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Coincidence and Fixed Point of Nonexpansive Type Mappings in 2-Metric Spaces

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Abstract

The aim of this paper is to prove a coincidence point theorem for a class of self mappings satisfying nonexpansive type condition under various conditions and a fixed point theorem is also obtained. Our results extend and generalize the corresponding result of Singh and Chandrashekhar [7].

1. Introduction and Preliminaries

The concept of 2-metric space was introduced by Gähler [2, 3, 4] whose abstract properties were suggested by the area function in Euclidean space. Employing various contractive conditions Iséki [5] setout the tradition of proving fixed point theorems in 2-metric spaces. Later on, Naidu and Prasad [6] contributed few fixed point theorems in 2-metric spaces introducing the concept of weak commutativity. Recently, Singh and Chandrashekhar [7] proved a fixed point theorem in 2-metric space for nonexpansive

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type mappings. They obtained the following result:

Theorem 1.1. *Let* (X, d) *be a 2-metric space and* $T : X \to X$ *be a self mapping satisfying the following nonexpansive type condition*:

$$d(Tx, Ty, u)$$
$$\leq a \max\left\{d(x, y, u), d(x, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\right\}$$
$$+ b \max\{d(x, Tx, u), d(y, Ty, u)\} + c[d(x, Ty, u) + d(y, Tx, u)], \quad (1)$$

for all $x, y, u \in X$, *where* a, b, c *are real numbers such that* $a + b + 2c = 1$ *and* $a \geq 0, b > 0, c > 0$. *Then T has a unique fixed point and T is continuous at the fixed point.*

Our condition is an extension of that of Ćirić [1] (see also [8]). Also, we will show that our condition (2) includes the above condition (1).

Now we give some definitions which are used frequently to prove our main results.

Definition 1.1. Gähler defined 2-metric space as follows:

A 2-metric on a set X with at least three points is a non-negative real-valued mapping $d : X \times X \times X \to R$ satisfying the following properties:

(1) To each pair of points a, b with $a \neq b$ in X there is a point $c \in X$ such that $d(a, b, c) \neq 0$.

(2) $d(a, b, c) = 0$, if at least two of the points are equal,

(3) $d(a, b, c) = d(b, c, a) = d(a, c, b)$,

(4) $d(a, b, c) \leq d(a, b, u) + d(a, u, c) + d(u, b, c)$ for all $a, b, c, u \in X$.

The pair (X, d) is called a 2-*metric space*.

Definition 1.2. The sequence $\{x_n\}$ is convergent to $x \in X$ and x is the limit of this sequence if $\lim_{n\to\infty} d(x_n, x, u) = 0$ for each $u \in X$.

Definition 1.3. A sequence $\{x_n\}$ is called *Cauchy sequence* if $\lim_{n, m\to\infty} d(x_n, x_m, u) = 0$ for all $u \in X$. A 2-metric space in which every Cauchy sequence is convergent is called *complete*.

Definition 1.4. Let f and g be two self mappings of a 2-metric space (X, d). Then f and g are said to be *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n, u) = 0$ for each $u \in X$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

Let Ψ be a set of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and strictly increasing.

(ψ_2) $\psi(t) = 0$ if and only of $t = 0$.

Let Φ_u be a set of all continuous functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(φ_1) φ is continuous.

(φ_2) $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) \geq 0$.

Let Φ be set of all lower continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) < t$ for all $t > 0$.

In 2014, Ansari [10] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 1.5 [10]**.** Let $F : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous mapping. Then it is called a *C-class function* if it satisfies the following conditions:

(F_1) $F(s, t) \leq s$, for all $(s, t) \in \mathbb{R}_+^2$.

(F_2) $F(s, t) = s$ implies that $s = 0$, or $t = 0$, for all $(s, t) \in \mathbb{R}_+^2$.

Note for some F we have that $F(0, 0) = 0$.

We denote C-class functions as $\mathcal{C}$.

