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Exploring the Statistical Treatments, Different Methods of Parameter Estimation, and Practical Applications of a New Probability Model

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Abstract

This paper proposes a new three-parameter generalized Fréchet distribution using the MTI transformation scheme. We refer to the proposed model as MTI-Fréchet (MTIF) distribution. Several statistical treatments of the MTIF distribution, including survival, hazard rate, and quantile functions, moments, incomplete moments, moment-generating function, probability-weighted moment, and Renyi entropy, are derived. The study adopts four methods of parameter estimation to estimate the parameters of the MTIF distribution, followed by a simulation experiment to investigate the performance of the parameter estimates based on the four methods. The simulation results suggest that the MPS is the most appropriate estimation method for estimating the parameters of the MTIF distribution. The flexibility of the proposed MTIF distribution in practical data fitting is illustrated using two data sets. The results obtained from some model selection criteria and goodness-of-fit test statistics revealed that the MTIF distribution offered a better fit for the data sets compared to the classical Fréchet distribution and other generalized Fréchet distributions.

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1 Introduction

The development of tractable statistical distributions for modeling time-to-event datasets have enormously increased in recent times. Due to the complexity and non-monotonic nature of most datasets, classical distributions have proved abortive in providing an adequate fit. Hence, the advent of several transformation schemes to enhance the flexibility of the existing classical distributions in modeling complex datasets. Details of some recent transformation schemes can be found in the works of Kavya and Manoharan (2020), Nasiru and Abubakar (2022), Kharazmi et al. (2022), Lone et al. (2022), Ubaka and Ewere (2023), Chesneau and Opone (2023), Opone and Chesneau (2024), Diab et al. (2025), and Alsalafi et al. (2025).

The Fréchet distribution is one of such statistical distributions that has been generalized using different transformation schemes. The distribution, also referred to as the inverted Weibull distribution, is known for its suitability in modelling extreme events. The density function of the Fréchet distribution is specified by

$$f(y;\xi,\lambda)=\xi\lambda^{\xi-1}\gamma^{-(\xi+1)}e^{-\left(\frac{\lambda}{\gamma}\right)^{\xi}},\qquad y>0,\xi,\lambda>0, \tag{1}$$

and the distribution function obtained as:

$$F(y;\xi,\lambda)=e^{-\left(\frac{\lambda}{y}\right)^{\xi}},\qquad y>0,\xi,\lambda>0. \tag{2}$$

Taking into account the distribution function in equation (2) as a baseline distribution, many generalizations of the Fréchet distribution have been developed in the literature. For instance, Nadarajah and Kotz (2003) established the exponentiated Fréchet distribution, Nadarajah and Gupta (2004) introduced the beta-Fréchet distribution, Krishna et al. (2013) developed the Marshall-Olkin Fréchet distribution, Mahmoud and Mandouh (2013) proposed the transmuted Fréchet distribution, Mead (2014) studied the Kumaraswamy-Fréchet distribution, Nasiru et al. (2019) developed the alpha power transformed Fréchet distribution, Al-Sobhi (2021) generated the modified kies-Fréchet distribution, Aldahlan (2022) constructed the sine-Fréchet distribution, Mosilhy and Eledum (2022) derived the cubic transmuted Fréchet distribution, Ocloo et al. (2022) pioneered the harmonic mixture Fréchet distribution, Moloy et al. (2024) studied the quartic transmuted Fréchet distribution, etc.

In this paper, we employ the MTI transformation scheme to introduce another novel generalized Fréchet distribution called the MTI-Fréchet (MTIF) distribution. Notable statistical treatments are derived, including survival, hazard rate, and quantile functions, moments, incomplete moments, moment-generating function, probability-weighted moment, and Renyi entropy. Four different methods of parameter estimation are employed, such as the maximum likelihood, ordinary least squares, weighted least squares, and maximum product spacing estimators, to estimate the parameters of the proposed MTIF distribution. Finally, two datasets are utilized to illustrate the relevance of the proposed model over some well-known generalized Fréchet distributions in modeling real-world datasets.

2 The MTI-Frechet (MTIF) Distribution

The MTI transformation scheme has been developed by Lone et al. (2022) with the CDF specified by

$$F_{MTI}\left(y;\psi,\beta\right)=\frac{\beta G\left(y;\psi\right)}{\beta-\log\beta\overline{G}\left(y;\psi\right)},\qquad y>0,\beta>0,\tag{3}$$

and the associated PDF is defined as

$$f_{MTI}\left(y;\psi,\beta\right)=\frac{\beta\left(\beta-\log\beta\right)g\left(y;\psi\right)}{\left[\beta-\log\beta\overline{G}\left(y;\psi\right)\right]^{2}}\qquad y>0,\beta>0.\tag{4}$$

By inserting the CDF of the Fréchet distribution defined in equation (2) into equation (3), the CDF of the MTI-Fréchet distribution is obtained as

$$F_{MTI}\left(y;\xi,\lambda,\beta\right)=\frac{\beta e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\beta-\log\beta\left[1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right]},\qquad y>0,\beta,\lambda,\xi>0.\tag{5}$$

Taking the first derivative of equation (5) yields the corresponding PDF as

$$f_{MTI}\left(y;\xi,\lambda,\beta\right)=\frac{\beta\xi\lambda^{\xi}\left(\beta-\log\beta\right)y^{-(\xi+1)}e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\left[\beta-\log\beta\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right]^{2}},\qquad y>0,\beta,\lambda,\xi>0.\tag{6}$$

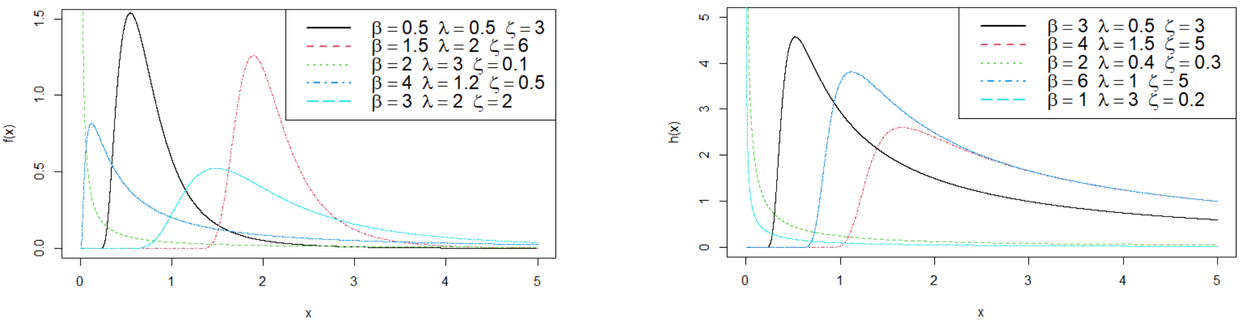
The survival and hazard rate functions of the MTIF distribution are obtained, respectively, by an algebraic manipulation of equations (5) and (6) as:

$$\begin{aligned}S_{MTIF}\left(y;\xi,\lambda,\beta\right)&=1-F_{MTIF}\left(y;\xi,\lambda,\beta\right)\\&=\frac{\left(1-\frac{\log\beta}{\beta}\right)\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)}{\left[1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right]},\qquad y>0,\beta,\lambda,\xi>0,\end{aligned}\tag{7}$$

and

$$h_{MTIF}(y;\xi,\lambda,\beta) = \frac{f_{MTIF}(y;\xi,\lambda,\beta)}{S_{MTIF}(y;\xi,\lambda,\beta)}$$
$$= \frac{\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right)y^{-(\xi+1)}e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\left(1-\frac{\log\beta}{\beta}\right)\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\left(1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right)}, \qquad y>0. \tag{8}$$

The graphical representation of the density function and the hazard rate function of the MTIF distribution for selected parameter values is shown in Figure 1.



