0 + \frac{F_0}{4a_0}\frac{\varphi}{U}\{1 +$$

$$+ \frac{P}{P^2+1}\{a_1[Pcos\alpha + sin\alpha] + a_2[Pcos(\alpha-\beta) + sin(\alpha-\beta)]\} +$$

$$+ \frac{P}{P^2+4}b_0\left\{cos\alpha[Pcos\alpha + 2sin\alpha] + \frac{2}{P}\right\} +$$

$$+ \frac{P}{P^2+4}\frac{b_1}{2}\left\{[Pcos(2\alpha-\beta) + 2sin(2\alpha-\beta)] + \frac{P^2+4}{P}cos\beta\right\} +$$

$$+ \frac{P}{P^2+4}b_2\left\{cos(\alpha-\beta)[Pcos(\alpha-\beta) + 2sin(\alpha-\beta)] + \frac{2}{P}\right\}\Bigg\}. \qquad (A3)$$

In this solution formula, the angle of revolution (α) is the independent variable while the precession angle $\beta < 0$, is an assigned parameter. Table A1 collects the numerical values of the constants.

Table A1: Environmental and astronomical orbital factors that determined the average annual temperature of the Earth, in 1975, in the Northern and Southern hemispheres.

Tropospheric environmental factors			Astronomical Orbital Factors		
			$t_0(s)$	8.766 10^4	
T_{0NL} (°C)	- 273.16		$t_1(s)$	2.261 10^9	
T_0 (°C)	-70		β_0	-0.22362	
φ	0.633		e	+0.0167	
F_0 (W/m^2)	1361.25		$a_0 = (1-e^2)^2$	+0.999442	
	Northern Hemisphere	Southern Hemisphere	$b_0 = e^2$	+0.000279	
T_M (°C)	14.220314	14.078418	$a_1 = 2e$	+0.0334	
T_1 (°C)	8.9	21.5	$b_2 = -2/\pi(1-2/\pi)\delta^2$	-0.038751	
T_2 (°C)	18.9	6.0		Northern Hemisphere	Southern Hemisphere
U; $W/(m^2\,°C)$	2.503600	2.520997088	δ	+0.40928	-0.40928
U_0; $W/(m^2\,°C)$	2.668331	2.420103281	$b_1 = -2/\pi(1-2/\pi)\delta$	-0.004967	+0.004967
P	0.938265	1.041690	$a_2 = -(1-2/\pi)\delta$	-0.148724	+0.148724
$E(1-\epsilon)$	0.465146	0.469073			

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