Example 1.1 [10]**.** The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

(1) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$;

(2) $F(s, t) = ms,\ 0 < m < 1,\ F(s, t) = s \Rightarrow s = 0;$

(3) $F(s, t) = \dfrac{s}{(1+t)^r};\ r \in (0, \infty),\ F(s, t) = s \Rightarrow s = 0$ or $t = 0;$

(4) $F(s, t) = \log(t + a^s)/(1+t),\ a > 1,\ F(s, t) = s \Rightarrow s = 0$ or $t = 0;$

(5) $F(s, t) = \ln(1 + a^s)/2,\ a > e,\ F(s, 1) = s \Rightarrow s = 0;$

(6) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l,\ l > 1,\ r \in (0, \infty),\ F(s, t) = s \Rightarrow t = 0;$

(7) $F(s, t) = s \log_{t+a} a,\ a > 1,\ F(s, t) = s \Rightarrow s = 0$ or $t = 0;$

(8) $F(s, t) = s - \left(\dfrac{1+s}{2+s}\right)\left(\dfrac{t}{1+t}\right),\ F(s, t) = s \Rightarrow t = 0;$

(9) $F(s, t) = s\beta(s),\ \beta : [0, \infty) \to [0, 1),$ and is continuous, $F(s, t) = s \Rightarrow s = 0;$

(10) $F(s, t) = s - \dfrac{t}{k+t},\ F(s, t) = s \Rightarrow t = 0;$

(11) $F(s, t) = s - \varphi(s),\quad F(s, t) = s \Rightarrow s = 0,$ here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0;$

(12) $F(s, t) = sh(s, t),\ F(s, t) = s \Rightarrow s = 0;$ here $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0;$

(13) $F(s, t) = s - \left(\dfrac{2+t}{1+t}\right)t,\ F(s, t) = s \Rightarrow t = 0.$

(14) $F(s, t) = \sqrt[n]{\ln(1 + s^n)},\ F(s, t) = s \Rightarrow s = 0;$

(15) $F(s, t) = \phi(s),\quad F(s, t) = s \Rightarrow s = 0,$ here $\phi : [0, \infty) \to [0, \infty)$ is a upper semicontinuous function such that $\phi(0) = 0,$ and $\phi(t) < t$ for $t > 0,$

(16) $F(s, t) = \dfrac{s}{(1+s)^r};\ r \in (0, \infty),\ F(s, t) = s \Rightarrow s = 0;$

(17) $F(s, t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function, $F(s, t) = s \Rightarrow s = 0$;

(18) $F(s, t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x} + t} dx$, where Γ is the Euler Gamma function.

Let Ψ be a set of all non-decreasing continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\psi(0) = 0$.

Let Φ be a set of all continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, such $\phi(0) \geq 0$.

Let Φ_u denote the class of the functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) φ continuous;

(b) $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Let Ψ be a set of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(ψ_1) ψ is continuous and strictly increasing.

(ψ_2) $\psi(t) = 0$ if and only of $t = 0$.

In this paper, we introduce a new class of self mappings satisfying the following nonexpansive type condition:

$$\psi(d(Tx, Ty, u))$$
$$\leq F\left(\begin{matrix}\psi\left(\frac{1}{a(x, y) + b(x, y) + c(x, y) + h(x, y)}\left[\begin{matrix} a(x, y)\max\{d(fx, fy, u), d(fy, Tx, u)\} \\ + b(x, y)\max\{d(fx, Tx, u), d(fy, Ty, u), d(y, Tx, u)\} \\ + c(x, y)d(fx, Ty, u) + h(x, y)d(fy, Tx, u)\end{matrix}\right]\right), \\ \varphi\left(\frac{1}{a(x, y) + b(x, y) + c(x, y) + h(x, y)}\left[\begin{matrix} a(x, y)\max\{d(fx, fy, u), d(fy, Tx, u)\} \\ + b(x, y)\max\{d(fx, Tx, u), d(fy, Ty, u), d(y, Tx, u)\} \\ + c(x, y)d(fx, Ty, u) + h(x, y)d(fy, Tx, u)\end{matrix}\right]\right)\end{matrix}\right) \tag{2}$$

for all $x, y, u \in X$, where $\varphi \in \Phi_u$, $\psi \in \Psi$, $F \in \mathbb{C}$ and $a(x, y) + b(x, y) + c(x, y) + h(x, y) > 0$ and $h(x, y) \geq c(x, y)$.