(a) The PDF of the MTIF distribution (b) The HRF of the MTIF distribution

Figure 1: The PDF and HRF of the MTIF distribution for selected values of the parameters.

The plots in Figure 1 suggest that the PDF of the MTIF distribution captures a decreasing or right-skewed shaped property, while its hazard rate includes a decreasing and inverted bathtub-shaped property. These unique features make the MTIF distribution suitable for analyzing datasets with heavy-tailed and non-monotonic properties.

2.1 The Quantile Function

The quantile function, also known as the inverted cumulative distribution function, is useful for generating random samples from a probability model. It is denoted by $Q_y(p) = F^{-1}(p)$, $0 < p < 1$. From the CDF defined in equation (5), the quantile function of MTIF distribution is obtained as follows

$$\frac{\beta e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\beta - \log\beta\left[1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right]} = p,$$

$$\beta e^{-\left(\frac{\lambda}{y}\right)^{\xi}} = p\left[\beta - \log\beta\left(1 - e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right],$$

$$e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\left[\beta - p\log\beta\right] = p\left[\beta - \log\beta\right],$$

$$e^{-\left(\frac{\lambda}{y}\right)^{\xi}} = \frac{p\left[\beta - \log\beta\right]}{\left[\beta - p\log\beta\right]}.$$

Taking the natural logarithm of both sides of the equation yields,

$$Q_y(p) = \lambda\left[-\log\left\{\frac{p\left[\beta - \log\beta\right]}{\left[\beta - p\log\beta\right]}\right\}\right]^{-\frac{1}{\xi}}, \qquad 0 < p < 1. \tag{9}$$

Some useful measures of partition, such as the lower quartile, median, and upper quartile of the MTIF distribution, can be generated from equation (9), respectively, as

$$Q_y\left(\frac{1}{4}, \xi, \lambda, \beta\right) = \lambda\left[-\log\left\{\frac{\left[\beta - \log\beta\right]}{\left[4\beta - \log\beta\right]}\right\}\right]^{-\frac{1}{\xi}},$$

$$Q_y\left(\frac{1}{2}, \xi, \lambda, \beta\right) = \lambda\left[-\log\left\{\frac{\left[\beta - \log\beta\right]}{\left[2\beta - \log\beta\right]}\right\}\right]^{-\frac{1}{\xi}},$$

and

$$Q_y\left(\frac{3}{4}, \xi, \lambda, \beta\right) = \lambda\left[-\log\left\{\frac{3\left[\beta - \log\beta\right]}{\left[4\beta - 3\log\beta\right]}\right\}\right]^{-\frac{1}{\xi}}.$$

2.2 The Moments, Incomplete Moment and Moment Generating Function

Suppose a random variable Y is associated with a known probability distribution with PDF $f(y)$, then the rth ordinary moment of Y is defined by

$$E\left[Y^r\right] = \int_{-\infty}^{\infty} y^r f\left(y\right) dy, \qquad r = 1, 2, 3, 4. \tag{10}$$

By substituting the density function in equation (6) into equation (10), the rth ordinary moment of a random variable Y following the MTIF distribution is constructed as follows

$$E\left[Y^r\right] = \int_{0}^{\infty} y^r \frac{\xi\lambda^{\xi}\left(1 - \frac{\log\beta}{\beta}\right) y^{-(\xi+1)} e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\left\{1 - \frac{\log\beta}{\beta}\left(1 - e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right\}^2} dy. \tag{11}$$

The expression in equation (11) can be evaluated using the generalized binomial series expansion

$$(1+y)^{-s}=\sum_{k=0}^{\infty}\binom{s+k-1}{k}y^{k}. \tag{12}$$

Thus,

$$\left\{1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right\}^{-2}=\sum_{j=0}^{\infty}(j+1)\left(\frac{\log\beta}{\beta}\right)\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)^{j},$$

$$\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)^{j}=\sum_{k=0}^{j}\binom{j}{k}(-1)^{k}e^{-k\left(\frac{\lambda}{y}\right)^{\xi}},$$

so that equation (11) now becomes

$$E\left[Y^{r}\right]=\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{j}(j+1)\left(\frac{\log\beta}{\beta}\right)^{j}\binom{j}{k}(-1)^{k}\int_{0}^{\infty}y^{r-\xi-1}e^{-(k+1)\left(\frac{\lambda}{y}\right)^{\xi}}dy. \tag{13}$$

Resolving the integral part of equation (13),

let $w=(k+1)\left(\frac{\lambda}{y}\right)^{\xi}$, $\quad y=\lambda\left(\frac{w}{(k+1)}\right)^{-\frac{1}{\xi}}$, $\quad dy=\frac{-\lambda}{\zeta(k+1)}\left(\frac{w}{(k+1)}\right)^{-\frac{1}{\xi}-1}dw$. Since $y\in(0,\infty)$, then as $y{\to}0$, $w{\to}\infty$. Also, as $y{\to}\infty$, $w{\to}0$. Hence, $w\in(\infty,\ 0)$.

That is,

$$\begin{aligned}\int_{0}^{\infty}y^{r-\xi-1}e^{-(k+1)\left(\frac{\lambda}{y}\right)^{\xi}}dy&=-\int_{\infty}^{0}\left(\lambda\left(\frac{w}{(k+l)}\right)^{-\frac{1}{\xi}}\right)^{r-\xi-1}e^{-w}\frac{\lambda}{\xi\,(k+1)}\left(\frac{w}{k+1}\right)^{\frac{-1}{\xi}-1}dw,\\&=\frac{\lambda^{2}}{\xi(k+1)}\int_{0}^{\infty}\left(\frac{w}{k+1}\right)^{\frac{-r}{\xi}}e^{-w}dw,\\&=\frac{\lambda^{2}(k+1)^{\frac{r}{\xi}}}{\xi(k+1)}\int_{0}^{\infty}w^{\frac{-r}{\xi}}e^{-w}dw,\\&=\frac{\lambda^{2}}{\xi}(k+1)^{\frac{r}{\xi}-1}\Gamma\left(1-\frac{r}{\xi}\right).\end{aligned}$$

The rth ordinary moment of the MTIF distribution is thus, obtained as

$$E\left[Y^{r}\right]=\lambda^{2+\xi}\left(1-\frac{\log\beta}{\beta}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{j}(j+1)\left(\frac{\log\beta}{\beta}\right)^{j}\binom{j}{k}(k+1)^{\frac{r}{\xi}-1}(-1)^{k}\Gamma\left(1-\frac{r}{\xi}\right). \tag{14}$$

The first four ordinary moments of the MTIF distribution are derived from equation (14) when $r=1$, 2, 3, and 4, respectively.