2. Main Results

Theorem 2.1. *Let* (X, d) *be a 2-metric space. Let T, f be self mappings of X satisfying nonexpansive type condition* (2). *Let* $T(X) \subseteq f(X)$ *and either*

(a) *X is complete and f is surjective, or,*

(b) *X is complete, f is continuous and the pair* (T, f) *is compatible.*

(c) *Either* $f(X)$ *is complete, or* $T(X)$.

Then f and T have a coincidence point in X. Further, the coincidence point is unique, that is, $f_p = f_q$, *whenever* $f_p = T_p$ *and* $f_q = T_q$; $p, q \in X$.

Proof. Let $x = x_0$ be an arbitrary point in X. Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} such that $y_{n+1} = fx_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots$.

On applying inequality (2) and taking $a(x_n, x_{n+1}) = a$, $b(x_n, x_{n+1}) = b$, $c(x_n, x_{n+1}) = c$ and $h(x_n, x_{n+1}) = h$, we get

$$\psi(d(fx_{n+2}, fx_{n+1}, fx_n))$$

$$= \psi(d(Tx_{n+1}, Tx_n, fx_n))$$

$$\leq F\left(\begin{array}{c}\psi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c} a\max\{d(fx_{n+1}, fx_n, fx_n), d(fx_{n+1}, Tx_{n+1}, fx_n)\} \\ +b\max\{d(fx_{n+1}, Tx_{n+1}, fx_n), d(fx_n, Tx_n, fx_n), d(fx_n, Tx_{n+1}, fx_n)\} \\ +cd(fx_{n+1}, Tx_n, fx_n) + hd(fx_n, Tx_{n+1}, fx_n)\end{array}\right]\right), \\ \varphi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c} a\max\{d(fx_{n+1}, fx_n, fx_n), d(fx_{n+1}, Tx_{n+1}, fx_n)\} \\ +b\max\{d(fx_{n+1}, Tx_{n+1}, fx_n), d(fx_n, Tx_n, fx_n), d(fx_n, Tx_{n+1}, fx_n)\} \\ +cd(fx_{n+1}, Tx_n, fx_n) + hd(fx_n, Tx_{n+1}, fx_n)\end{array}\right]\right)\end{array}\right),$$

$$= F\left(\psi\left(\frac{a+b}{a+b+c+h} d(fx_{n+1}, Tx_{n+1}, fx_n)\right), \varphi\left(\frac{a+b}{a+b+c+h} d(fx_{n+1}, Tx_{n+1}, fx_n)\right)\right)$$

$$= F\left(\psi\left(\frac{a+b}{a+b+c+h} d(fx_{n+2}, fx_{n+1}, fx_n)\right), \varphi\left(\frac{a+b}{a+b+c+h} d(fx_{n+2}, fx_{n+1}, fx_n)\right)\right)$$

$$\leq \psi\left(\frac{a+b}{a+b+c+h} d(fx_{n+2}, fx_{n+1}, fx_n)\right) \leq \psi(d(fx_{n+2}, fx_{n+1}, fx_n)).$$

So,

$$\psi\left(\frac{a+b}{a+b+c+h}d(fx_{n+2}, fx_{n+1}, fx_n)\right) = 0,$$

or

$$\varphi\left(\frac{a+b}{a+b+c+h}d(fx_{n+2}, fx_{n+1}, fx_n)\right) = 0$$

thus

$$d(fx_{n+2}, fx_{n+1}, fx_n) = 0. \tag{3}$$

On applying inequality (2) again and using triangular inequality and (3), we get

$$\psi(d(Tx_n, Tx_{n+1}, u))$$

$$\leq F\left(\begin{array}{c}\psi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, fx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}\\ +cd(fx_n, Tx_{n+1}, u)+hd(fx_{n+1}, Tx_n, u)\end{array}\right]\right),\\ \varphi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, fx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}\\ +cd(fx_n, Tx_{n+1}, u)+hd(fx_{n+1}, Tx_n, u)\end{array}\right]\right)\end{array}\right),$$

$$= F\left(\begin{array}{c}\psi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +cd(fx_n, Tx_{n+1}, Tx_n)+cd(fx_n, Tx_n, u)\\ +cd(Tx_{n+1}, Tx_n, u)\end{array}\right]\right),\\ \varphi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +cd(fx_n, Tx_{n+1}, Tx_n)+cd(fx_n, Tx_n, u)\\ +cd(Tx_{n+1}, Tx_n, u)\end{array}\right]\right)\end{array}\right)$$

$$= F\left(\begin{array}{c}\psi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +c[d(fx_n, Tx_n, u)+d(fx_{n+1}, Tx_{n+1}, u)]\end{array}\right]\right),\\ \varphi\left(\frac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\}\\ +c[d(fx_n, Tx_n, u)+d(fx_{n+1}, Tx_{n+1}, u)]\end{array}\right]\right)\end{array}\right)$$

$$\leq \psi\left(\frac{1}{a+b+c+h}\begin{bmatrix} a\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ + b\max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ + c[d(fx_n, Tx_n, u) + d(fx_{n+1}, Tx_{n+1}, u)] \end{bmatrix}\right). \tag{4}$$

Suppose that, for some n, $d(fx_{n+1}, Tx_{n+1}, u) > d(fx_n, Tx_n, u)$, then from (3), we have

$$\psi(d(fx_{n+1}, Tx_{n+1}, u))$$

$$= \psi(d(Tx_n, Tx_{n+1}, u))$$

$$\leq F\left(\begin{array}{l} \psi\left(\frac{1}{a+b+c+h}\begin{bmatrix} ad(fx_{n+1}, Tx_{n+1}, u) + bd(fx_{n+1}, Tx_{n+1}, u) \\ + c[d(fx_{n+1}, Tx_{n+1}, u) + d(fx_{n+1}, Tx_{n+1}, u)] \end{bmatrix}\right), \\ \varphi\left(\frac{1}{a+b+c+h}\begin{bmatrix} ad(fx_{n+1}, Tx_{n+1}, u) + bd(fx_{n+1}, Tx_{n+1}, u) \\ + c[d(fx_{n+1}, Tx_{n+1}, u) + d(fx_{n+1}, Tx_{n+1}, u)] \end{bmatrix}\right) \end{array}\right)$$

$$\leq \psi\left(\frac{1}{a+b+c+h}\begin{bmatrix} ad(fx_{n+1}, Tx_{n+1}, u) + bd(fx_{n+1}, Tx_{n+1}, u) \\ + c[d(fx_{n+1}, Tx_{n+1}, u) + d(fx_{n+1}, Tx_{n+1}, u)] \end{bmatrix}\right)$$

$$= \psi\left(\frac{a+b+2c}{a+b+c+h} d(fx_{n+1}, Tx_{n+1}, u)\right)$$

$$\leq \psi(d(fx_{n+1}, Tx_{n+1}, u)).$$

So,

$$\psi\left(\frac{a+b}{a+b+c+h} d(fx_{n+2}, fx_{n+1}, u)\right) = 0,$$

or

$$\varphi\left(\frac{a+b}{a+b+c+h} d(fx_{n+2}, fx_{n+1}, u)\right) = 0,$$

thus $d(fx_{n+2}, fx_{n+1}, u) = 0$, a contradiction. Hence we must have, $d(fx_{n+1}, Tx_{n+1}, u) \leq d(fx_n, Tx_n, u)$, or equivalently,

$$d(Tx_n, Tx_{n+1}, u) \leq d(Tx_{n-1}, Tx_n, u). \tag{5}$$

On applying inequality (2) again and evaluating a, b, c at (x_{n-1}, x_n), we have