Furthermore, the incomplete moment of the MTIF distribution can be constructed from the expression:

$$m_r\left(t\right) = \int_0^t y^r f\left(y\right) dy. \tag{15}$$

Following the same procedure of generating the ordinary moment, we have the following:

$$m_r\left(t\right) = \xi\lambda^{\xi}\left(1 - \frac{\log\beta}{\beta}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\left(j+1\right)\left(\frac{\log\beta}{\beta}\right)^j\left(\begin{array}{c} j \\ k \end{array}\right)\left(-1\right)^k\int_0^t y^{r-\xi-l}e^{-(k+1)\left(\frac{\lambda}{y}\right)^{\xi}}dy. \tag{16}$$

Again, resolving the integral part of equation (16), since $y \in (0,t)$, we have $y{\rightarrow}0$, $w{\rightarrow}\infty$. Also, as $y{\rightarrow}t$, $w{\rightarrow}\left(k+1\right)\left(\frac{\lambda}{t}\right)^{\xi}$. Hence, $w \in \left(\infty, \left(k+1\right)\left(\frac{\lambda}{t}\right)^{\xi}\right)$.

so that,

$$\begin{aligned}
\int_0^t y^{r-\xi-1}e^{-(k+1)\left(\frac{\lambda}{y}\right)^{\xi}}dy &= -\int_{\infty}^{(k+1)\left(\frac{\lambda}{t}\right)^{\xi}}\left(\lambda\left(\frac{w}{k+1}\right)^{-\frac{1}{\xi}}\right)^{r-\xi-1}e^{-w}\frac{\lambda}{\xi\left(k+1\right)}\left(\frac{w}{k+1}\right)^{-\frac{1}{\xi}-1}dw,\\
&= \int_{(k+1)\left(\frac{\lambda}{t}\right)^{\xi}}^{\infty}\left(\lambda\left(\frac{w}{k+1}\right)^{-\frac{1}{\xi}}\right)^{r-\xi-1}e^{-w}\frac{\lambda}{\xi\left(k+1\right)}\left(\frac{w}{k+1}\right)^{-\frac{1}{\xi}-1}dw,\\
&= \frac{\lambda^2}{\xi}\left(k+1\right)^{\frac{r}{\xi}-1}\int_{(k+1)\left(\frac{\lambda}{t}\right)^{\xi}}^{\infty}w^{-\frac{r}{\xi}}e^{-w}dw,\\
&= \frac{\lambda^2}{\xi}\left(k+1\right)^{\frac{r}{\xi}-1}\Gamma\left(\left(k+1\right)\left(\frac{\lambda}{t}\right)^{\xi}, 1 - \frac{r}{\xi}\right).
\end{aligned}$$

Hence, the rth incomplete moment of the MTIF distribution is derived as

$$m_r\left(t\right) = \lambda^{2+\xi}\left(1 - \frac{\log\beta}{\beta}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\left(j+1\right)\left(\frac{\log\beta}{\beta}\right)^j\left(\begin{array}{c} j \\ k \end{array}\right)\left(-1\right)^k\left(k+1\right)^{\frac{r}{\xi}-1}\Gamma\left(\left(k+1\right)\left(\frac{\lambda}{t}\right)^{\xi}, 1 - \frac{r}{\xi}\right). \tag{17}$$

The moment generating function of the MTIF distribution can be obtained by introducing the Maclaurin series expansion of the exponential function to the rth ordinary moment defined in equation (14) as

$$\begin{aligned}
M_Y\left(t\right) &= E\left[e^{ty}\right] = \int_{-\infty}^{\infty} e^{ty} f\left(y\right) dy,\\
&= \int_0^{\infty} e^{ty}\frac{\xi\lambda^{\xi}\left(1 - \frac{\log\beta}{\beta}\right)y^{-(\xi+1)}e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\left\{1 - \frac{\log\beta}{\beta}\left(1 - e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right\}^2}dy,\\
&= \lambda^{2+\xi}\left(1 - \frac{\log\beta}{\beta}\right)\sum_{j,m=0}^{\infty}\sum_{k=0}^{j}\left(j+1\right)\left(\frac{\log\beta}{\beta}\right)^j\left(\begin{array}{c} j \\ k \end{array}\right)\left(k+l\right)^{\frac{m}{\xi}-1}\frac{\left(-1\right)^k t^m}{m!}\Gamma\left(1 - \frac{m}{\xi}\right).
\end{aligned} \tag{18}$$

It is important to note that the first four raw moments of the MTIF distribution can also be obtained from equation (18) by taking its first four derivatives, respectively, and evaluating the expression at $t = 0$. That is, $E[Y] = M_y'(0)$, $E[Y^2] = M_y''(0)$, $E[Y^3] = M_y'''(0)$, and $E[Y^4] = M_y''''(0)$.

2.3 The Probability-Weighted Moments

Greenwood et al. (1979) have defined the Probability Weighted Moments (PWMs) of a random variable Y with the density function, $f(y)$, as

$$\rho_{q,r} = E\left[f(y)\, F^q(y)\right] = \int_{-\infty}^{\infty} y^r f(y)\, F^q(y)\, dy. \tag{19}$$

Direct substitution of the PDF and CDF of the MTIF distribution into equation (19), we derive the PWMs of the MTIF distribution as follows:

$$\rho_{q,r} = \int_0^{\infty} y^r \frac{\beta^{-q}\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right) y^{-(\xi+1)} e^{-(q+1)\left(\frac{\lambda}{y}\right)^{\xi}}}{\left\{1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right\}^{2+q}} dy. \tag{20}$$

Now, using the generalized binomial series expansion in equation (20), we have

$$\left\{1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)\right\}^{-(2+q)} = \sum_{j=0}^{\infty}\binom{q+j+1}{j}\left(\frac{\log\beta}{\beta}\right)^j\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)^j,$$

$$\left(1-e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right)^j = \sum_{k=0}^{j}\binom{j}{k}(-1)^k e^{-k\left(\frac{\lambda}{y}\right)^{\xi}}.$$

Inserting these expressions into equation (20), yields

$$\rho_{q,r} = \beta^{-q}\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right)\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{q+j+1}{j}\binom{j}{k}\left(\frac{\log\beta}{\beta}\right)^j(-1)^k\int_0^{\infty} y^{r-\xi-1} e^{-(q+k+1)\left(\frac{\lambda}{y}\right)^{\xi}} dy. \tag{21}$$

Evaluating the integral part,

let $w = (q+k+1)\left(\frac{\lambda}{y}\right)^{\xi}$, which implies that $y = \lambda\left[\frac{w}{(q+k+1)}\right]^{-\frac{1}{\xi}}$ and $dy = \frac{-\lambda}{\xi(q+k+1)}\left[\frac{w}{(q+k+1)}\right]^{-\frac{1}{\xi}-1} dw$.

Note that $y \in (0, \infty)$. As $y \to 0$, $w \to \infty$. Also, as $y \to \infty$, $w \to 0$. Hence, $w \in (\infty, 0)$.