$$\psi(d(y_n, y_{n+1}, u))$$

$$= \psi(d(Tx_{n-1}, Tx_n, u))$$

$$\leq F\left(\begin{array}{c}\psi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_{n-1}, fx_n, u), d(fx_n, Tx_{n-1}, u)\} \\ +b\max\{d(fx_{n-1}, Tx_{n-1}, u), d(fx_n, Tx_n, u), d(fx_n, Tx_{n-1}, u)\} \\ +cd(fx_{n-1}, Tx_n, u)+hd(fx_n, Tx_{n-1}, u)\end{array}\right]\right), \\ \varphi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(fx_{n-1}, fx_n, u), d(fx_n, Tx_n, u)\} \\ +b\max\{d(fx_{n-1}, Tx_{n-1}, u), d(fx_n, Tx_n, u), d(fx_n, Tx_{n-1}, u)\} \\ +cd(fx_{n-1}, Tx_n, u)+hd(fx_n, Tx_{n-1}, u)\end{array}\right]\right)\end{array}\right)$$

$$= F\left(\begin{array}{c}\psi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\} \\ +b\max\{d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\} \\ +cd(Tx_{n-2}, Tx_n, u)\end{array}\right]\right), \\ \varphi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}a\max\{d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\} \\ +b\max\{d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\} \\ +cd(Tx_{n-2}, Tx_n, u)\end{array}\right]\right)\end{array}\right)$$

$$= F\left(\begin{array}{c}\psi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}ad(Tx_{n-2}, Tx_{n-1}, u)+bd(Tx_{n-2}, Tx_{n-1}, u) \\ +cd(Tx_{n-2}, Tx_n, u)\end{array}\right]\right), \\ \varphi\left(\dfrac{1}{a+b+c+h}\left[\begin{array}{c}ad(Tx_{n-2}, Tx_{n-1}, u)+bd(Tx_{n-2}, Tx_{n-1}, u) \\ +cd(Tx_{n-2}, Tx_n, u)\end{array}\right]\right)\end{array}\right).$$

On applying inequality (2) again and using (3), (5) and by triangular inequality, we get

$$\psi(d(Tx_{n-2}, Tx_n, u))$$

$$\leq F\left(\begin{array}{c}\psi\left(\dfrac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\{d(fx_{n-2}, fx_n, u), d(fx_n, Tx_n, u)\} \\ +\bar{b}\max\{d(fx_{n-2}, Tx_{n-2}, u), d(fx_n, Tx_n, u), d(fx_n, Tx_{n-2}, u)\} \\ +\bar{c}[d(fx_{n-2}, Tx_n, u)+\bar{h}d(fx_n, Tx_{n-2}, u)]\end{array}\right]\right), \\ \varphi(\cdots)\end{array}\right)$$

$$= F\left(\begin{array}{c}\psi\left(\dfrac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\{d(Tx_{n-3}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\} \\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u), d(Tx_{n-1}, Tx_{n-2}, u)\} \\ +\bar{c}[d(Tx_{n-3}, Tx_n, u)+d(Tx_{n-1}, Tx_{n-2}, u)]\end{array}\right]\right), \\ \varphi(\cdots)\end{array}\right)$$

$$\leq F\left(\begin{array}{c}\psi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\left\{\begin{array}{c}d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1})\\ +d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-2}, Tx_{n-1}, u),\\ d(Tx_{n-1}, Tx_n, u)\end{array}\right\}\\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u), d(Tx_{n-1}, Tx_{n-2}, u)\}\\ +\bar{c}[d(Tx_{n-3}, Tx_{n-2}, Tx_n)+d(Tx_{n-3}, Tx_{n-2}, u)\\ +d(Tx_{n-2}, Tx_n, u)+d(Tx_{n-1}, Tx_{n-2}, u)]\end{array}\right]\right),\\ \varphi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\left\{\begin{array}{c}d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1})\\ +d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-2}, Tx_{n-1}, u),\\ d(Tx_{n-1}, Tx_n, u)\end{array}\right\}\\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u), d(Tx_{n-1}, Tx_{n-2}, u)\}\\ +\bar{c}[d(Tx_{n-3}, Tx_{n-2}, Tx_n)+d(Tx_{n-3}, Tx_{n-2}, u)\\ +d(Tx_{n-2}, Tx_n, u)+d(Tx_{n-1}, Tx_{n-2}, u)]\end{array}\right]\right)\end{array}\right)$$