Consequently,

$$\begin{aligned}\int_0^\infty y^{r-\xi-1} e^{-(q+k+1)\left(\frac{\lambda}{y}\right)^\xi} dy &= -\int_\infty^0 \left[\lambda\left(\frac{w}{q+k+1}\right)^{-\frac{1}{\xi}}\right]^{r-\xi-1} e^{-w} \frac{\lambda}{\xi(q+k+1)}\left[\frac{w}{q+k+1}\right]^{-\frac{1}{\xi}-1} dw,\\ &= \frac{\lambda^{r-\xi}}{\xi(q+k+1)^{1-\frac{r}{\xi}}}\int_0^\infty w^{\frac{-r}{\xi}} e^{-w} dw,\\ &= \frac{\lambda^{r-\xi}}{\xi(q+k+1)^{1-\frac{r}{\xi}}}\Gamma\left(1-\frac{r}{\xi}\right).\end{aligned}$$

Finally, the PWM of the MTIF distribution is expressed as:

$$\rho_{q,r} = \frac{\beta^{-q}\lambda^r\left(1-\frac{\log\beta}{\beta}\right)}{(q+k+1)^{1-\frac{r}{\xi}}}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\begin{pmatrix} q+j+1 \\ j\end{pmatrix}\begin{pmatrix} j \\ k\end{pmatrix}\left(\frac{\log\beta}{\beta}\right)^j(-1)^k\,\Gamma\left(1-\frac{r}{\xi}\right). \tag{22}$$

2.4 The Renyi Entropy

Let Y be a random variable following a probability distribution, the entropy of Y describes the degree of uncertainty associated with Y. Renyi (1961) has defined the Renyi entropy of Y as

$$R(\Psi) = \frac{1}{1-\Psi}\log\int_{-\infty}^{\infty} f^{\Psi}(y)\,dy, \qquad \Psi > 0, \Psi \neq 1. \tag{23}$$

Again, by substituting the PDF of the MTIF distribution into equation (23), we have

$$R(\Psi) = \frac{1}{1-\Psi}\log\int_0^\infty \frac{\left[\xi\lambda^\xi\left(1-\frac{\log\beta}{\beta}\right)\right]^\Psi y^{-\Psi(\xi+1)} e^{-\Psi\left(\frac{\lambda}{y}\right)^\xi}}{\left[1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^\xi}\right)\right]^{2\Psi}} dy. \tag{24}$$

Now,

$$\left[1-\frac{\log\beta}{\beta}\left(1-e^{-\left(\frac{\lambda}{y}\right)^\xi}\right)\right]^{-2\Psi} = \sum_{j=0}^{\infty}\begin{pmatrix} 2\Psi+j-1 \\ j\end{pmatrix}\left(\frac{\log\beta}{\beta}\right)^j\left(1-e^{-\left(\frac{\lambda}{y}\right)^\xi}\right)^j,$$

$$\left(1-e^{-\left(\frac{\lambda}{y}\right)^\xi}\right)^j = \sum_{k=0}^{j}\begin{pmatrix} j \\ k\end{pmatrix}(-1)^k e^{-k\left(\frac{\lambda}{y}\right)^\xi}.$$

Inserting these expressions into equation (24), yields

$$R(\Psi) = \frac{1}{1-\Psi}\log\left[\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right)\right]^{\Psi}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{2\Psi+j-l}{j}\binom{j}{k}\left(\frac{\log\beta}{\beta}\right)^{j}(-1)^{k} \times \int_{0}^{\infty} y^{-\Psi(\xi+1)} e^{-(\Psi+k)\left(\frac{\lambda}{y}\right)^{\xi}}dy. \tag{25}$$

The integral part of equation (25) is further evaluated as

$$\int_{0}^{\infty} y^{-\Psi(\xi+1)} e^{-(\Psi+k)\left(\frac{\lambda}{y}\right)^{\xi}}dy = \frac{\lambda^{1-\Psi(\xi+1)}}{\xi(\Psi+k)^{\frac{(\Psi+1)}{\xi}+\Psi}}\Gamma\left(\Psi+\frac{(\Psi-1)}{\xi}\right), \tag{26}$$

so that the Renyi entropy of the MTIF distribution now becomes

$$R(\Psi) = \frac{1}{1-\Psi}\log\left[\xi\lambda^{\xi}\left(1-\frac{\log\beta}{\beta}\right)\right]^{\Psi}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{2\Psi+j-l}{j}\binom{j}{k}\left(\frac{\log\beta}{\beta}\right)^{j}(-1)^{k} \times \frac{\lambda^{1-\Psi(\xi+1)}}{\xi(\Psi+k)^{\frac{(\Psi+1)}{\xi}+\Psi}}\Gamma\left(\Psi+\frac{(\Psi-1)}{\xi}\right). \tag{27}$$

3 Methods of Parameter Estimation and Simulation Experiment

This section is dedicated to exploring different methods of parameter estimation applicable to the unknown parameters of the MTIF distribution. In particular, maximum likelihood, ordinary least squares, weighted least squares, and maximum product spacing estimators are investigated to determine the best estimation method for the unknown parameters of the MTIF distribution.

3.1 Maximum Likelihood Estimator(MLE)

Let $(y_1, y_2, \ldots, y_n)$ be random samples from the MTIF distribution with PDF defined in equation (6), then the log-likelihood function of Y is derived by

$$\ell_{\text{mle}}(y_i;\xi,\lambda,\beta) = n\log_e(\beta) + n\log_e(\xi) + n\xi\log_e(\lambda) + n\log_e(\beta - log_e\beta) - (\xi+1)\sum_{i=1}^{n}\log_e(y_i) - \sum_{i=1}^{n}\left(\frac{\lambda}{y_i}\right)^{\xi} - 2\sum_{i=1}^{n}\log_e\left[\beta - \beta\log_e\left(1-e^{-\left(\frac{\lambda}{y_i}\right)^{\xi}}\right)\right]. \tag{28}$$

Taking the partial derivative of the function in equation (28) in terms of the parameters ξ, λ, and β, the estimates $\hat{\xi}$, $\hat{\lambda}$, and $\hat{\beta}$ are, respectively, obtained as the solution of the system of non-linear equations $\frac{\partial l_{mle}(y_i;\xi,\lambda,\beta)}{\partial\xi} = 0$, $\frac{\partial l_{mle}(y_i;\xi,\lambda,\beta)}{\partial\lambda} = 0$, and $\frac{\partial l_{mle}(y_i;\xi,\lambda,\beta)}{\partial\beta} = 0$.

3.2 Ordinary Least Squares Estimator (OLS)

Swain (1988) developed an alternative method of parameter estimation known as the ordinary least squares estimation. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be a set of ordered statistics with corresponding CDF $F(Y_i)$, and $y_{(1)}, y_{(2)}, \ldots, y_{(n)}$ be the ordered observed values. The ordinary least squares estimates is obtained by differentiating the function

$$\ell_{\text{ols}}(y_i;\psi) = \sum_{i=1}^{n} \left[F(Y_i;\psi) - \mathbb{E}[F(Y_i;\psi)]\right]^2 \qquad \psi = (\xi, \lambda, \beta)^T, \tag{29}$$

with respect to the parameters ξ, λ, and β, where $E[F(Y_i;\psi)] = \frac{i}{n+1}$. Substituting the CDF of the MTIF distribution into equation (29), we derive the ordinary least squares estimates from the function

$$\ell_{\text{ols}}\left(y_i;\xi,\lambda,\beta\right) = \sum_{i=1}^{n} \left[\frac{\beta e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\beta - \log\beta\left[1 - e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right]} - \frac{i}{n+1}\right]^2. \tag{30}$$

That is, the estimates $\hat{\xi}_{ols}, \hat{\lambda}_{ols}$, and $\hat{\beta}_{ols}$ are the solution of the system of non-linear equations $\frac{\partial l_{ols}(y_i;\xi,\lambda,\beta)}{\partial \xi} = 0$, $\frac{\partial l_{ols}(y_i;\xi,\lambda,\beta)}{\partial \lambda} = 0$, and $\frac{\partial l_{ols}(y_i;\xi,\lambda,\beta)}{\partial \beta} = 0$, respectively.

3.3 Weighted Least Squares Estimator(WLS)

Similar to the OLS method, Swain (1988) also developed the weighted least squares estimation method. In this method, the weighted least square estimates $\hat{\xi}_{wls}$, $\hat{\lambda}_{wls}$, and $\hat{\beta}_{wls}$ of the parameters of the MTIF distribution are derived by taking the partial derivative of the function

$$\begin{aligned}\ell_{\text{wls}}((y_i;\xi,\lambda,\beta) &= \sum_{i=1}^{n} W_i \left[F(Y_i;\psi) - \frac{i}{n+1}\right]^2, \\ &= \sum_{i=1}^{n} W_i \left[\frac{\beta e^{-\left(\frac{\lambda}{y}\right)^{\xi}}}{\beta - \log\beta\left[1 - e^{-\left(\frac{\lambda}{y}\right)^{\xi}}\right]} - \frac{i}{n+1}\right]^2,\end{aligned} \tag{31}$$

in terms of the parameters ξ, λ, and β, where $W_i = \frac{1}{\text{Var}[F(Y_i;\xi,\lambda,\beta)]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$. The estimates $\hat{\xi}_{wls}$, $\hat{\lambda}_{wls}$, and $\hat{\beta}_{wls}$ are obtained as the solution of the system of equations $\frac{\partial l_{wls}(y_i;\xi,\lambda,\beta)}{\partial \xi} = 0$, $\frac{\partial l_{wls}(y_i;\xi,\lambda,\beta)}{\partial \lambda} = 0$, and $\frac{\partial l_{wls}(y_i;\xi,\lambda,\beta)}{\partial \beta} = 0$, respectively.

3.4 Maximum Product Spacing Estimator (MPS)

Cheng and Amin (1979) derived the maximum product spacing method, which suggests that the differences between the values of the distribution function at consecutive observed values should be identically distributed. Let the distribution function follow the MTIF distribution, then by taking the partial derivative of the function

$$\ell_{\text{mps}}\left(y_i;\xi,\lambda,\beta\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F_{\text{MTIF}}(Y_{(i)};\xi,\lambda,\beta)-F_{\text{MTIF}}(Y_{(i-1)};\xi,\lambda,\beta)\right], \quad (32)$$

with respect to the parameters ξ, λ, and β, the estimates $\hat{\xi}_{mps}$, $\hat{\lambda}_{mps}$, and $\hat{\beta}_{mps}$ are obtained as the solution of the system of equations $\frac{\partial l_{mps}(y_i;\xi,\lambda,\beta)}{\partial \xi}=0$, $\frac{\partial l_{mps}(y_i;\xi,\lambda,\beta)}{\partial \lambda}=0$, and $\frac{\partial l_{mps}(y_i;\xi,\lambda,\beta)}{\partial \beta}=0$, respectively.

3.5 Simulation experiment

In this subsection, a Monte Carlo simulation experiment is carried out to examine the level of precision of the parameter estimates of the MTIF distribution. The simulation experiment is based on the four methods of parameter estimation in order to determine the best method suitable for the parameters of the MTIF distribution. Random samples of size $n = 30, 75, 200, 500$, and 1000, are generated from the MTIF distribution using the quantile function defined in equation (9). The experiment is repeated 1000 times, and the mean estimate, absolute bias, and root mean square error of the parameter estimates are computed as follows;

1. mean estimate $=\frac{1}{1000}\sum_{i=1}^{1000}\hat{\psi}_i$
2. absolute bias $=\frac{1}{1000}\sum_{i=1}^{1000}\left|\hat{\psi}_i-\psi\right|$
3. root mean square error (RMSE) $=\sqrt{\frac{1}{1000}\sum_{i=1}^{1000}(\hat{\psi}_i-\psi)^2}$

Tables 1–3 provide the simulation results of the mean estimate, absolute bias, and root mean square error of the parameter estimates of the MTIF distribution, respectively.

Table 1: Mean Estimates of the parameters of the MTIF distribution

Parameter		**n**	**MLE**	**OLS**	**WLS**	**MPS**
$\beta = 0.5$, $\lambda = 0.2$, $\xi = 0.3$	β	30	0.8842	1.1429	1.0748	0.8075
		75	0.7711	0.9285	0.8827	0.6771
		200	0.6103	0.7301	0.6540	0.5731
		500	0.5344	0.5748	0.5436	0.5261
		1000	0.5149	0.5272	0.5182	0.5117
	λ	30	1.4135	2.1153	1.9492	1.2377
		75	0.9024	1.2794	1.1496	0.7460
		200	0.5092	0.7684	0.5986	0.4408
		500	0.3028	0.4131	0.3292	0.2901
		1000	0.2413	0.2817	0.2499	0.2390
	ξ	30	0.3202	0.2983	0.3022	0.3484
		75	0.3067	0.2970	0.2986	0.3207
		200	0.3013	0.2970	0.2980	0.3083
		500	0.2995	0.2977	0.2985	0.3030
		1000	0.2998	0.2985	0.2991	0.3018
$\beta = 0.8$, $\lambda = 0.4$, $\xi = 0.6$	β	30	1.2008	1.3888	1.3798	1.0557
		75	1.2268	1.3899	1.3604	1.0705
		200	1.0632	1.2356	1.1632	0.9505
		500	0.9412	1.0422	0.9697	0.8763
		1000	0.8718	0.9456	0.8906	0.8391
	λ	30	0.4645	0.4792	0.4868	0.4372
		75	0.4731	0.4869	0.4886	0.4410
		200	0.4514	0.4736	0.4667	0.4227
		500	0.4323	0.4523	0.4402	0.4139
		1000	0.4235	0.4390	0.4287	0.4113
	ξ	30	0.6769	0.6370	0.6388	0.7435
		75	0.6315	0.6154	0.6155	0.6663
		200	0.6110	0.6038	0.6039	0.6304
		500	0.6047	0.6003	0.6011	0.6153
		1000	0.6002	0.5971	0.5982	0.6069

Table 2: Absolute bias of the parameters of the MTIF distribution

Parameter		**N**	**MLE**	**OLS**	**WLS**	**MPS**
$\beta = 0.5$						
$\lambda = 0.2$	β	30	0.6133	0.9672	0.8871	0.5239
$\xi = 0.3$		75	0.4696	0.6889	0.6176	0.3651
		200	0.2365	0.4060	0.2971	0.1939
		500	0.1238	0.1928	0.1418	0.1149
		1000	0.0805	0.1127	0.0885	0.0776
	λ	30	1.3522	2.0896	1.9215	1.1522
		75	0.8381	1.2381	1.1022	0.6652
		200	0.4067	0.6976	0.5096	0.3282
		500	0.1821	0.3147	0.2174	0.1653
		1000	0.1046	0.1623	0.1181	0.1001
	ξ	30	0.0582	0.0669	0.0645	0.0677
		75	0.0426	0.0506	0.0469	0.0435
		200	0.0274	0.0349	0.0301	0.0267
		500	0.0177	0.0227	0.0191	0.0173
		1000	0.0124	0.0157	0.0132	0.0123
$\beta = 0.8$						
$\lambda = 0.4$	β	30	0.8286	1.0946	1.0596	0.7010
$\xi = 0.6$		75	0.7526	0.9668	0.9048	0.6158
		200	0.5013	0.7134	0.6095	0.4070
		500	0.3206	0.4498	0.3534	0.2700
		1000	0.1971	0.2958	0.2196	0.1765
	λ	30	0.2982	0.3648	0.6571	0.2549
		75	0.2563	0.3061	0.2880	0.2232
		200	0.1887	0.2387	0.2112	0.1651
		500	0.1380	0.1773	0.1498	0.1252
		1000	0.0988	0.1305	0.1066	0.0921
	ξ	30	0.1318	0.1406	0.1349	0.1670
		75	0.0909	0.1038	0.0966	0.0993
		200	0.0611	0.0741	0.0669	0.0626
		500	0.0435	0.0535	0.0461	0.0435
		1000	0.0301	0.0381	0.0321	0.0300