$$\leq F\left(\begin{array}{c}\psi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\left\{\begin{array}{c}d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1})+d(Tx_{n-3}, Tx_{n-2}, u)\\ +d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\end{array}\right\}\\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u), d(Tx_{n-1}, Tx_{n-2}, u)\}\\ +\bar{c}[d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1})+d(Tx_{n-3}, Tx_{n-1}, Tx_n)\\ +d(Tx_{n-2}, Tx_{n-1}, Tx_n)+d(Tx_{n-3}, Tx_{n-2}, u)]\end{array}\right]\right),\\ \varphi(\cdots)\end{array}\right)$$

$$= F\left(\begin{array}{c}\psi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\{d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-2}, Tx_{n-1}, u), d(Tx_{n-1}, Tx_n, u)\}\\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u), d(Tx_{n-1}, Tx_{n-2}, u)\}\\ +\bar{c}[d(Tx_{n-3}, Tx_{n-1}, Tx_n)+d(Tx_{n-3}, Tx_{n-2}, u)\\ +d(Tx_n, Tx_{n-1}, u)+d(Tx_{n-2}, Tx_{n-1}, u)]\end{array}\right]\right),\\ \varphi(\cdots)\end{array}\right)$$

$$\leq F\left(\begin{array}{c}\psi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}\left[\begin{array}{c}\bar{a}\max\{2d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-3}, Tx_{n-2}, u)\}\\ +\bar{b}\max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-3}, Tx_{n-2}, u)\}\\ +\bar{c}[d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-3}, Tx_{n-2}, u)\\ +d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-3}, Tx_{n-2}, u)+d(Tx_{n-3}, Tx_{n-1}, Tx_n)]\end{array}\right]\right),\\ \varphi(\cdots)\end{array}\right)$$

$$\leq F\left(\psi\left(\frac{2(\bar{a}+\bar{b}+\bar{c})-\bar{b}}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}d(Tx_{n-3}, Tx_{n-2}, u)\right), \varphi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}d(Tx_{n-3}, Tx_{n-2}, u)\right)\right)$$

$$\leq F\left(\psi\left(\frac{2-\bar{b}}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}(Tx_{n-3}, Tx_{n-2}, u)\right), \varphi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}}d(Tx_{n-3}, Tx_{n-2}, u)\right)\right).$$

This implies that

$$\psi(d(Tx_{n-2}, Tx_n, u)) \leq F\left(\psi\left(\frac{2-\bar{b}}{\bar{a}+\bar{b}+\bar{c}+\bar{h}} d(Tx_{n-3}, Tx_{n-2}, u)\right),\right.$$

$$\left.\varphi\left(\frac{1}{\bar{a}+\bar{b}+\bar{c}+\bar{h}} d(Tx_{n-3}, Tx_{n-2}, u)\right)\right)$$

$$\leq \psi\left(\frac{2-\bar{b}}{\bar{a}+\bar{b}+\bar{c}+\bar{h}} d(Tx_{n-3}, Tx_{n-2}, u)\right),$$

where $\bar{a}, \bar{b}, \bar{c}$ are evaluated at (x_{n-2}, x_n).

At the bottom line of the above inequality, $d(Tx_{n-3}, Tx_{n-1}, Tx_n) = 0.$

Because, let $d(Tx_{n-3}, Tx_{n-1}, Tx_n) \neq 0,$ then applying (4), we get

$$\begin{aligned} d(Tx_{n-3}, Tx_{n-1}, Tx_n) &= d(Tx_{n-1}, Tx_n, Tx_{n-3}) \\ &\leq a \max\{d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}), d(fx_n, Tx_n, Tx_{n-3})\} \\ &\quad + b \max\{d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}), d(fx_n, Tx_n, Tx_{n-3})\} \\ &\quad + c[d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}) + d(fx_n, Tx_n, Tx_{n-3})] \\ &\leq a \max\{d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}), d(Tx_{n-1}, Tx_n, Tx_{n-3})\} \\ &\quad + b \max\{d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}), d(Tx_{n-1}, Tx_n, Tx_{n-3})\} \\ &\quad + c[d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}) + d(Tx_{n-1}, Tx_n, Tx_{n-3})] \\ &= (a+b+c) d(Tx_{n-1}, Tx_n, Tx_{n-3}) \\ &< d(Tx_{n-1}, Tx_n, Tx_{n-3}). \end{aligned}$$