Table 3: Root mean square error (RMSE) of the parameters of the MTIF distribution

Parameter		**n**	**MLE**	**OLS**	**WLS**	**MPS**
$\beta = 0.5$						
$\lambda = 0.2$	β	30	0.9746	1.3214	1.2477	0.8623
$\xi = 0.3$		75	0.8121	1.0492	0.9814	0.6439
		200	0.4442	0.7214	0.5513	0.3465
		500	0.1773	0.3559	0.2299	0.1577
		1000	0.1088	0.1589	0.1209	0.1037
	λ	30	3.5629	4.7984	4.5810	3.1524
		75	1.7946	2.3087	2.1732	1.4695
		200	0.9728	1.4196	1.1257	0.7508
		500	0.3368	0.6890	0.4933	0.2924
		1000	0.1649	0.2997	0.1946	0.1544
	ξ	30	0.0761	0.0851	0.0820	0.0910
		75	0.0532	0.0612	0.0578	0.0554
		200	0.0334	0.0425	0.0372	0.0338
		500	0.0223	0.0285	0.0241	0.0218
		1000	0.0156	0.0197	0.0166	0.0154
$\beta = 0.8$						
$\lambda = 0.4$	β	30	1.0967	1.3163	1.2908	0.9604
$\xi = 0.6$		75	1.0405	1.2299	1.1839	0.8854
		200	0.7793	1.0180	0.9145	0.6511
		500	0.5522	0.7241	0.5913	0.4562
		1000	0.3361	0.5146	0.3782	0.2866
	λ	30	0.3924	0.4518	0.4507	0.3418
		75	0.3192	0.3610	0.3453	0.2825
		200	0.2383	0.2879	0.2637	0.2082
		500	0.1829	0.2261	0.1952	0.1648
		1000	0.1326	0.1738	0.1422	0.1211
	ξ	30	0.1794	0.1928	0.1799	0.2234
		75	0.1152	0.1312	0.1212	0.1289
		200	0.0767	0.0914	0.0826	0.0799
		500	0.0538	0.0652	0.0571	0.0544
		1000	0.0375	0.0470	0.0399	0.0372

The result in Table 1 shows that as the sample size n increases, the mean estimate tends to the value of the population parameter. Also, from Tables 2 and 3, the values of the absolute bias and root mean square error decrease as the sample size n increases. These features were consistent for all four methods of estimation. In terms of the performance of the methods, we observed that the maximum product spacing (MPS), which had the smallest values of the absolute bias and root mean square error as the sample size n increases, is the best method for estimating the parameters of the MTIF distribution. This is followed by the maximum likelihood, weighted least squares, and lastly, the ordinary least squares methods.

4 Data fitting

Demonstrating the potential of the proposed model in real-life data fitting is one of the crucial task in developing statistical distributions. In this section, the relevance of the MTIF distribution is illustrated using two real-life datasets. Some existing generalizations of the Fréchet distribution, such as the transmuted Fréchet (TFD), alpha power transformed Fréchet (APTFD), exponentiated Fréchet (ExFD), Sine Fréchet (SFD), and the classical Fréchet distribution, are employed to fit the datasets alongside the proposed MTIF distribution. Model comparison is based on the maximized log-likelihood (LogL), Akaike information criterion (AIC), Komolgorov-Smirnov ($K - S$), Anderson Darling (A^*), and Crammer-von Mises (W^*) test statistics.

Dataset I: This dataset comprises of the trade share data reported in Bantan et al. (2021). The data was also utilized by Chesneau and Opone (2022) to illustrate the flexibility of the power continuous Bernoulli distribution. The data are given as: 0.140501976, 0.156622976, 0.157703221, 0.160405084, 0.160815045, 0.22145839, 0.299405932, 0.31307286, 0.324612707, 0.324745566, 0.329479247, 0.330021679, 0.337879002, 0.339706242, 0.352317631, 0.358856708, 0.393250912, 0.41760394, 0.425837249, 0.43557933, 0.442142904, 0.444374621, 0.450546652, 0.4557693, 0.46834656, 0.473254889, 0.484600782, 0.488949597, 0.509590268, 0.517664552, 0.527773321, 0.534684658, 0.543337107, 0.544243515, 0.550812602, 0.552722335, 0.56064254, 0.56074965, 0.567130983, 0.575274825, 0.582814276, 0.603035331, 0.605031252, 0.613616884, 0.626079738, 0.639484167, 0.646913528, 0.651203632, 0.681555152, 0.699432909, 0.704819918, 0.729232311, 0.742971599, 0.745497823, 0.779847085, 0.798375845, 0.814710021, 0.822956383, 0.830238342, 0.834204197, and 0.979355395. The data has a skewness value of 0.006 and a kurtosis value of 2.553, which suggests approximately symmetric and platykurtic properties, respectively. Figure 2 describes an exploratory visualization of the property of the dataset in terms of the boxplot and the total time on test (TTT) plot.

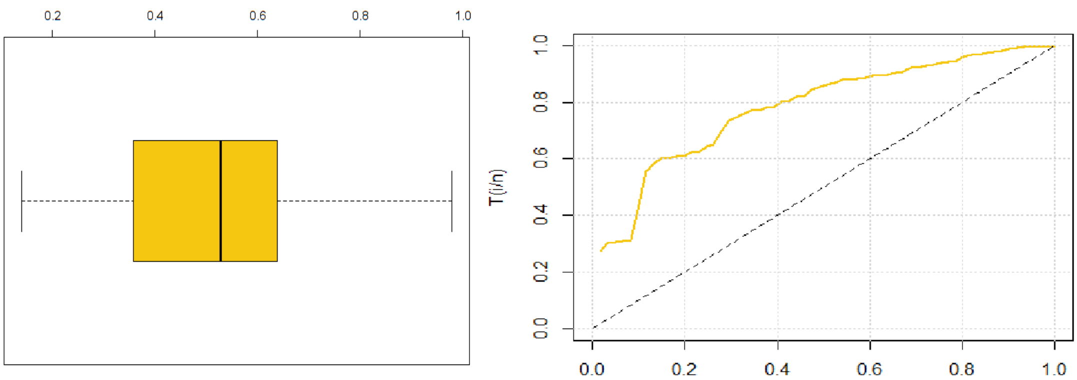


Figure 2: The boxplot and TTT plot for the trade share data.

From the figure, the boxplot suggests that the dataset is approximately symmetric and outlier-free, while the TTT plot reveals that the data exhibits an increasing failure rate property. Table 4 provides the fitting results of the distributions for the trade share data.

Table 4: Summary Result for Trade Share Data

Models	**Estimates**	**-LogL**	**AIC**	**K-S** (p-value)	**A*** (p-value)	**W*** (p-value)
MTIFD	$\beta = 0.0112$ $\lambda = 0.1136$ $\xi = 4.0769$	-9.2675	-12.5349	0.0902 (0.6701)	0.0880 (0.6484)	1.0409 (0.3362)
TFD	$\theta = 0.6731$ $\lambda = 0.3057$ $\xi = 2.1008$	2.0195	10.0391	0.1661 (0.0613)	0.4804 (0.0443)	3.2732 (0.0201)
APTFD	$\alpha = 50.8153$ $\lambda = 0.2397$ $\xi = 2.4737$	-1.3953	3.2094	0.1478 (0.1254)	0.3175 (0.1206)	2.4947 (0.0501)
ExFD	$\alpha = 0.5228$ $\lambda = 0.5199$ $\xi = 1.8844$	4.6759	15.3519	0.1767 (0.0389)	0.6279 (0.0188)	3.9322 (0.0095)
SFD	$\lambda = 0.4853$ $\xi = 1.5048$	-1.2452	1.5096	0.1548 (0.0966)	0.4325 (0.0591)	2.8514 (0.0327)
FD	$\lambda = 0.3686$ $\xi = 1.8843$	4.6759	13.3519	0.1767 (0.0388)	0.6276 (0.0188)	3.9313 (0.0095)

An appropriate model suitable for fitting a given dataset can be investigated by following the model having the least value in terms of the $-LogL$, AIC, $K-S$, A^*, and W^* with the maximum corresponding p-value. Clearly, the results in Table 4 indicates that the proposed MTIF distribution satisfies the above criteria and thus, becomes the most appropriate model for fitting the trade share data.

The fitting abilities of the distributions are also investigated through the fitted density and empirical cdf, and probability-probability (P-P) plots as shown in Figures 3 and 4, respectively.

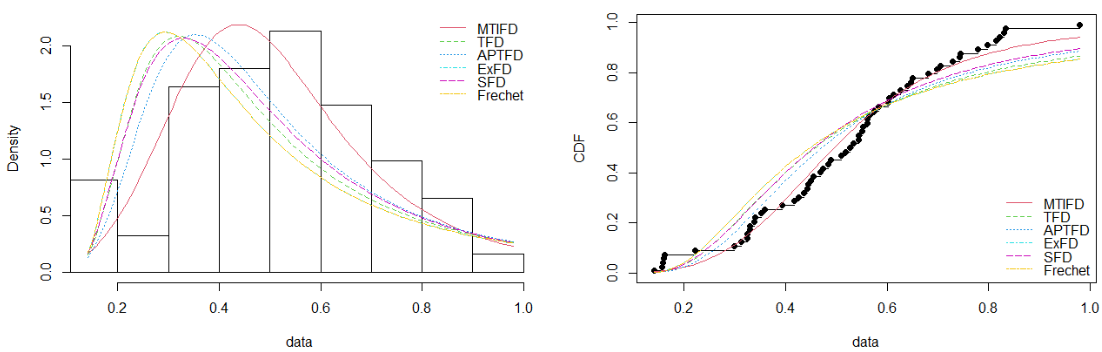


Figure 3: The fitted density and empirical cdf of the trade share data.

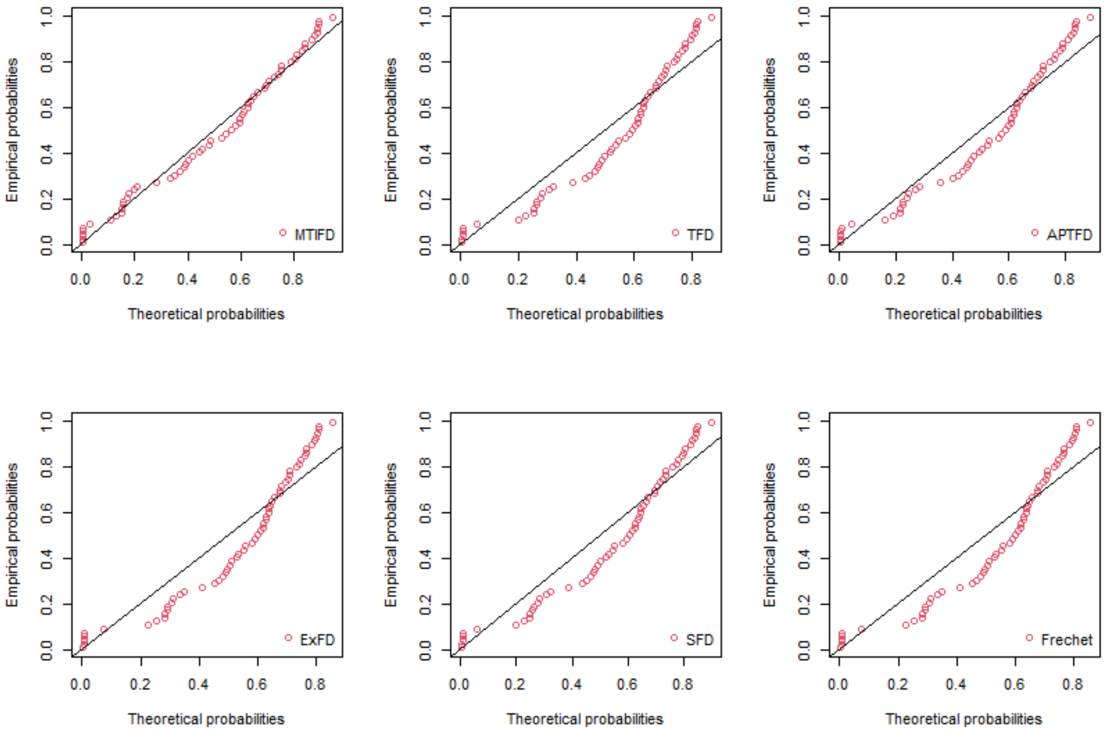


Figure 4: The Probability-Probability (P-P) of the trade share data.

Dataset II: The second dataset holds the waiting time (in minutes) before service of 100 bank customers. This data was first utilized by Ghitany et al. (2008) to demonstrate the usefulness of the Lindley distribution, while Chesneau et al. (2022) transformed the data into a unit interval data and used it to illustrate the flexibility of the transmuted continuous Bernoulli distribution. The dataset are as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9 ,1.9, 2.1, 2.6, 2.7,2.9, 3.1, 3.2, 3.3 ,3.5, 3.6, 4.0, 4.1, 4.2, 4.2,4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9,5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3,6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0,8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6,9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5,11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9,14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0,19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5. The skewness value of the dataset is 1.4727, while the kurtosis value is 5.5402. This information explains that the dataset has right-skewed and leptokurtic properties, respectively. Figure 5 provides an exploratory visualization of the property of the dataset in terms of the boxplot and the total time on test (TTT) plot.

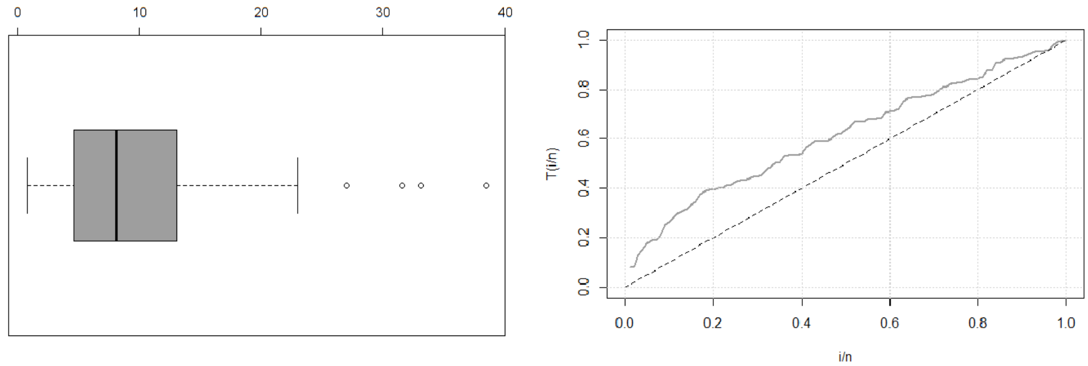


Figure 5: The boxplot and TTT plot for the trade share data.