A contradiction. Thus, $d(Tx_{n-3}, Tx_{n-1}, Tx_n) = 0.$

Using (5), (5) and (6), we get

$$\begin{aligned} &d(Tx_{n-1}, Tx_n, u) \\ &= d(y_n, y_{n+1}, u) \\ &\leq ad(Tx_{n-2}, Tx_{n-1}, u) + bd(Tx_{n-2}, Tx_{n-1}, u) + c[(2-\bar{b}) d(Tx_{n-3}, Tx_{n-2}, u)] \end{aligned}$$

$$\leq ad(Tx_{n-3}, Tx_{n-2}, u) + bd(Tx_{n-3}, Tx_{n-2}, u) + c(2 - \overline{b})d(Tx_{n-3}, Tx_{n-2}, u)$$

$$= (a + b + 2c)d(Tx_{n-3}, Tx_{n-2}, u) - \overline{b}cd(Tx_{n-3}, Tx_{n-2}, u)$$

$$\leq (1 - \overline{b}c)d(Tx_{n-3}, Tx_{n-2}, u)$$

$$\leq (1 - \beta\gamma)d(Tx_{n-3}, Tx_{n-2}, u)$$

$$\leq (1 - \beta\gamma)^{\frac{n}{2}} d(y_0, y_1, u).$$

Hence $\{y_n\}$ is a Cauchy sequence.

For case (a) and (b), suppose that X is complete. Then Cauchy sequence $\{y_n\}$ will converge to a point p in X.

Case (a): Since f is surjective, there exists a point z in X such that $p = fz$.

Now applying inequality (2), we get

$$d(fz, Tz, u)$$

$$\leq d(fz, y_{n+1}, u) + d(fz, Tz, y_{n+1}) + d(Tz, y_{n+1}, u)$$

$$\leq d(fz, y_{n+1}, u) + d(fz, Tz, y_{n+1}) + d(Tx_n, Tz, u)$$

$$\leq d(fz, y_{n+1}, u) + d(fz, Tz, y_{n+1}) + a(x, y)\max\{d(fx_n, fz, u), d(fz, Tz, u)\}$$

$$+ b(x, y)\max\{d(fx_n, Tx_n, u), d(fz, Tz, u), d(fz, Tx_n, u)\}$$

$$+ c(x, y)[d(fx_n, Tz, u) + d(fz, Tx_n, u)]$$

$$\leq \sup_{x, y \in X} \{a(x, y) + c(x, y)\} \cdot \max\{\max\{d(fx_n, fz, u), d(fz, Tz, u)\}, d(fz, fx_{n+1}, u)\}$$

$$+ \sup_{x, y \in X} \{b(x, y) + c(x, y)\} \cdot \max\left\{\begin{matrix} \max\{d(fx_n, fx_{n+1}, u), d(fz, Tz, u), d(fz, fx_{n+1}, u)\}, \\ d(fx_n, Tz, u) + d(fz, fx_{n+1}, u) \end{matrix}\right\}.$$

Taking the limit as $n \to \infty$, we have

$$d(fz, Tz, u) \leq \sup x, y \in X(b + c)d(fz, Tz, u) < d(fz, Tz, u),$$

implies that $fz = Tz$.

Case (b): Since f is continuous and f, T are compatible, we have

$$\lim_{n\to\infty} fy_n = fp, \ \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_{n+1} = p,$$

and hence

$$\lim_{n\to\infty} d(fTx_n, Tfx_n, u) = 0. \tag{7}$$

By above results, we have

$$\begin{aligned}
& d(fp, Tp, u) \\
& \leq d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) + d(fy_{n+1}, u, Tp) \\
& \leq d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) + d(Tp, Tfx_n, u) \\
& \leq d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) \\
& \quad + a\max\{d(ffx_n, fp, u), d(fp, Tp, u)\} \\
& \quad + b\max\{d(ffx_n, Tfx_n, u), d(fp, Tp, u), d(fp, Tfx_n, u)\} \\
& \quad + c[d(ffx_n, Tp, u) + d(fp, Tfx_n, u)] \\
& \leq d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) \\
& \quad + \sup_{x, y\in X}[a(x, y) + b(x, y) + c(x, y)](\max\left\{\begin{matrix} d(ffx_n, fp, u), d(fp, Tp, u), \\ d(ffx_n, Tfx_n, u), d(fp, Tp, u), \\ d(fp, Tfx_n, u) \end{matrix}\right\} \\
& \quad + d(ffx_n, Tfx_n, u), d(fp, Tp, u), d(fp, Tfx_n, u)).
\end{aligned}$$