From the boxplot in Figure 5, the waiting time dataset is skewed to the right and there are presence of outliers, while the TTT plot shows that the data exhibits an increasing failure rate property. The fitting results of the distributions for the waiting time dataset is displayed in Table 5.

Table 5: Summary Results for the Waiting Time Data

Models	**Estimates**	**-LogL**	**AIC**	**K-S** (p-value)	**A*** (p-value)	**W*** (p-value)
MTIFD	$\beta = 0.0184$ $\lambda = 0.7117$ $\xi = 2.2494$	319.2996	644.6993	0.0505 (0.9606)	0.0346 (0.9591)	0.3322 (0.9120)
TFD	$\theta = -0.7579$ $\lambda = 3.4743$ $\xi = 1.2824$	330.6362	667.2723	0.1036 (0.2336)	0.2875 (0.1466)	2.1126 (0.0798)
APTFD	$\alpha = 94.9943$ $\lambda = 2.1297$ $\xi = 1.4874$	326.3056	658.6111	0.0842 (0.4770)	0.1537 (0.3789)	1.3066 (0.2302)
ExFD	$\alpha = 7.9435$ $\lambda = 0.8451$ $\xi = 1.1629$	334.381	674.762	0.1167 (0.1315)	0.4275 (0.0609)	2.8994 (0.0311)
SFD	$\lambda = 7.8602$ $\xi = 0.9154$	326.4818	656.9636	0.0831 (0.4939)	0.2149 (0.2405)	1.5167 (0.1726)
FD	$\lambda = 5.0232$ $\xi = 1.1631$	334.381	672.762	0.1167 (0.1313)	0.4266 (0.0613)	2.8901 (0.0312)

The fitted density and empirical cdf, and probability-probability (P-P) plots of the distributions for the waiting time data are also investigated in Figures 6 and 7, respectively.

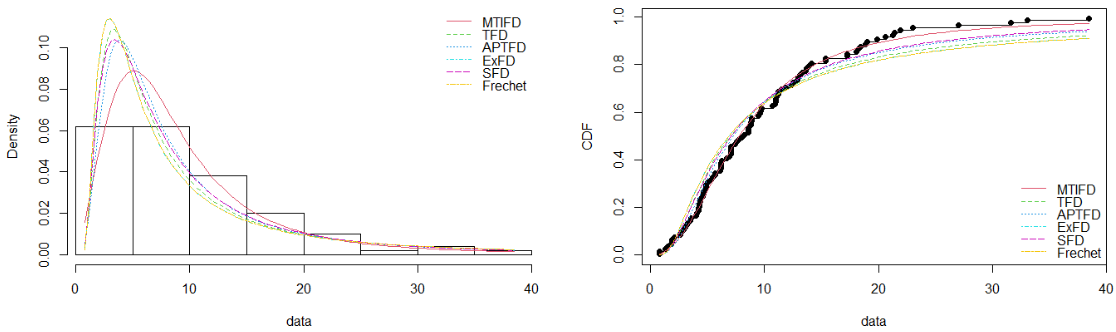


Figure 6: The fitted density and empirical cdf of the waiting time data.

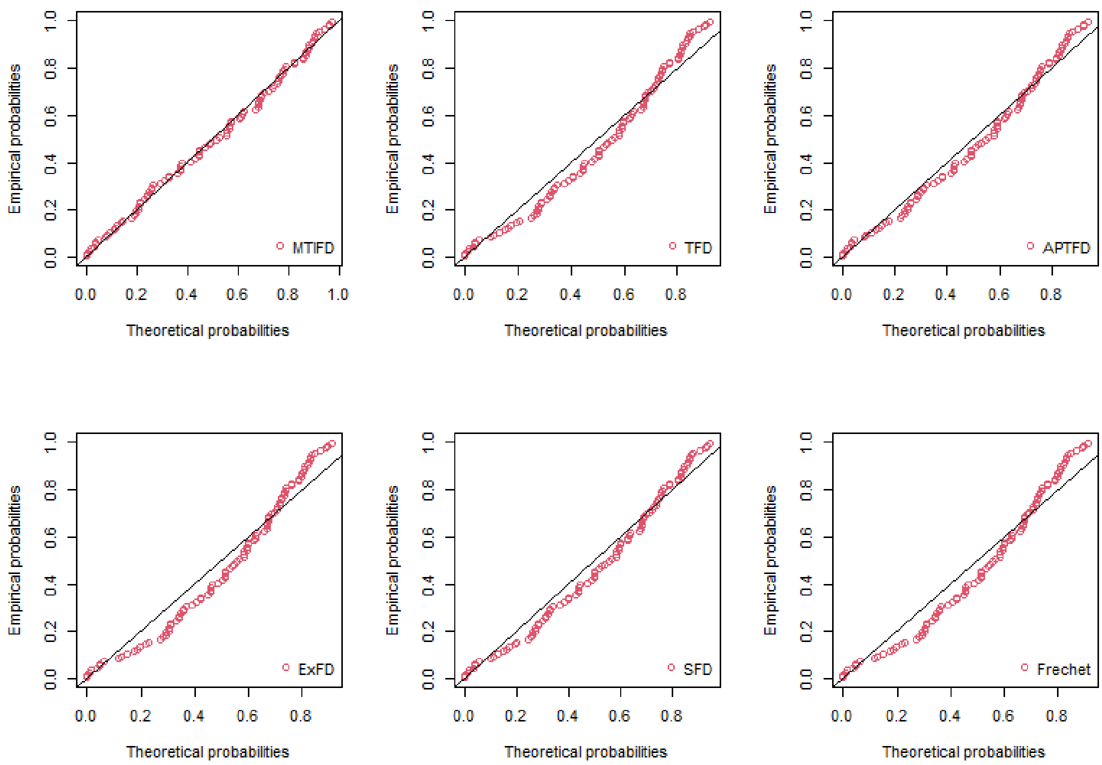


Figure 7: The Probability-Probability (P-P) of the waiting time data.

5 Conclusion

In this paper, the MTI transformation scheme has been employed to generalize the Fréchet distribution. We have referred to the proposed model as the MTI-Fréchet (MTIF) distribution. Some important statistical treatments of the MTIF distribution, which include the survival, hazard rate, and quantile functions, moments, incomplete moments, moment generating function, probability weighted moment, and Renyi entropy, were studied. Graphical plots of the density function and hazard rate function revealed that the MTIF distribution exhibits decreasing, right-skewed, and inverted bathtub-shaped properties. Four methods of parameter estimation were considered to estimate the parameters of the MTIF distribution, and a simulation experiment was conducted to examine the performance of the parameter estimates based on the four methods. The results from the experiment suggested that the maximum product spacing method was the appropriate method for estimating the parameters of the MTIF distribution. In a practical scenario, two datasets comprising of the trade share and waiting time of 100 bank customers were employed to illustrate the applicability of the proposed MTIF distribution. The summary results from the fitting of the two datasets favored the MTIF distribution over the competing generalized Fréchet distributions. The superiority claim was further supported through graphical goodness-of-fit test statistics such as the fitted densities, empirical cdfs, and probability-probability (p-p) plots.

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