Now we have

$$d(ffx_n, Tfx_n, u) \leq d(ffx_n, fTx_n, u) + d(fTx_n, Tfx_n, u) + d(ffx_n, Tfx_n, Tfx_n).$$

Using the continuity of f and the compatibility of f and T, it follows that

$$\lim_{n\to\infty} d(ffx_n, Tfx_n, u) = 0, \quad \lim_{n\to\infty} d(ffx_n, fTx_n, u) = 0. \tag{8}$$

$\lim_{n\to\infty} ffx_n = fp$, implies that $\lim_{n\to\infty} Tfx_n = fp$.

Taking limit as $n \to \infty$ and using the inequality (6) and (7), we get

$$d(fp, Tp, u) \leq \sup_{x, y \in X} [a(x, y) + b(x, y) + c(x, y)] d(fp, Tp, u), \text{ implies that } fp = Tp.$$

Case (c): In this case, $p \in f(X)$. Let $z \in f^{-1}p$, then $p = fz$, and the proof is completed by Case (a).

To establish uniqueness, suppose that q is another coincidence point of f and T. Then from (2) with a, b, c evaluated at (p, q), we have

$$\begin{aligned} d(Tp, Tq, u) &\leq a \max\{d(fp, fq, u), d(fp, Tq, u)\} \\ &\quad + b \max\{d(fp, Tp, u), d(fq, Tq, u), d(fq, Tp, u)\} \\ &\quad + c[d(fp, Tq, u) + d(fq, Tp, u)] \\ &\leq (a + b + 2c) d(Tp, Tq, u). \end{aligned}$$

Hence $Tp = Tq$.

Corollary 2.1. *Let (X, d) be a complete 2-metric space and T be a self map of X satisfying* (2) *with $f = I$, the identity mapping on X. Then T has a unique fixed point and at this fixed point T is continuous.*

Proof. The existence and uniqueness of the fixed point comes from Theorem 2.1 by setting $f = I$. To prove continuity at the unique fixed point p, we apply inequality (2), where a, b, c are evaluated at (y_n, p).

$$\begin{aligned} d(Ty_n, p, u) &= d(Ty_n, Tp, u) \\ &\leq a \max\{d(y_n, p, u), d(p, Tp, u)\} \\ &\quad + b \max\{d(y_n, Ty_n, u), d(p, Tp, u), d(p, Ty_n, u)\} \\ &\quad + c[d(y_n, Tp, u) + d(p, Ty_n, u)]. \end{aligned}$$

Taking limit as $n \to \infty$ yields

$$\lim_{n \to \infty} d(Ty_n, p, u) \leq (b + c) \lim_{n \to \infty} d(p, Ty_n, u) < \lim_{n \to \infty} d(p, Ty_n, u).$$

a contradiction. Therefore, $\lim_{n \to \infty} Ty_n = p = Tp$.

Remark 2.1. Our condition (2) includes condition (1) of [7] if we define, with $f = I$ the identity mapping,

$$m(x, y, u) = \max\left\{d(x, y, u), d(x, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\right\}.$$

For each $x, y \in X$ such that

$$m(x, y, u) = \max\{d(x, Tx, u), d(y, Ty, u)\},$$

define $a(x, y) = 0, b(x, y) = a + b, c(x, y) = c.$

For each $x, y \in X$ such that

$$m(x, y, u) = \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)],$$

define $a(x, y) = 0, b(x, y) = b, c(x, y) = a + 2c.$

Hence our Theorem 2.1 is a proper generalization of [7].

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